THE MULTIPLICITY OF RATES

September 1999

The Multiplicity of Rates

$$r_{t_0, t_1, t_2}$$

define

t_0 = time of commitment

t_2 = time money repayed

t_1 = time money lent

Definition:

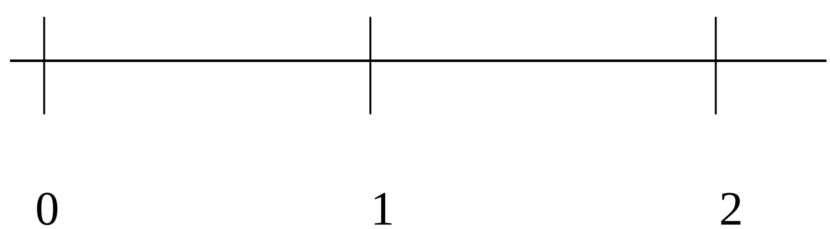
> **Spot rate is the rate of interest on a bond with one payment and where commitment and lending date are the same.**
>
> **Forward rate is the rate of interest on a loan where the commitment date and date money is lent are different.**

Relationship of Forwards and Spots

Equivalent :

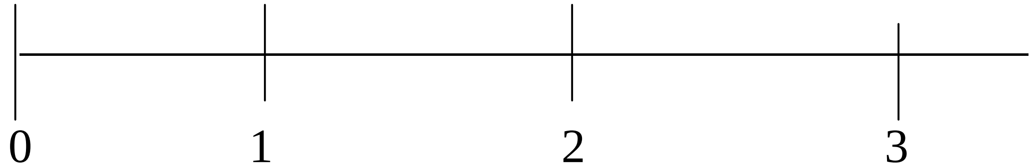
$$(1+\frac{r_{001}}{2})(1+\frac{r_{012}}{2}) = (1+\frac{r_{002}}{2})^2$$

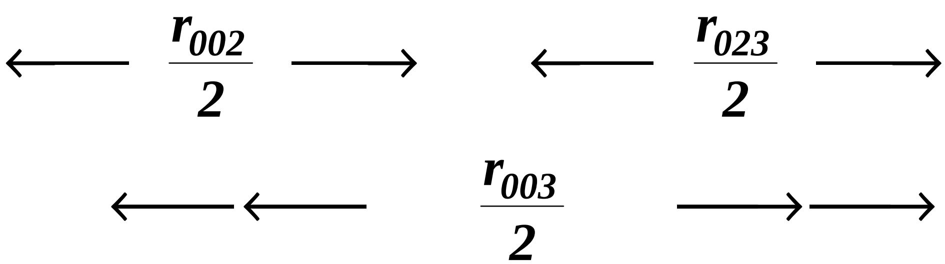
$$1+\frac{r_{012}}{2} = \frac{(1+\frac{r_{002}}{2})^2}{(1+\frac{r_{001}}{2})}$$



$$\leftarrow \quad \frac{r_{001}}{2} \quad \rightarrow \quad \leftarrow \quad \frac{r_{012}}{2} \quad \rightarrow$$

$$\leftarrow \quad \leftarrow \quad \frac{r_{002}}{2} \quad \rightarrow \quad \rightarrow$$





$$\left(1+\frac{r_{003}}{2}\right)^3=\left(1+\frac{r_{002}}{2}\right)^2\left(1+\frac{r_{023}}{2}\right)$$

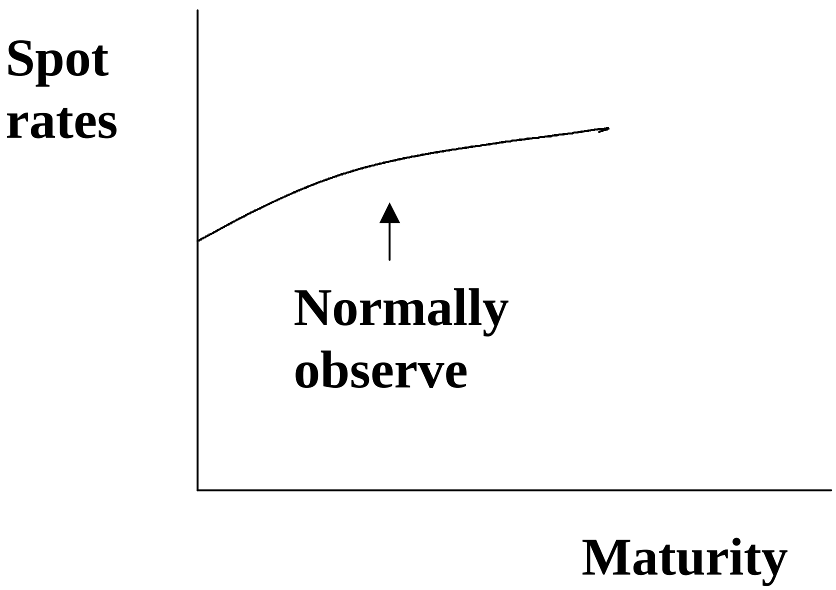
$$1+\frac{r_{023}}{2}=\frac{\left(1+\frac{r_{003}}{2}\right)^3}{\left(1+\frac{r_{002}}{2}\right)^2}$$

Pure discount bond is a bond with single repayment date a zero.

0	1	2	3	4
-90	**+100**			
-82		**+100**		
-76			**+100**	
-100				**+140**

$$\textit{discount function} = D(t) = \frac{1}{\left(1 + \frac{r_{00t}}{2}\right)^{t}}$$

What Determines Spot Rates or Term Structure Theories?



We talk about term structure theories to explain this curve

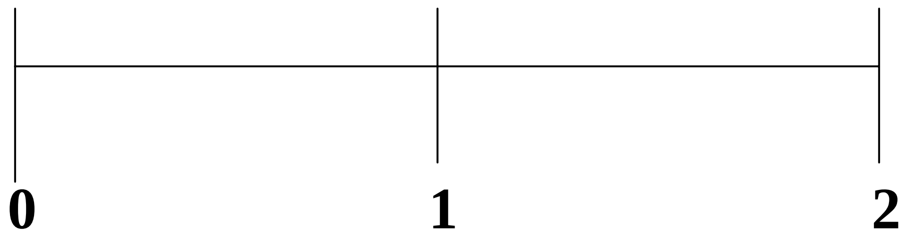
- **a. Pure expectations**
- **b. Liquidity**
- **c. Preferred habitat**
- **d. Market segmentation**

The importance of these from an investment point of view is: do we earn an excess return on average by holding long bonds?

- **(1) Expectations theory = No term premium**
- **(2) Liquidity = Positive term Premium**
- **(3) Preferred habitat = ?**
- **(4) Market Segmentation = ?**

How does this come about?

1. Expectations Theory



$$\longleftarrow \quad \frac{r_{001}}{2} \quad \longrightarrow$$

$$\longleftarrow \qquad \frac{r_{002}}{2} \qquad \longrightarrow$$

We know

$$\left(1+\frac{r_{002}}{2}\right)^2 = \left(\frac{1+r_{001}}{2}\right)\left(\frac{1+r_{012}}{2}\right)$$

$$\textit{expectations theory}\ \ \frac{\overline{r_{112}}}{2} = \frac{r_{012}}{2} = \frac{(1+\frac{r_{002}}{2})^2}{(1+\frac{r_{001}}{2})}$$

(1) one period return

a. One period bond earns

$$\frac{r_{001}}{2}$$

b. two period sold after one period

$$\frac{\overline{r_{112}}}{2}$$

At 1, if expectations realized then must offer and sell for

$$\frac{\left(1+\frac{r_{002}}{2}\right)^2}{\left(1+\frac{\overline{r_{112}}}{2}\right)} = \frac{\left(1+\frac{r_{002}}{2}\right)^2}{\left(1+\frac{r_{012}}{2}\right)} = \left(1+\frac{r_{001}}{2}\right)$$

$$\textit{first period return}\ \frac{r_{001}}{2}$$

<u>Law of one Price</u>: Two identical items can't sell at different prices.

Time	Bond A	Bond B	Bond C
0			
1	100	10	15
2		110	115

$$(100)\, x_a + 10\, x_B = 15$$

$$110\, x_B = 115$$

$$X_B = \frac{23}{22} \qquad X_A = \frac{1}{22}$$

check

$$100(\frac{1}{22}) + 10(\frac{23}{22}) = 15$$

$$110(\frac{23}{22}) = 115$$

Thus

$$\begin{bmatrix} 1\ Bond\ A \\ 23\ Bond\ B \end{bmatrix} (is\ equivalent\ to) \begin{bmatrix} 22 \\ Bond\ C \end{bmatrix}$$

implication

$$P_C = \frac{1}{22} P_A + \frac{23}{22} P_B$$

<u>Note</u>:

A coupon paying bond can always be viewed as a portfolio of pure discount instruments.

To see this, it is convenient to assume all pure discount instruments mature at $1.

Consider

Time	**Price of pure discount instruments**	**Maturity value**
1	P_1	**1**
2	P_2	**1**
3	P_3	**1**
4	P_4	**1**

Consider Bond with cash flows as follows:

Time	cf
1	10
2	10
3	10
4	110

Then this bond can be viewed as equivalent to 10 of 1, 10 of 2, 10 of 3, and 110 of 4.

Price is $10P_1 + 10P_2 + 10P_3 + 110P_4$

Since $$P_t = \frac{1}{\left(1 + \frac{r_{00t}}{2}\right)^t}$$

$$Price = \frac{10}{\left(1 + \frac{r_{001}}{2}\right)} + \frac{10}{\left(1 + \frac{r_{002}}{2}\right)^2} + \frac{10}{\left(1 + \frac{r_{003}}{2}\right)^3} + \frac{110}{\left(1 + \frac{r_{004}}{2}\right)^4}$$

Implication:

Bonds must be priced as if each cash flow is discounted at spot rate (if law of one price holds).

Thus spot rates are the basic building blocks of bond valuation.

Two ways to spot mispriced bonds:

1. **Directly looking for swaps.**

2. **Indirectly by computing spot rates and comparing "equilibrium" price to traded price.**

<u>Swap Example</u>:

The prior example was an example of a swap.

A second example:

Time	Bond A	Bond B	Bond C
0			
1	100	5	10
2		105	110
Price	92	96	103

Have three bonds and two time periods; thus can always create one bond from other two. Which is chosen is arbitrary. Assume matching cash flows of c.

$$x_A = amount\ in\ A$$

$$x_B = amount\ in\ B$$

To match cash flows:

$$100\ x_A + 5x_B = 10$$

$$105\ x_B = 110$$

$$x_B = \frac{110}{105}$$

$$100x_B = 10 - 5\left(\frac{110}{105}\right)$$

$$x_A = \frac{5}{105}$$

Check ...

$$100\left(\frac{5}{105}\right) + 5\left(\frac{110}{105}\right) = 10$$

$$105\left(\frac{110}{105}\right) = 110$$

Combination Costs ...

$$92\left(\frac{5}{105}\right) + 96\left(\frac{110}{105}\right) = 104.9524$$

Since C costs 103, have an arbitrage.
Holders of A and B should sell and buy C.

<u>Estimating Spot Rates</u>

<u>Note</u>:

In what follows I will be using price to stand for invoice price or what one pays for the bond. Invoice price is quoted price plus accrued interest.

$$P = \frac{cf(1)}{(1 + \frac{r_{001}}{2})} + \frac{cf(2)}{(1 + \frac{r_{002}}{2})^2} + \frac{cf(3)}{(1 + \frac{r_{003}}{2})^3}$$

$$D(t) = \frac{1}{(1 + \frac{r_{00t}}{2})^t}$$

$$P = CF(1)D(1) + CF(2)D(2) + CF(3)D(3)$$

Problem is to estimate D(t)

Using zeros

- **observe two period zero at $920**

$$920 = \frac{1000}{\left(1 + \frac{r_{002}}{2}\right)^{2}}$$

$$\frac{r_{002}}{2} = \left(\frac{1000}{920}\right)^{\frac{1}{2}} - 1$$

$$r_{002} = 8.514$$

Using sequence of bonds

Bond		1	2	3
A	-930	1000		
B	-800	100	1100	
C	-700	80	80	1080

use bond A to determine r_{001} or D(1)

+930 = 1000 *D(1)*

$$D(1) = \frac{930}{1000}$$

Use bond B and D(1) to determine r_{002} *or D(2)*

$$800 = 100D(1) + 1100D(2)$$

$$800 = 93 + 1100D(2)$$

$$800 = 100\left(\frac{930}{1000}\right) + 1100D(2)$$

$$D(2) = \frac{707}{1100}$$

Use bond C and D(1) and D(2) to obtain D(3)

$$700 = 80D(1) + 80D(2) + 1080D(3)$$

$$700 = 80(\frac{930}{1000}) + 80(\frac{707}{1100}) + 1080D(3)$$

$$700 = 74.4 + 51.42 + 1080D(3)$$

$$700 = 125.82 + 1080D(3)$$

$$D(3) = \frac{574.18}{1080}$$

Problem is sensitive to bonds chosen

Law of one price may not hold perfectly because

(1) Non synchronous trading

(2) Bid-ask spread

(3) Bonds out of equilibrium

<u>Solution: Use average estimate</u>

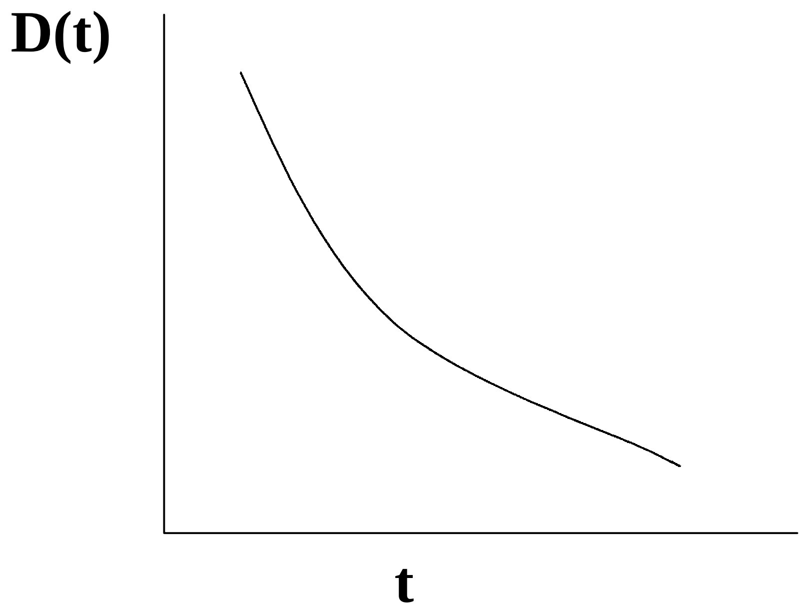
$$P_i = cf(1)D(1) + cf(2)D(2) + cf(3)D(3) + e_i$$

Estimate regression

Important often to constrain so conforms to what we KNOW must be true.

$$D(t) > D(t+1)$$

Problem with this procedure is that it depends on bonds having same coupon dates. Alternative is to estimate a function.



assume smooth and concave from above

Example:

$$D(t) = C_o + C_1 t + C_2 t^2$$

$$P_i = \sum_t cf(t) D(t) + e_i$$

$$P_i = \sum_t cf(t)(c_0 + c_1 t + C_2 t^2) + e_i$$

$$P_i = C_0 [\sum cf(t)] + c_1 [\sum cf(t)(t)] + c_2 [\sum cf(t) t^2] + e_i$$

$$estimate \;\; c_0 , c_1 , c_2$$

Putting in Tax Term

historically

Adjusted cash flows to after tax

$$P_i = cf(1)D(1) + cf(2)D(2) + cf(3)D(3) + (100 - P_0)t + e_i$$

currently

For individuals mainly postponement thus differential taxation function of maturity as well as price relative to Par

clientele effects

?

Review

I. Terms

- **a. Spot rate**
- **b. Forward rate**
- **c. Pure discount bond or zero**
- **d. Law of one price**

II. Concepts

- **a. That default free bonds should be priced by calculating present value of cash flows at the spot rate**
- **b. Bond swaps**
- **c. Bond arbitrage**
- **d. Consequences of term structure theories**

III. Calculations

- **a. Spot rates**
- **b. Forward rates**
- **c. Bond swaps**
- **d. Checking for arbitrage**

Problems

I. Given the following invoice prices and cash flows, what are the spot rates and forward rates?

Period	Invoice Price	Cash flows 1	2	3
0	960	1000		
1	920		1000	
2	880			1000

$$960 = \frac{1000}{(1 + \frac{r_{001}}{2})}$$

Answer:

$$\frac{r_{001}}{2} = \frac{1000}{960} - 1$$

$$r_{001} = 8.33$$

$$920 = \frac{1000}{(1 + \frac{r_{002}}{2})^2}$$

$$\frac{r_{002}}{2} = \left(\frac{1000}{920}\right)^{\frac{1}{2}} - 1$$

$$r_{002} = 8.51$$

$$880 = \frac{1000}{(1 + \frac{r_{003}}{2})^3}$$

$$\frac{r_{003}}{2} = \left(\frac{1000}{880}\right)^{\frac{1}{3}} - 1$$

$$r_{003} = 8.71$$

$$1 + \frac{r_{012}}{2} = \frac{\left(\frac{1000}{920}\right)}{\left(\frac{1000}{960}\right)} = \frac{960}{920}$$

$$r_{012} = 8.70$$

$$1 + \frac{r_{023}}{2} = \frac{\left(\frac{1000}{880}\right)}{\left(\frac{1000}{920}\right)} = \frac{920}{880}$$

$$r_{023} = 9.09$$

2. Consider the following:

Bond	Invoice Price	Cash flows 1	Cash flows 2
A	100	5	105
B	102	6	106
C	95	102	

Is there an arbitrage? What is the swap, if any?

Answer:

Use B and C to match the cash flows of A.

$$x_C\ 102 + x_B\ 6 = 5$$

$$x_B\ 106 = 105$$

$$x_B = \frac{105}{106}$$

$$102\ x_C = 5 - 6\left(\frac{105}{106}\right)$$

$$x_C = \frac{-100}{(106)(102)}$$

Cost of replicating portfolio:

$$102\left(\frac{105}{106}\right) - 95\,\frac{(100)}{(106)(102)} = 100.1591$$

Thus A is cheaper. Swap is to sell B and buy A and C in proportions shown. Note x_C is negative.

Accrued Interest Calculations

Conventions

Different markets have different conventions for calculating accrued interest. The method used in each market is denoted by specifying the method of calculating two values:

- ***d*** **is the number of days from the previous coupon payment date to settlement date (or from issue date to settlement date, if the next coupon payment if the first); that is, the number of days over which interest has accrued.**

- **A_y is the assumed number of days in one year.**

The particular convention used can be Actual/Actual, Actual/365, Actual/360, or 30E/360.

In the name of the convention, the first part of the name denotes the method of computing *d*; and the second part of the name denotes the method of calculating A_y.

Computing the Accrued Interest

Once *d* and A_y have been calculated, accrued interest is computed by the formula:

$$I_A = C\frac{d}{A_y}$$

I_A is the accrued interest
C is the annual coupon payment

Calculating *d*

The three methods of calculating *d* are:

1. "Actual" - Calculate the actual number of days from the previous coupon payment date to the settlement date.

Examples:

There are 10 days from 1/3/93 to 1/13/93
There are 41 days from 1/3/93 to 2/13/93
There are 31 days from 1/1/93 to 2/1/93
There are 28 days from 2/1/93 to 3/1/93
There are 29 days from 2/1/92 to 3/1/92

2. "30" - Calculate the number of days from the previous coupon payment date to the settlement date by assuming 30-day months, as follows:

Let the two dates by $M_1/D_1/Y_1$ and $M_2/D_2/Y_2$.
If D_1 is 31, change it to 30.
If D_2 is 31, change it to 30.
Then $d = 360(Y_2-Y_1) + 30(M_2-M_1) + (D_2-D_1)$.

(Note: According to this formula, February will always appear to be a 30-day month).

Examples:

There are 10 days from 1/3/93 to 1/13/93
There are 30 days from 1/1/93 to 2/1/93
There are 30 days from 2/1/93 to 3/1/93
There are 30 days from 2/1/92 to 3/1/92
There are 4 days from 2/27/93 to 3/1/93
There are 0 days from 5/30/93 to 5/31/93
There are 2 days from 5/29/93 to 5/31/93

This "30" method is used in the U.S. agencies, corporates, and municipals markets.

3. "30E" - Calculate the number of days from the previous coupon payment date to the settlement date by assuming 30-day months, as follows:

Let the two dates by $M_1/D_1/Y_1$ and $M_2/D_2/Y_2$.
If D_1 is 31, change it to 30.
If D_2 is 31, change it to 30.
Then $d = 360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$.

(Note: According to this formula, February will always appear to be a 30-day month.)

Examples:

There are 30 days from 2/1/93 to 3/1/93
There are 30 days from 2/1/92 to 3/1/92
There are 4 days from 2/27/93 to 3/1/93
There are 0 days from 5/30/93 to 5/31/93
There is 1 day from 5/29/93 to 5/31/93

This method, slightly different from the "30" method above, is used in the Eurobond market and many European government and corporate markets.

Calculating A_y

The three method of calculating A_y are:

(1) "365" - A_y is equal to 365.

(2) "360" - A_y is equal to 360.

(3) "Actual - A_y is equal to the number of days in the current coupon period times the number of coupon payments per year. For a semi-annual coupon, the number of days in the coupon period can range from 181 to 184, so A_y can range from 362 to 368.

Examples:

(1) A 6.75% bond paying semi-annually is traded to settle on July 2, 1993. The previous coupon paid on March 15, 1993, and the next coupon pays on September 15, 1993. If the bond accrues according to the Actual/Actual convention, what is the accrued interest at settlement?

There are 184 actual days in the current coupon period, and there are 109 actual days from the last coupon payment to settlement. Therefore, the accrued interest is:

$$\frac{109}{2x194}\ x\ 6.75 = 1.999321\ per\ 100\ of\ face\ value$$

(2) Assume that the bond in example (1) accrues according to the 30/360 convention, instead of Actual/Actual. What would be the accrued interest?

There are 107 "30" days from the last coupon payment to settlement. Therefore, the accrued interest is:

$$\frac{107}{360}\ x\ 675 = 2.006250\ per\ 100\ of\ face\ value$$

Accrued Interest - Market Conventions

Instrument Type	Accrual Convention	Coupons per Year	Notes
Domestic:			
US Govt. Treasury Bonds	Actual/Actual	2	
US Govt. Agency Bonds	30/360	2	
Corporate Bonds	30/360	2	
Municipal Bonds	30/360	2	
GNMA/FNMA/FLHMC	Actual/360	12	
International:			
Eurobonds	30E/360	1	
German Govt. and Corp. Bonds	30E/360	1	
Italian Govt. and Corp. Bonds	30E/360	2	
Japanese Govt. and Corp. Bonds	Actual/365	2	1
Swiss Govt. and Corp. Bonds	30/360	1	
UK Govt. and Corp. Bonds	Actual/365	2	

Notes:

1. In the Japanese markets, February 29 is never counted.