SOLVING FOR THE

EFFICIENT FRONTIER

Fall 2002

Recall T is point on the efficient frontier in most counter-clockwise directions. Equation of line Tangent at T is:

$$\bar{R}_i = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_i$$

Moving in most counter-clockwise directions is equivalent to maximizing slope. Thus, the objective is to find portfolio where:

$$\theta = \frac{\bar{R}_T - R_F}{\sigma_T}$$

is maximum.

$$\Theta = \frac{\Sigma X_i \bar{R}_i - 1 R_F}{\left[\Sigma X_i^2 \sigma_i^2 + \Sigma\Sigma X_i X_j \sigma_{ij}\right]^{1/2}}$$

Subject to $\Sigma X_i = 1$

$$\Theta = \frac{\Sigma X_i \bar{R}_i - \left[\Sigma X_i\right] R_F}{\left[\Sigma X_i^2 \sigma_i^2 + \Sigma\Sigma X_i X_j \sigma_{ij}\right]^{1/2}}$$

$$= \frac{\Sigma X_i \left(\bar{R}_i - R_F\right)}{\left[\Sigma X_i^2 \sigma_i^2 + \Sigma\Sigma X_i X_j \sigma_{ij}\right]^{1/2}}$$

Θ can be written as:

$$\Theta=\left(\overline{R}_T-R_F\right)\left(\sigma^2\right)^{-1/2}$$

recall how to take a derivative using the chain rule namely:

$$d\Theta=f\left(X,Y\right)=XdY+YdX$$

and that

$$dX^N=NX^{N-1}$$

thus

$$\frac{d\Theta}{dX_k} = \left(\bar{R}_T - R_F\right)\left(-\tfrac{1}{2}\right)\left(\sigma^2\right)^{-3/2}\frac{d\sigma^2}{dX_k} + \frac{d\left[\bar{R}_T - R_F\right]\left(\sigma^2\right)^{-1/2}}{dX_k}$$

or

$$\frac{d\Theta}{dX_k} = \left(\bar{R}_T - R_F\right)\left(-\tfrac{1}{2}\right)\left(\sigma^2\right)^{-3/2}\left[2X_k\sigma_k^2 + 2\sum_{k\neq i} X_k\sigma_{ik}\right] + \left(\bar{R}_k - R_F\right)\left(\sigma^2\right)^{-1/2}$$

$$\frac{d\Theta}{dX_k} = -\frac{\overline{R}_T - R_F}{\sigma^2}\left[X_k\sigma_k^2 + \sum_{k\neq i} X_k\sigma_{ik}\right] + \left(\overline{R}_k - R_F\right)$$

$$\left(\overline{R}_k - R_F\right) = \lambda\left[X_k\sigma_k^2 + \sum_{k\neq i} X_k\sigma_{ik}\right]$$

where:

$$\lambda = \frac{\overline{R}_T - R_F}{\sigma^2}$$

First order conditions

$$\bar{R}_k - R_F = \lambda\left[X_1\sigma_{1k} + X_2\sigma_{2k} + \ldots X_k\sigma_k^2 + \ldots + X_N\sigma_{Nk}\right]$$

for all i

where $$\lambda = \frac{\bar{R}_T - R_F}{\sigma_T^2}$$

define $Z_i = \lambda X_i$ **then**

$$\bar{R}_k - R_F = Z_1\sigma_{ik} + Z_2\sigma_{2k} + \ldots Z_k\sigma_k^2 + \ldots + Z_N\sigma_{Nk}$$ **all k**

Example 3 securities:

$$\bar{R}_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{21} + Z_3\sigma_{31}$$

$$\bar{R}_2 - R_F = Z_1\sigma_{21} + Z_2\sigma_2^2 + Z_3\sigma_{32}$$

$$\bar{R}_3 - R_F = Z_1\sigma_{31} + Z_2\sigma_{23} + Z_3\sigma_3^2$$

X's are $$X_i = \frac{Z_i}{\Sigma_j Z_j}$$

$$\overline{R} = \begin{pmatrix} \overline{R}_1 \\ \overline{R}_2 \\ \overline{R}_3 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

$$\overline{R} - R_F 1 = \Sigma Z$$

and

$$Z = \Sigma^{-1}\overline{R} + R_F \Sigma^{-1} 1$$

Typical Element:

$$Z_i = C_0 + C_1 R_F$$

Thus

Short Sales Not Allowed

Minimize σ^2

Subject to:

1. $\Sigma X_i = 1$

2. $\Sigma X_i \bar{R}_i = \bar{R}_p$

3. $\Sigma X_i \geq 0$

Quadratic Programming Problem

Since have squared and cross product terms.