OPTION VALUATION

September 1999

Essentially there are two models for pricing options

a. **Black Scholes Model**
b **Binomial option Pricing Model**

→For equities, usual model is Black Scholes. For most bond options there are problems that eliminate the Black Scholes model from consideration.

1. **Recall that a bond's volatility is a function of Duration. The Volatility is generally considered to be a direct function of Duration. As time passes Duration declines. But B.S. assumes constant volatility.**

2. **Black Scholes assumes the evolution of stock prices is a stationary process. But bond prices must converge to par at maturity so process must change.**

3. **Black Scholes assumes a constant short rate. But assuming a constant risk-free rate and simultaneously that long rates changes does not make much sense.**

→Where are the Black Scholes assumptions not badly violated:

1. **A short term option on a long instrument since duration will not change very much over the life of the option....**

2. **An option on a future. In this case the deliverable instrument is a "constant" maturity bond whose duration is fairly stable.**

In these two cases Black Scholes can be made to work well. For future one uses a variation in Black Scholes formula called Black Model. In other cases a binomial model is needed.

There are a lot of binomial models. There are a number of ways we can model changes in interest rates or discount functions. The basic characteristics that drive these models are:

1. **That they be arbitrage free or "No free lunch." However we assume interest rates or discount function evolve, we can't find a strategy that always has a higher return no matter what.**

2. **No memory. An up movement followed by a down movement is the same as a down movement followed by an up movement, e.g., how we get somewhere is unimportant. Reason for this assumption is computational. It results in manageable problems.**

The Black Scholes Model

$$C = S\,N(d_1) - Ee^{-rt}\,N(d_2)$$

$$d_1 = \frac{\ln(S_0/E) + (r + 1/2\sigma^2)t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{\ln(S_0/E) + (r - 1/2\sigma^2)t}{\sigma\sqrt{t}}$$

S_0 = Current price of security
E = Exercise Price
r = interest rate (continuously compounded)
t = time to maturity in fraction of year
σ = Standard deviation of returns (continuously compounded)
N( ) = Cumulative Normal

(Black model is same as above except S_0 is replaced by Futures price times e^{-rt}.)

Some Definitions

1. Hedge ratio $\delta = \frac{\partial C}{\partial S} = N(d_1) > 0$

2. $\gamma = \frac{\partial^2 c}{\partial^2 s} = \frac{\frac{1}{S}}{\sigma \sqrt{t}} n(d_1)$

3. Put is valued using Put Call Parity

BINOMIAL OPTION PRICING

(SINGLE STATE MODELS)

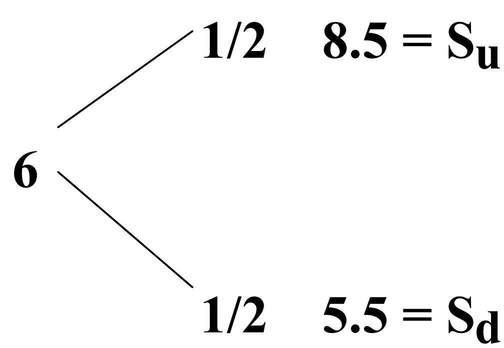
EXAMPLE

ASSUME

$R_{01} = 6\%$

$R_{02} = 7\%$

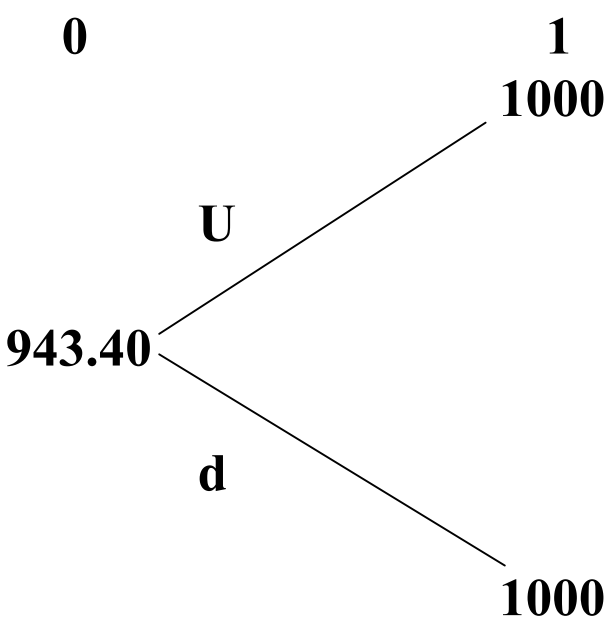
Assume one period rate can evolve to



ZEROS and their PRICES

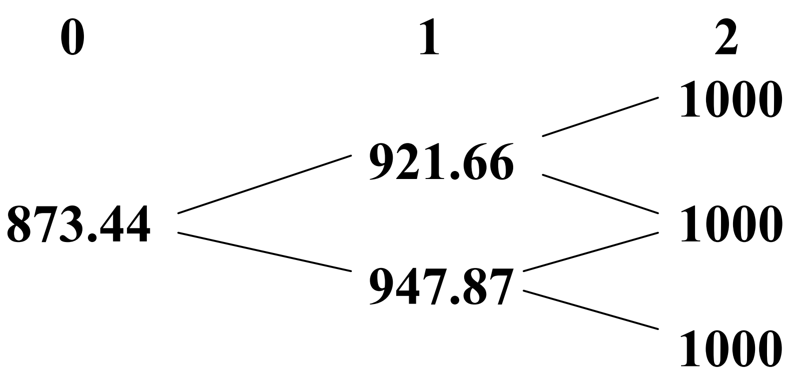
One Period

Price	S_u	S_d
$\frac{1000}{1.06}$	1000	1000



Two Period

Price	S_u	S_d
$\frac{1000}{(1.07)^2}$	$\frac{1000}{1.085}$	$\frac{1000}{1.055}$



NOTE

1. **Earlier mentioned Binomial option has to have "no memory."
   This is why tree has three points at time 2.**

2. **Check on "no free lunch."**

   873.44 x 1.06 = $925.85

   Thus if investor buys two period bond can be better or worse than one period.

3. **Note [½ (921.66) + ½ (947.87)]/1.06 (e.g., present value of bond prices at 1 does not equal $873.44). There is a risk premium.**

Pricing option using replication

Consider option to buy one year zero next year for $930

Next year price is $921.66 or $947.87

0

$17.87

Question: What is value?

Can buy at time zero a two year pure discount and a one year pure discount.

Let

X_1 = fraction of one year pure discount

X_2 = fraction of two year pure discount

Construct portfolio with same payoff as option

At time one a then one year zero will be worth either \$921.66 or \$947.87 and thus this will be the value of a time zero two period pure discount instrument

$$X_1\,(1000) + X_2\,(921.66) = 0$$

$$X_1\,(1000) + X_2\,(947.87) = 17.87$$

Solving

$$X_1 = -.62839$$

$$X_2 = +.68180$$

The price of this portfolio is

$$(-.62839) \times (943.40) + (.68180)\,(873.44) = \$2.69$$

Since this is equivalent it should cost the same or have arbitrage.

NOTE

1. **Probabilities were never used in valuing option.**

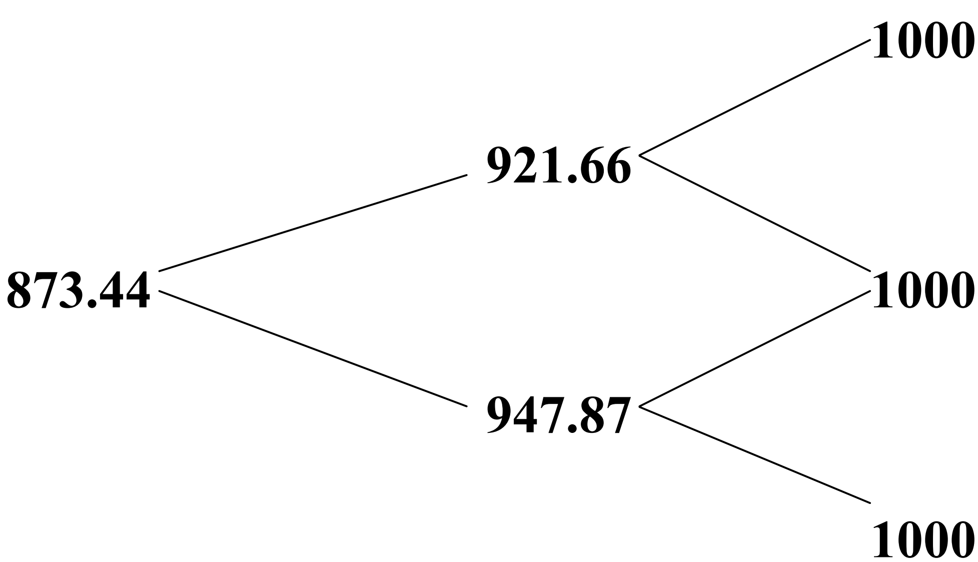
2. **Logically, if up movement was more likely option would be worth less so probabilities must enter indirectly. If up movement is more likely bond would be worth less and working through analysis to obtain option value would result in a lower value.**

3. **Preceding was used to value call, but could be used to value any instrument whose value depended on interest rate movements.**

Alternative way of pricing option
(risk neutral)

The basic idea of risk neutral price is as follows. We did not use characteristics of investors (accept prefer more to less) in deriving valuation. Therefore must hold for all investors. One group that is convenient is risk neutral. The advantage of using this group is they only care about expected value.

Recall tree for two year bond

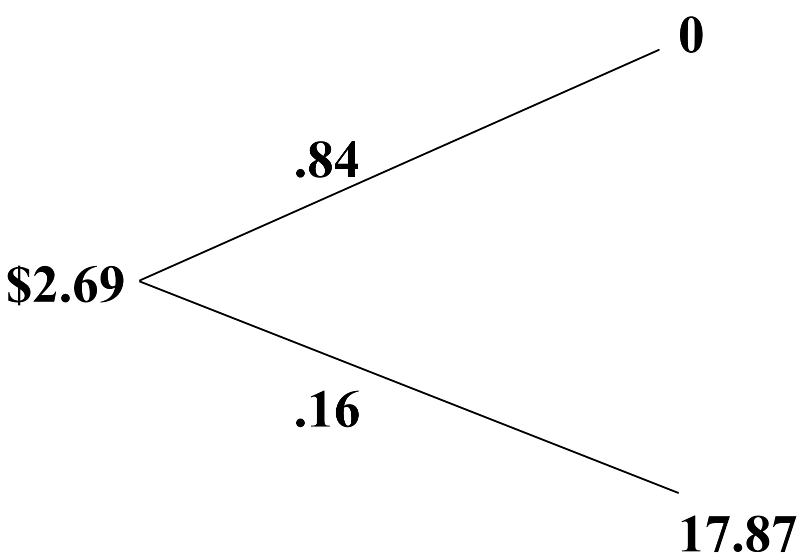


What probabilities will make expected value at one; give a price of $873.44 at zero

$$\frac{\$921.6\ P_1 + 947.87(1 - P_1)}{1.06} = \$873.44$$

$$P_1 = .84$$

Now consider option



If risk neutral value is

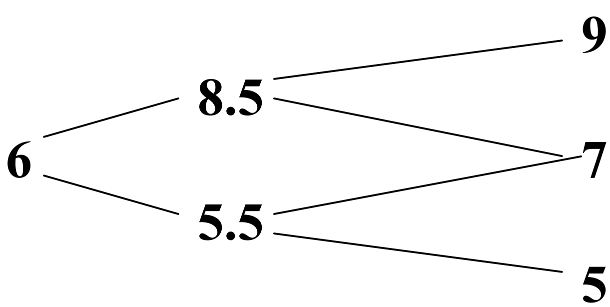
$$\frac{(.84)\ \ 0 + .16(17.87)}{(1.06)} = \$2.69$$

Assume

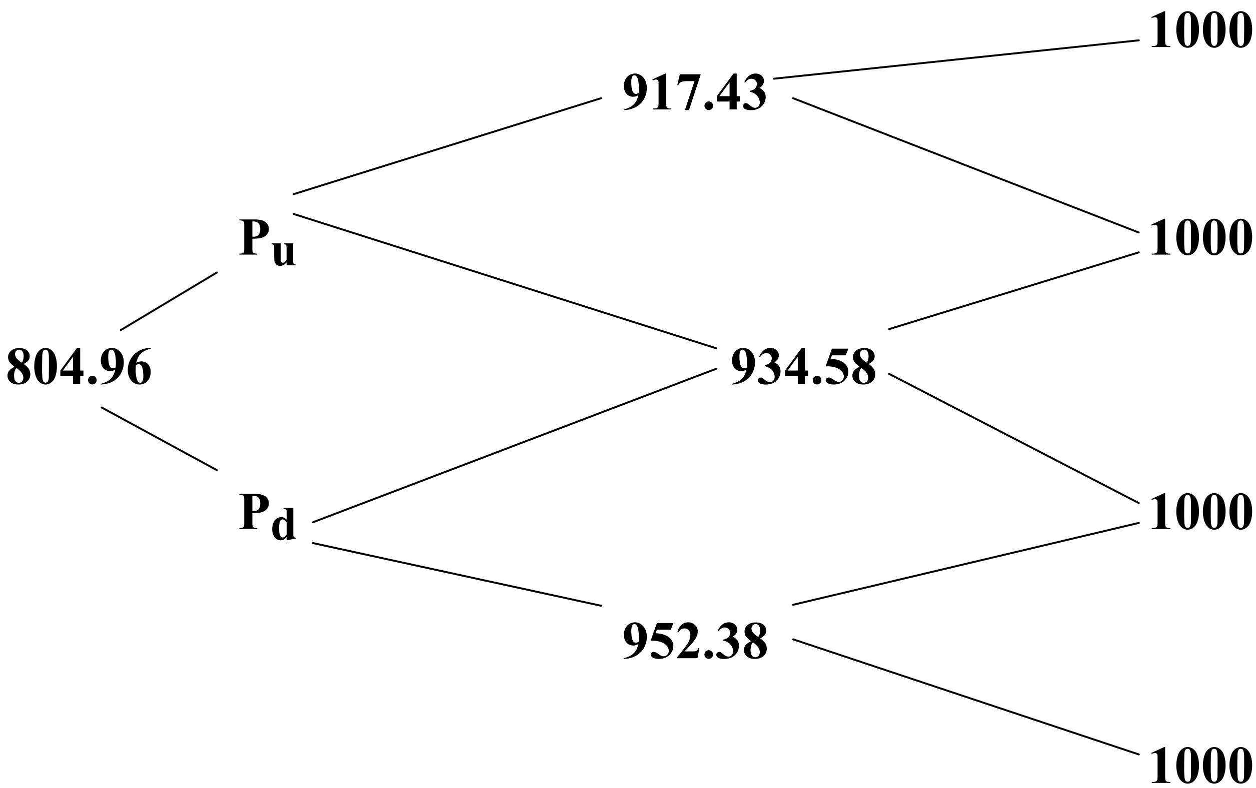
$$r_{003} = 7.5\%$$

0 1 2

Rates



Prices

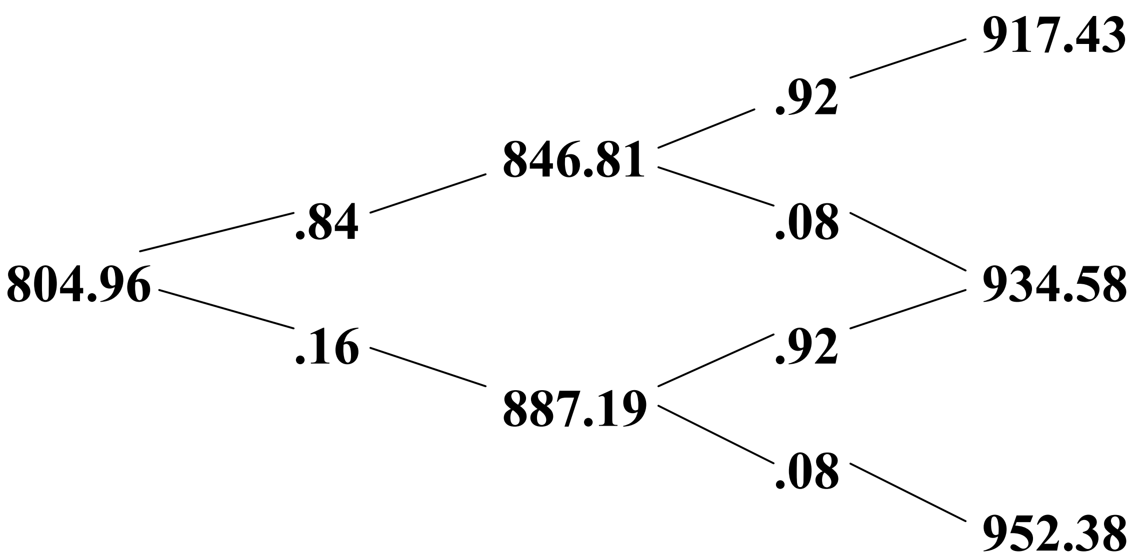


Solving for risk neutral probabilities

$$P_u = \frac{[P(917.43) + (1 - P)(934.58)]}{(1.085)}$$

$$P_d = \frac{[P(934.58) + (1 - P)(952.38)]}{(1.055)}$$

$$804.96 = \frac{.84\,P_u + .16\,P_d}{(1.06)}$$



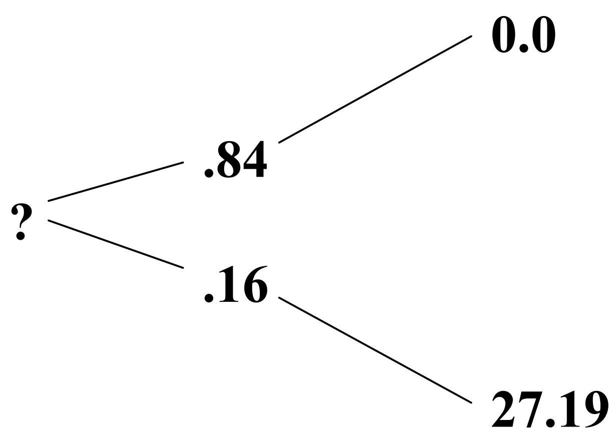
Note:

1. **One factor model. Everything depends on evolution of six-month rates.**

2. **Can change frequency of up and down to month or day.**

Examples

1. **Consider option to buy two year zero in one year at $860.**



$$\frac{27.19 \text{ x } .16}{1.06} = \$4.10$$

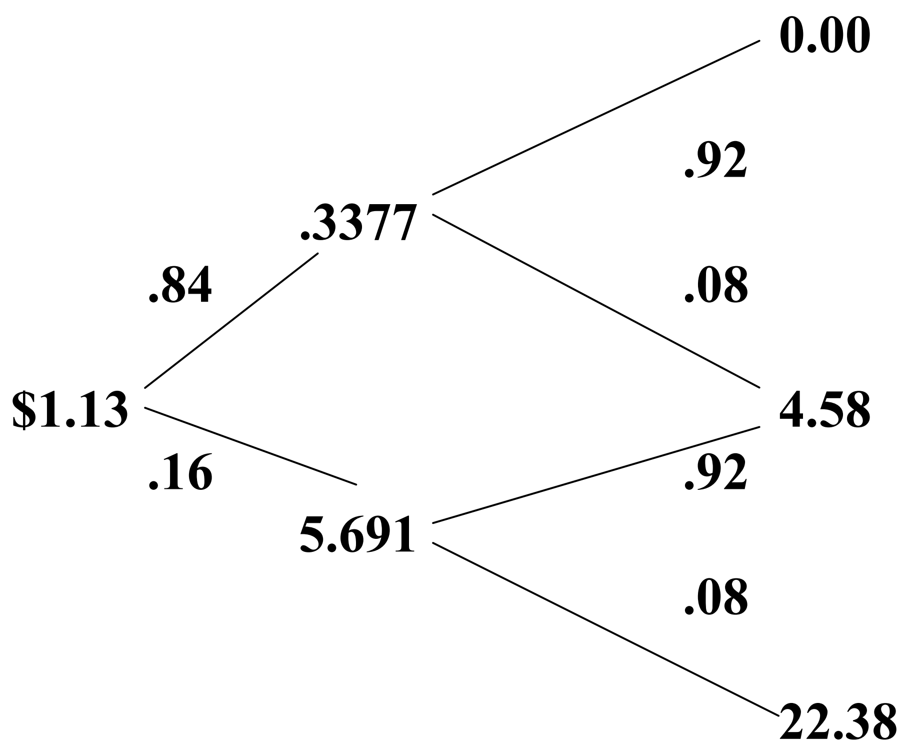
$$x_1 \, 1000 + x_2 \, 921.66 = 0$$

$$x_1 \, 1000 + x_2 \, 947.87 = 27.19$$

$$x_2 \quad 1.03739$$

$$x_1 = -.9561212$$

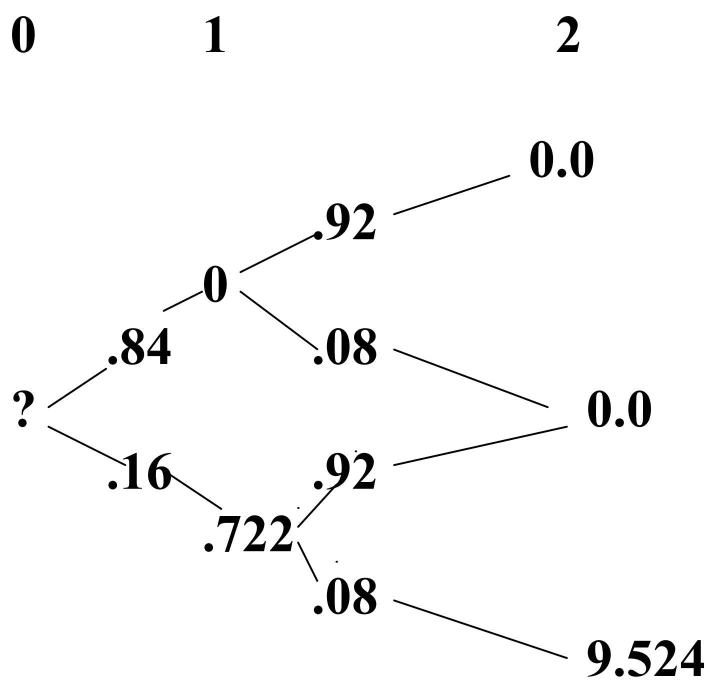
2. Consider option to buy one year zero in two years at $930.



$$\frac{4.58 \times .08}{1.085} = .3377$$

$$\frac{4.58 \times .92 + 22.38 \times .08}{1.055} = 5.691$$

3. Consider a 3 year bond callable at 1000 paying a coupon of 6% with call protection until time 2.



value = .109

Non callable

$$P = \frac{60}{1.06} + \frac{60}{(1.07)^2} + \frac{1060}{(1.075)^3}$$

P = $962.27

Price = 962.27 - .109 = 962.161

Caps, Floors and Collars

- **issued with floating rate note or floating rate mortgages**

- **cap fixes a maximum interest rate, e.g., floating rate note can't exceed 8%**

- **cap is like a call option, e.g., an 8% cap on 6-month LIBOR can be valued as if cap pays (actual LIBOR - 8%) x (days in period / 360)**

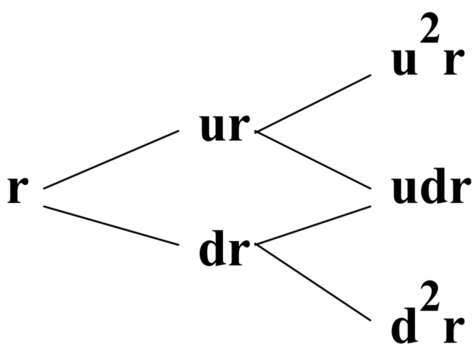
$$\text{(actual LIBOR} - 8\%) \times \left(\frac{\text{days in period}}{360}\right)$$

- **floor fixes minimum rate, like a put option**

- **collar fixes both maximum and minimum rate**

<u>Need Process of Spot changes</u>

- **Must be consistent with current rates**
- **Arbitrage free**



one choice original Solomon Model

$$u = e^{m\,t + \sigma\sqrt{t}}$$

$$d = e^{m\,t - \sigma\sqrt{t}}$$

Note:

1. Order of up and down doesn't matter

$$ud = e^{m1+\sigma} \times e^{m2-\sigma} = e^{m1+m2}$$

$$du = e^{m1-\sigma} \times e^{m2+\sigma} = e^{m1+m2}$$

They fix

(1) Probabilities = 1/2
(2) σ which is volatility of short term rate

Note:

1. **Negative rates can't occur.**

2. **Rate changes proportional to level.**

3. **Does not account for term structure of volatility.**

Assume observe following rates

$r_{001} = 10$ $\quad$ $P_1 = 909.09$

$r_{002} = 10\ 1/2$ $\quad$ $P_2 = 818.98$

$r_{003} = 11$ $\quad$ $P_3 = 731.19$

1. **Estimate volatility at .2 corresponds to 20% per year and up and down equally likely.**

2. **Imply m**

$$818.98 = \frac{\left[\frac{1/2(1000)}{1 + ru} + \frac{1/2(1000)}{1 + rd}\right]}{(1.10)}$$

Solving for m_1 yields

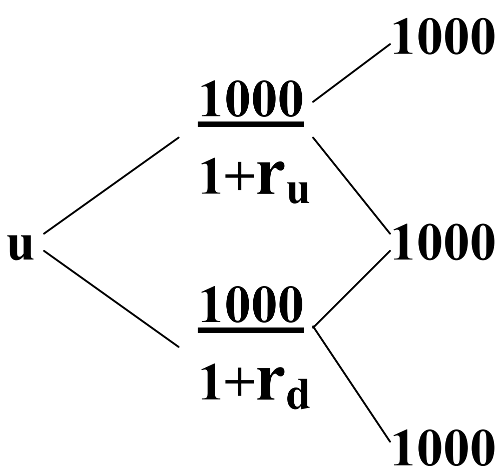
$$m_1 = .07958$$

$$r_u = .13226$$

$$r_d = \underline{.08866}$$

$$22092$$

$$1/2\ r_u + 1/2\ r_d = \text{Forward}$$



$$up = \frac{\frac{.5(1000)}{(1 + u^{2} r)} + \frac{.5(1000)}{(1 + udr)}}{1 + ur}$$

$$down = \frac{\frac{.5(l000)}{(1 + udr)} + \frac{.5(1000)}{1 + d^{2} r}}{(1 + dr)}$$

$$731.19 = \frac{.5(up) + (.5)down}{(1.10)}$$

M_2 = .1556

r_{uu} = 17.429

r_{ud} = 11.68

r_{dd} = 7.83

.1556
.<u>07958</u>
.23518

Black Derman Toy

(Goldman Sachs)

1. **Fix term structure volatility e.g. allow one year rates to vary more than two year rates etc. This is principal advantage of the model. Empirical evidence supports short rate varies more than long.**

2. **Fix probabilities**

Assume Prior Example

$r_{001} = 10$ $\quad P_1 = 909.09$

$r_{002} = 10.5 \quad \Rightarrow \quad P_2 = 818.98$

$r_{003} = 11.0$ $\quad P_3 = 731.19$

Estimate

1. $\sigma_1 = .19 \qquad \sigma_2 = .18$

2. up + down = 1/2

3. expectations theory

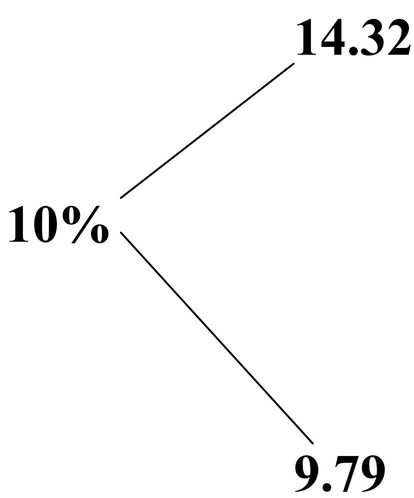
$$u_1 = e^{m_1 + \sigma_1}$$

$$d_1 = e^{m_2 - \sigma_1}$$

Calculating Second Period Spots

$$.19 = \frac{1}{2}\ln\left(\frac{u_1}{d_1}\right)$$

$$\frac{1}{(1+r002)^2} = \frac{1}{(1+r001)} \text{ x } \frac{\left(\frac{1}{(1+u_1 r)}\right)+\left(\frac{1}{(1+d_1 r}\right)}{2}$$



CALCULATES SPOT RATES IN PERIOD 3

$$\frac{1}{(1+r003)^{3}} = \frac{1}{(1+r001)} \frac{\left[\frac{1}{1+u2r}\right]+\left[\frac{1}{1+d2r}\right]}{2}$$

$$.5\ln(\frac{u2}{d2}) = .18$$

$$\frac{1}{1+u2} = \left[\frac{1}{[1+u]}\right]\left[\frac{\frac{1}{(1+uu)}+\frac{1}{(1+ud)}}{2}\right]$$

$$\frac{1}{(1+d2)} = \left[\frac{1}{(1+d)}\right] x \left[\frac{\frac{1}{(1+ud)}+\frac{1}{(1+dd)}}{2}\right]$$

$$uu.dd = (ud)^2$$

Valuing a Callable Bond

Interest rates

				25.53
			21.79	
		19.42		19.48
	14.32		16.06	
10		13.77		14.86
	9.79		11.83	
		9.76		11.34
			8.72	
				8.65

Prices of Five-Year Non-Callable 13.5 Annual Coupon Bond

					100
				90.42	
			87.20		100
		87.65		94.99	
	93.19		95.13		100
102.87		98.43		98.82	
	106.12		101.83		100
		107.59		101.94	
			107.34		100
				104.46	
					100

Value of Call Option

		102	**101**	**100**	
					0
				0	
			0		**0**
		0		**0**	
	.1667		**0**		**0**
1.311		**.382**		**0**	
	2.7176		**.8674**		**0**
		5.5861		**1.94**	
			6.3417		**0**
				4.4639	
					0

Sinking Fund Bond

		4.7
	6.4	
8		**9.6**
	11.9	
		15.5

Coupon Bond Prices for 12% coupon

			100
		106.97	
	109.57		**100**
108		**102.19**	
	99.71		**100**
		96.97	
			100

Two-Year European Option

		6.97
	4.30	
2.44		2.19
	0.979	
		0

One-Year European Option

9.57

4.44

0

Value of sinking fund
= 108 - 1/3 (2.446) + 1/3 (4.44) = 105.70

I. Terms

a. **Callable bonds**

b. **Caps**

c. **Collars**

II. Concepts

a. **Reason for use of binomial models for bonds**

b. **Put call parity**

c. **Risk neutral valuation**

d. **Pricing by replication**

III. Calculations

a. Option value using risk neutral valuation

b. Option value using street models

c. Valuing callable bond

COMPUTATION OF INTEREST RATES USING THE SOLOMON MODEL

assumputions used in computing values

$r_{01} := 0.1$ $r_{03} := 0.11$ $r_{05} := 0.12$ $s := .2$

$r_{02} := 0.105$ $r_{04} := 0.115$

pure discount prices

$p2 := \frac{100}{(1 + r_{02})^2}$ $p3 := \frac{100}{(1 + r_{03})^3}$ $p4 := \frac{100}{(1 + r_{04})^4}$ $p5 := \frac{100}{(1 + r_{05})^5}$

assumptions period 1

$x21 := 100$

$x22 := 100$

$x23 := 100$

guesses period 1

$m1 := .1$

$x11 := 95$

$x12 := 100$

solving for secound period prices and spots

given

$$x11 = \frac{.5 \cdot x21 + .5 \cdot x22}{1 + r_{01} \cdot \exp(m1 + s)}$$

$$p2 = \frac{.5 \cdot x11 + .5 \cdot x12}{1 + r_{01}}$$

$$x12 = \frac{.5 \cdot x22 + .5 \cdot x23}{1 + r_{01} \cdot \exp(m1 - s)}$$

$$\begin{pmatrix} y1 \\ y2 \\ y3 \end{pmatrix} := \text{find}(m1, x11, x12)$$

$y1 = 0.079531$

$y2 = 88.319643$

$y3 = 91.856848$

$m1 := y1$

$u := r_{01} \cdot \exp(m1 + s)$

$d := r_{01} \cdot \exp(m1 - s)$

$u = 0.134986 \quad d = 0.090484 \qquad m1 = 0.079531$

COMPUTING SPOTS AND PRICES FOR THREE PERIOD SPOTS

ASSUMPTIONS

$x31 := 100$

$x32 := 100$

$x33 := 100$

$x34 := 100$

GUESSES

$m2 := .1$

given

$$x21 = \frac{.5 \cdot x31 + .5 \cdot x32}{1 + r_{01} \cdot \exp(m2 + 2 \cdot s)}$$

$$x11 = \frac{.5 \cdot x21 + .5 \cdot x22}{1 + r_{01} \cdot \exp(m1 + s)}$$

$$x22 = \frac{.5 \cdot x32 + .5 \cdot x33}{1 + r_{01} \cdot \exp(m2)}$$

$$x12 = \frac{.5 \cdot x22 + .5 \cdot x23}{1 + r_{01} \cdot \exp(m1 - s)}$$

$$x23 = \frac{.5 \cdot x33 + .5 \cdot x34}{1 + r_{01} \cdot \exp(m2 - 2 \cdot s)}$$

$$p3 = \frac{.5 \cdot x11 + .5 \cdot x12}{1 + r_{01}}$$

$$\begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \\ y5 \\ y6 \end{bmatrix} := \text{find}(m2, x21, x22, x23, x11, x12)$$

$y1 = 0.15558$

$y2 = 85.157463$

$y3 = 89.538856$

$y4 = 92.737204$

$y5 = 77.145582$

$y6 = 83.716521$

$m2 := y1$

$uu := r_{01} \cdot \exp(m2 + 2 \cdot s)$

$ud := r_{01} \cdot \exp(m2)$

$dd := r_{01} \cdot \exp(m2 - 2 \cdot s)$

one period spots

$uu = 0.174295$

$ud = 0.116834$

$dd = 0.078316$

COMPUTES PRICES AND SPOTS FOR FOURTH PERIOD

ASSUMPTIONS

$x41 := 100$

$x42 := 100$

$x43 := 100$

$x44 := 100$

$x45 := 100$

GUESSES

$m3 := .1$

computing prices

given

$$x31 = \frac{.5 \cdot x41 + .5 \cdot x42}{1 + r_{01} \cdot \exp(m3 + 3 \cdot s)}$$

$$x32 = \frac{.5 \cdot x42 + .5 \cdot x43}{1 + r_{01} \cdot \exp(m3 + s)}$$

$$x33 = \frac{.5 \cdot x43 + .5 \cdot x44}{1 + r_{01} \cdot \exp(m3 - s)}$$

$$x34 = \frac{.5 \cdot x44 + .5 \cdot x45}{1 + r_{01} \cdot \exp(m3 - 3 \cdot s)}$$

$$x21 = \frac{.5 \cdot x31 + .5 \cdot x32}{1 + r_{01} \cdot \exp(m2 + 2 \cdot s)}$$

$$x22 = \frac{.5 \cdot x32 + .5 \cdot x33}{1 + r_{01} \cdot \exp(m2)}$$

$$x23 = \frac{.5 \cdot x33 + .5 \cdot x34}{1 + r_{01} \cdot \exp(m2 - 2 \cdot s)}$$

$$x11 = \frac{.5 \cdot x21 + .5 \cdot x22}{1 + r_{01} \cdot \exp(m1 + s)}$$

$$x12 = \frac{.5 \cdot x22 + .5 \cdot x23}{1 + r_{01} \cdot \exp(m1 - s)}$$

$$p4 = \frac{.5 \cdot x11 + .5 \cdot x12}{1 + r_{01}}$$

$$\begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \\ y5 \\ y6 \\ y7 \\ y8 \\ y9 \\ y10 \end{bmatrix} := \text{find}(m3, x31, x32, x33, x34, x21, x22, x23, x11, x12)$$

y1 = 0.230197

y2 = 81.342017

y3 = 86.673441

y4 = 90.656431

y5 = 93.537756

y6 = 71.53885

y7 = 79.389569

y8 = 85.408269

y9 = 66.649721

y10 = 75.68905

m3 := y1

one period spot rates

$uuu := r_{001} \cdot \exp(m3 + 3 \cdot s)$

$uud := r_{001} \cdot \exp(m3 + s)$

$udd := r_{001} \cdot \exp(m3 - s)$

$ddd := r_{001} \cdot \exp(m3 - 3 \cdot s)$

uuu = 0.229377
uud = 0.153756
udd = 0.103066
ddd = 0.069087

CALCULATES SPOTS AND RATES IN PERIOD 5

ASSUMPTIONS

x51 := 100

x52 := 100

x53 := 100

x54 := 100

x55 := 100

x56 := 100

GUESSES

m4 := .1

given

$$x41=\frac{.5\cdot x51+.5\cdot x52}{1+r_{01}\cdot \exp(m4+4\cdot s)}$$

$$x31=\frac{.5\cdot x41+.5\cdot x42}{1+r_{01}\cdot \exp(m3+3\cdot s)}$$

$$x21=\frac{.5\cdot x31+.5\cdot x32}{1+r_{01}\cdot \exp(m2+2\cdot s)}$$

$$x42=\frac{.5\cdot x52+.5\cdot x53}{1+r_{01}\cdot \exp(m4+2\cdot s)}$$

$$x32=\frac{.5\cdot x42+.5\cdot x43}{1+r_{01}\cdot \exp(m3+s)}$$

$$x22=\frac{.5\cdot x32+.5\cdot x33}{1+r_{01}\cdot \exp(m2)}$$

$$x43=\frac{.5\cdot x53+.5\cdot x54}{1+r_{01}\cdot \exp(m4)}$$

$$x33=\frac{.5\cdot x43+.5\cdot x44}{1+r_{01}\cdot \exp(m3-s)}$$

$$x23=\frac{.5\cdot x33+.5\cdot x34}{1+r_{01}\cdot \exp(m2-2\cdot s)}$$

$$x44=\frac{.5\cdot x54+.5\cdot x55}{1+r_{01}\cdot \exp(m4-2\cdot s)}$$

$$x34=\frac{.5\cdot x44+.5\cdot x45}{1+r_{01}\cdot \exp(m3-3\cdot s)}$$

$$x11=\frac{.5\cdot x21+.5\cdot x22}{1+r_{01}\cdot \exp(m1+s)}$$

$$x45=\frac{.5\cdot x55+.5\cdot x56}{1+r_{01}\cdot \exp(m4-4\cdot s)}$$

$$p5=\frac{.5\cdot x11+.5\cdot x12}{1+r_{01}}$$

$$x12=\frac{.5\cdot x22+.5\cdot x23}{1+r_{01}\cdot \exp(m1-s)}$$

$$\begin{bmatrix} y1\\ y2\\ y3\\ y4\\ y5\\ y6\\ y7\\ y8\\ y9\\ y10\\ y11\\ y12\\ y13\\ y14\\ y15 \end{bmatrix} := \text{find}(m4,x41,x42,x43,x44,x45,x31,x32,x33,x34,x21,x22,x23,x11,x12)$$

$y2 = 76.807962$ $y7 = 65.063372$ $y11 = 59.297112$ $y14 = 56.960386$ $y1 = 0.305086$

$y3 = 83.166858$ $y8 = 74.201257$ $y12 = 69.689791$ $y15 = 67.873522$ $m4 := y1$

$y4 = 88.053427$ $y9 = 81.46254$ $y13 = 78.091291$

$y5 = 91.663641$ $y10 = 86.95162$

$y6 = 94.254054$

computes spots

$uuuu := r_{001} \cdot \exp(m4 + 4 \cdot s)$

$uuud := r_{001} \cdot \exp(m4 + 2 \cdot s)$

$uudd := r_{001} \cdot \exp(m4)$

$uddd := r_{001} \cdot \exp(m4 - 2 \cdot s)$

$dddd := r_{001} \cdot \exp(m4 - 4 \cdot s)$

$uuuu = 0.301948$

$uuud = 0.202402$

$uudd = 0.135674$

$uddd = 0.090945$

$dddd = 0.060962$

calculates intrest rates for black dermon toy

BLACKDT

$r_{001} := .1$ $\sigma_1 := .20$

$r_{002} := .105$ $\sigma_2 := .19$

$r_{003} := .11$ $\sigma_3 := .18$

$r_{004} := .115$ $\sigma_4 := .17$

$r_{005} := .12$

calculates period 2 short rates $u := 1.5$ $d := .5$

given

$$\sigma_1 = .5 \cdot \ln\left(\frac{u}{d}\right)$$

$$\frac{1}{(1+r_{002})^2} = \frac{1}{(1+r_{001})} \cdot \left[\frac{1}{(1+u)} + \frac{1}{(1+d)}\right] \cdot .5$$

$$\begin{pmatrix} a \\ b \end{pmatrix} := \text{find}(u, d)$$

value of period 2 short rates

$a = 0.1323$ $b = 0.0887$

$u := a$ $d := b$

$$\text{check} := .5 \cdot \ln\left(\frac{u}{d}\right)$$

$\text{check} = 0.2$

$u2 := .12$ $d2 := .10$

calculates two period rates in one period

given

$$\frac{1}{\left(1+r_{003}\right)^3} = \left[\frac{1}{\left(1+r_{001}\right)}\right]\cdot\left[\frac{1}{(1+u2)^2}+\frac{1}{(1+d2)^2}\right]\cdot .5$$

$$.5\cdot\ln\left(\frac{u2}{d2}\right) = \sigma_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \text{find}(u2, d2)$$

$x_1 = 0.1374$ $x_2 = 0.094$ x1and x2 are two period rates

$uu := .3$

$ud := .2$

$dd := .1$

calculates one period rates in period 3

given

$$\frac{1}{\left(1+x_1\right)^2} = \left[\frac{1}{(1+u)}\right]\cdot\left[\frac{1}{(1+uu)}+\frac{1}{(1+ud)}\right]\cdot .5$$

$$\frac{1}{\left(1+x_2\right)^2} = \left[\frac{1}{(1+d)}\right]\cdot\left[\frac{1}{(1+ud)}+\frac{1}{(1+dd)}\right]\cdot .5$$

$$uu\cdot dd = ud^2$$

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} := \text{find}(uu, ud, dd)$$

$x_3 = 0.1688$ $x_4 = 0.1174$ $x_5 = 0.0817$

$uu := x_3$ $ud := x_4$ $dd := x_5$

calculates three period spot rates

$u3 := .14$ $d3 := .07$

given

$$\frac{1}{(1+r_{004})^4}=\left(\frac{1}{1+r_{001}}\right)\cdot\left[\left[\frac{1}{(1+u3)^3}\right]+\left[\frac{1}{(1+d3)^3}\right]\right]\cdot .5$$

$$.5\cdot\ln\left(\frac{u3}{d3}\right)=\sigma_3$$

$$\begin{pmatrix} x_6 \\ x_7 \end{pmatrix} := \mathrm{find}(u3, d3)$$

$x_6 = 0.1424$ $x_7 = 0.0993$ three period rates in one period

$uuu := .20$
$uud := .16$
$udd := .08$
$ddd := .04$

given

$$\frac{1}{(1+x_6)^3}=\left(\frac{1}{1+u}\right)\cdot\left[\left(\frac{1}{1+uu}\right)\cdot\left[\frac{1}{(1+uuu)}\right]\cdot .25+\left[\frac{1}{(1+uu)}\right]\cdot\left(\frac{1}{1+uud}\right)\cdot .25+\left[\frac{1}{(1+ud)}\right]\cdot\left[\frac{1}{(1+uud)}\right]\cdot .25+\left[\frac{1}{(1+ud)}\right]\cdot\left[\frac{1}{(1+udd)}\right]\cdot .25\right]$$

$$\frac{1}{(1+x_7)^3}=\left(\frac{1}{1+d}\right)\cdot\left[\left(\frac{1}{1+ud}\right)\cdot\left[\frac{1}{(1+uud)}\right]\cdot .25+\left[\frac{1}{(1+ud)}\right]\cdot\left(\frac{1}{1+udd}\right)\cdot .25+\left[\frac{1}{(1+dd)}\right]\cdot\left[\frac{1}{(1+udd)}\right]\cdot .25+\left[\frac{1}{(1+dd)}\right]\cdot\left[\frac{1}{(1+ddd)}\right]\cdot .25\right]$$

$$uuu\cdot udd=uud^2$$

$$uud\cdot ddd=udd^2$$

$$\begin{bmatrix} x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{bmatrix} := \text{find}(uuu, uud, udd, ddd)$$

$x_8 = 0.2082$ $x_9 = 0.1501$ $x_{10} = 0.1082$ $x_{11} = 0.078$

$uuu := x_8$ $uud := x_9$ $udd := x_{10}$ $ddd := x_{11}$

calculates four period rates in one period

$u4 := .15$

$d4 := .10$

given

$$\frac{1}{(1+r_{005})^5} = \left(\frac{1}{1+r_{001}}\right) \cdot \left[\left[\frac{1}{(1+u4)^4}\right] + \left[\frac{1}{(1+d4)^4}\right]\right] \cdot .5$$

$$.5 \cdot \ln\left(\frac{u4}{d4}\right) = \sigma_4$$

$$\begin{pmatrix} x_{12} \\ x_{13} \end{pmatrix} := \text{find}(u4, d4)$$

$x_{12} = 0.1473$ $x_{13} = 0.1048$ four period rates in one period

calculates the distribution of one period rates in period five

$uuuu := .20$
$uuud := .16$
$uudd := .08$
$uddd := .04$
$dddd := .02$

given

$$\frac{1}{(1+x_{12})^4}=\left(\frac{1}{1+u}\right)\cdot\left[\begin{array}{l}\left(\frac{1}{1+uu}\right)\cdot\left(\frac{1}{1+uuu}\right)\cdot\left(\frac{1}{1+uuuu}\right)\cdot .125+\left(\frac{1}{1+uu}\right)\cdot\left(\frac{1}{1+uuu}\right)\cdot\left(\frac{1}{1+uuud}\right)\cdot .125\ ...\\ +\left(\frac{1}{1+uu}\right)\cdot\left(\frac{1}{1+uud}\right)\cdot\left(\frac{1}{1+uuud}\right)\cdot .125+\left(\frac{1}{1+uu}\right)\cdot\left(\frac{1}{1+uud}\right)\cdot\left(\frac{1}{1+uudd}\right)\cdot .125\ ...\\ +\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+uud}\right)\cdot\left(\frac{1}{1+uuud}\right)\cdot .125+\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+uud}\right)\cdot\left(\frac{1}{1+uudd}\right)\cdot .125\ ...\\ +\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+udd}\right)\cdot\left(\frac{1}{1+uudd}\right)\cdot .125+\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+udd}\right)\cdot\left(\frac{1}{1+uddd}\right)\cdot .125\end{array}\right]$$

$$\frac{1}{(1+x_{13})^4}=\left(\frac{1}{1+d}\right)\cdot\left[\begin{array}{l}\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+uud}\right)\cdot\left(\frac{1}{1+uuud}\right)\cdot .125+\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+uud}\right)\cdot\left(\frac{1}{1+uudd}\right)\cdot .125\ ...\\ +\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+udd}\right)\cdot\left(\frac{1}{1+uudd}\right)\cdot .125+\left(\frac{1}{1+ud}\right)\cdot\left(\frac{1}{1+udd}\right)\cdot\left(\frac{1}{1+uddd}\right)\cdot .125\ ...\\ +\left(\frac{1}{1+dd}\right)\cdot\left(\frac{1}{1+udd}\right)\cdot\left(\frac{1}{1+uudd}\right)\cdot .125+\left(\frac{1}{1+dd}\right)\cdot\left(\frac{1}{1+udd}\right)\cdot\left(\frac{1}{1+uddd}\right)\cdot .125\ ...\\ +\left(\frac{1}{1+dd}\right)\cdot\left(\frac{1}{1+ddd}\right)\cdot\left(\frac{1}{1+uddd}\right)\cdot .125+\left(\frac{1}{1+dd}\right)\cdot\left(\frac{1}{1+ddd}\right)\cdot\left(\frac{1}{1+dddd}\right)\cdot .125\end{array}\right]$$

$$uuuu\cdot uudd=uuud^2$$

$$uuud\cdot uddd=uudd^2$$

$$uudd\cdot dddd=uddd^2$$

$$\begin{bmatrix}x_{14}\\x_{15}\\x_{16}\\x_{17}\\x_{18}\end{bmatrix}:=\mathrm{find}(uuuu,uuud,uudd,uddd,dddd)$$

$x_{14}=0.2482$ $\quad x_{15}=0.1851$ $\quad x_{16}=0.1381$ $\quad$ .04