PROTECTING AGAINST TERM STRUCTURE SHIFTS

(INVESTMENT MANAGEMENT WITH LIABILITY STREAM)

September 1999

BASIC IDEA

1 Manager is concerned with Liability Stream.

2. Concern is to maximize net worth.

Major Tools

(1) Cash flow matching or dedication

(2) Sensitivity matching or Immunization

CASH FLOW MATCHING

Observe liability stream $L_1, L_2, L_3, \ldots, L_T$

EXACT MATCHING
Cash flow

Minimize Cost
Subject to

1. **Meet Cash Flows**
2. **Do not issue Bonds**

Let

1. **P_i is Price of Bond i**
2. **N_i is Number of Bonds of Type i Purchased**
3. **L_t is Liabilities in Time t**
4. **r is Short term Interest Rate**
5. **cf(it) is Cash Flow of Bond i in Period t. Interest or principal and interest**
6. **S_t is Amount Invested in Short Term Bond in Period t**

MINIMIZE

$$\sum_i N_i P_i$$

SUBJECT TO

1. $\sum_i N_i cf(it) + S_{t-1}(1+r) - L_t - S_t = 0 \quad \text{all } t$

2. $N_i \geq 0 \text{ all } i$

3. $S_t \geq 0 \text{ all } t$

4. $S_{-1} = 0$

RISKS OF CASH FLOW MATCHING

1. Reinvestment Risk

2. Risk of disappearing security

    a. Call

    b. Sinking fund

    c. Default

Immunization

1. **Assume a manager is responsible for designing a portfolio that will have sufficient cash flows to meet a liability L_t is liability in period t.**
   Thus liabilities are L_1, L_2, L_3, L_4

2. **Sensitivity matching involves finding a Portfolio that**

   a) **at equilibrium prices the Portfolio costs the same as the Liability**

   b) **the value of the asset portfolio and Liability portfolio move in tandem**

3. **Need measures of how value of portfolio changes given change in factor or factors effecting it.**

Choices

(1) Duration
(2) Coefficients of factor model

Types of Duration measures

(1) Analytical
(2) Numerical
(3) Time Series Estimation

ANALYTICAL

SENSITIVITY

MEASURES

ANALYTICAL
MODELS OF SENSITIVITY TO INTEREST RATE CHANGE

1.	MACAULAY	CHANGE IN YIELD
2.	MACAULAY/FISHER AND WEIL	ADDITIVE CHANGE
3.	BIERWAG AND KAUFMAN	MULTIPLICATIVE
4.	BIERWAG	ADDITIVE PLUS MULTIPLICATIVE
5.	KHANG	SHORT MORE THAN LONG
6.	COX, INGERSOLL AND ROSS	SHORT GAUSS MARKOV AND EXPECTATIONS THEORY
7.	BRENNAN AND SCHWARTZ	SHORT AND LONG MARKOV AND EXPECTATIONS THEORY

I. MAUCAULAY $[D_i]$

$$P_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C+100}{(1+r)^N}$$

$$\frac{dP_0}{d(1+r)} = \frac{-C}{(1+r)^2} + \frac{-2C}{(1+r)^3} + \frac{-3C}{(1+r)^4} + \ldots + \frac{(C+100)(-N)}{(1+r)^{N+1}}$$

$$\frac{dP_0}{d(1+r)} = \frac{-1}{1+r} \sum_t \frac{tcf(t)}{(1+r)^t}$$

$$\frac{\frac{dP_0}{d(1+r)}}{P_0} = \frac{-1}{1+r} \left[\sum_t \frac{\frac{tcf(t)}{(1+r)^t}}{P_0}\right]$$

$$\%\ \text{price change} = \frac{dP_0}{P_0} = -D_1 \frac{d(1+r)}{(1+r)}$$

$$= \text{MINUS DURATION}_1 \times \%\text{change in } (1+r)$$

Modified Duration

$$\frac{dP_o}{P_o} = -D_1^M\, dr \qquad \text{where } D_1^M = \frac{D_1}{1+r}$$

Price change per interest rate change

$$dP = -D_1^M\, P\, dr$$

- **Modified Duration useful when worried about riskiness of different positions.**

- **Price change when hedging.**

- **If use different r's on different bonds such as yield to maturity, then duration is not additive.**

General Principles

**(1) The lower the interest rate,
the longer the duration.**

**(2) The lower the coupon rate,
the longer the duration.**

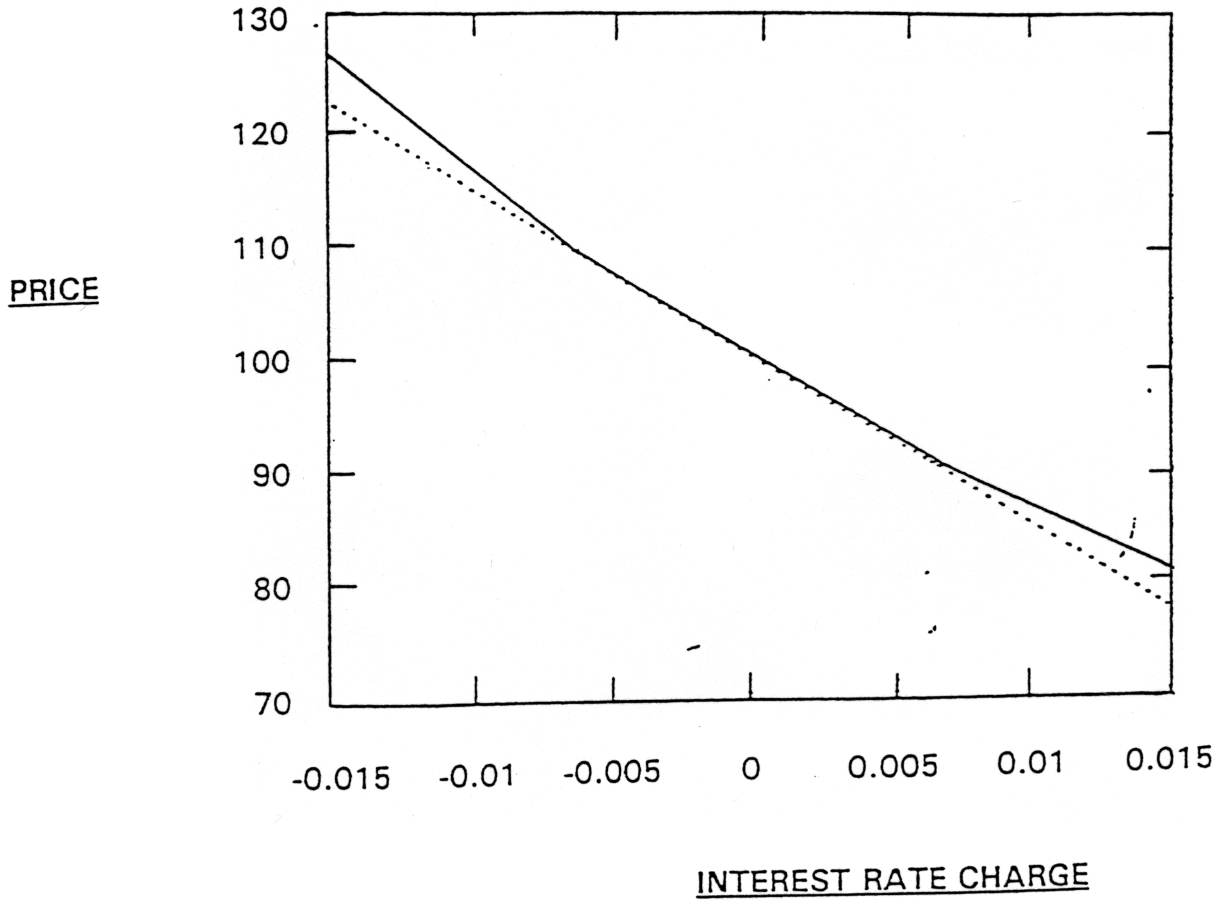
**(3) The greater the maturity,
the longer the duration in general.**

$$\textbf{Estimated Price} = \textbf{Original Price} - (\textbf{Duration})(P_0)\,\frac{dr}{1+r}$$

$$P_0 + dP = P_0 - D_1 \frac{dr}{1+r} P_0$$

<u>EXAMPLE 1</u>

Interest Rate	**10%**
Coupon	**10%**
Principal	**100**
Maturity	**15 years**
Semi-Annual Payment	
Price	**$100**
Duration	**8.07 years**



Actual Price ________

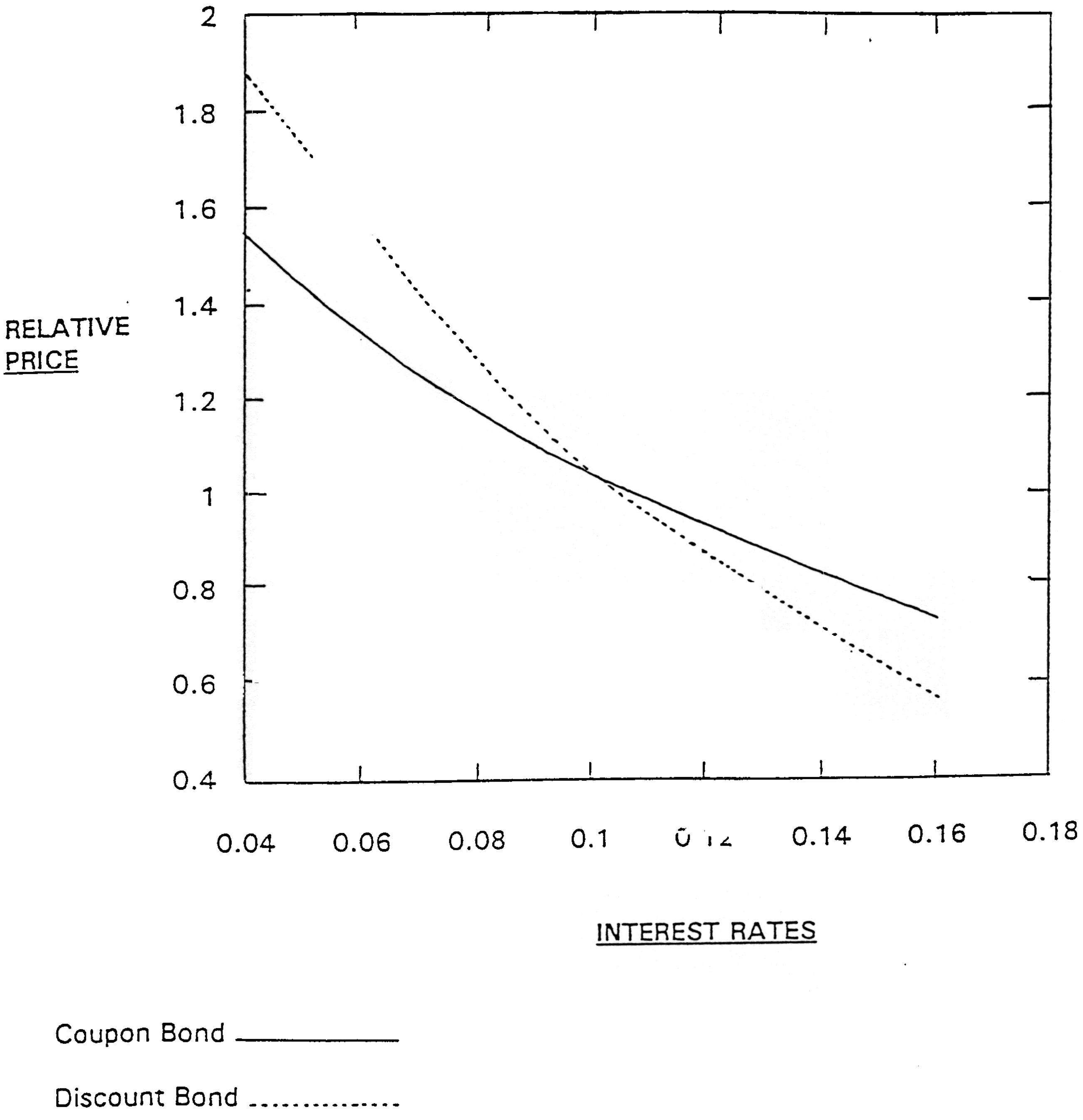
Implied Price ....

CHANGE IN INTEREST RATE	INTEREST RATE	TRUE PRICE	ESTIMATED PRICE
-0.015	0.035	127.588	123.059
-0.014	0.036	125.429	121.521
-0.013	0.037	123.322	119.984
-0.012	0.038	121.264	118.447
-0.011	0.039	119.254	116.91
-0.01	0.04	117.292	115.372
-0.009	0.041	115.376	113.835
-0.009	0.042	113.504	112.298
-0.007	0.043	111.675	110.761
-0.006	0.044	109.889	109.223
-0.005	0.045	108.144	107.686
-0.004	0.046	106.44	106.149
-0.003	0.047	104.774	104.612
-0.002	0.048	103.146	103.074
-0.001	0.049	101.555	101.537
0	0.05	100	100
.001	0.051	98.48	98.463
0.002	0.052	96.994	96.926
0.003	0.053	95.542	95.388
0.004	0.054	94.122	93.851
0.005	0.055	92.733	92.314
0.006	0.056	91.375	90.777
0.007	0.057	90.047	89.239
0.008	0.058	88.748	87.702
0.009	0.058	87.478	86.165
0.01	0.06	86.235	84.628
0.011	0.061	85.019	83.09
0.012	0.062	83.83	81.553
0.013	0.063	82.666	80.016
0.014	0.064	81.527	78.479
0.015	0.065	80.412	76.941

<u>EXAMPLE 2</u>

Coupon vs. Pure Discount:

Coupon	**10%**
Maturity	**10 years**
Initial Price	**$100**
Annual Payments	
Interest Rate	**10%**


RELATIVE PRICE
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.04
0.06
0.08
0.1
0.14
0.16
0.18
INTEREST RATES
Coupon Bond
Discount Bond

II. MACAULAY/FISHER AND WEIL [D_2]

$$P_0 = \frac{C}{(1+r_{01})} + \frac{C}{(1+r_{01})(1+r_{12})} + \frac{C}{(1+r_{01})(1+r_{12})(1+r_{23})} + \ldots$$

$$\frac{dp_0}{d(1+r_{01})} = \frac{-C}{(1+r_{01})^2} + \frac{-C}{(1+r_{01})^2(1+r_{12})} + -C\frac{\frac{d(1+r_{12})}{d(1+r_{01})}}{(1+r_{01})(1+r_{12})^2} + \ldots$$

ASSUME

$$\frac{d(1+r_{N-1N})}{(1+r_{N-1N})} = \frac{d(1+r_{01})}{(1+r_{01})}$$

$$\frac{dP_0}{d(1+r_{01})} = \frac{-C}{(1+r_{01})^2} + \frac{-2C}{(1+r_{01})^2(1+r_{12})} + \frac{-3C}{(1+r_{01})^2(1+r_{12})(1+r_{23})} +$$

$$= \frac{-1}{(1+r_{01})}\sum_t \frac{t(CF)(t)}{\prod_{i=1}^{t}(1+r_{i-1,i})}$$

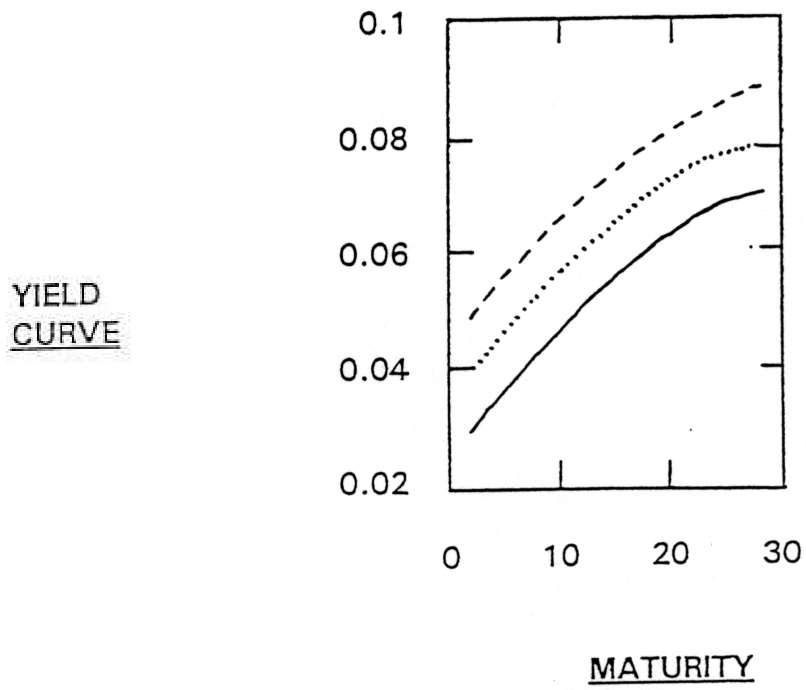
$$\frac{dP_0}{P_0} = -\left[\frac{\sum_t \frac{tCF(t)}{\prod_{i=1}^{t}(1+r_{i-1,i})}}{P_0}\right]\frac{d(1+r_{01})}{(1+r_{01})}$$

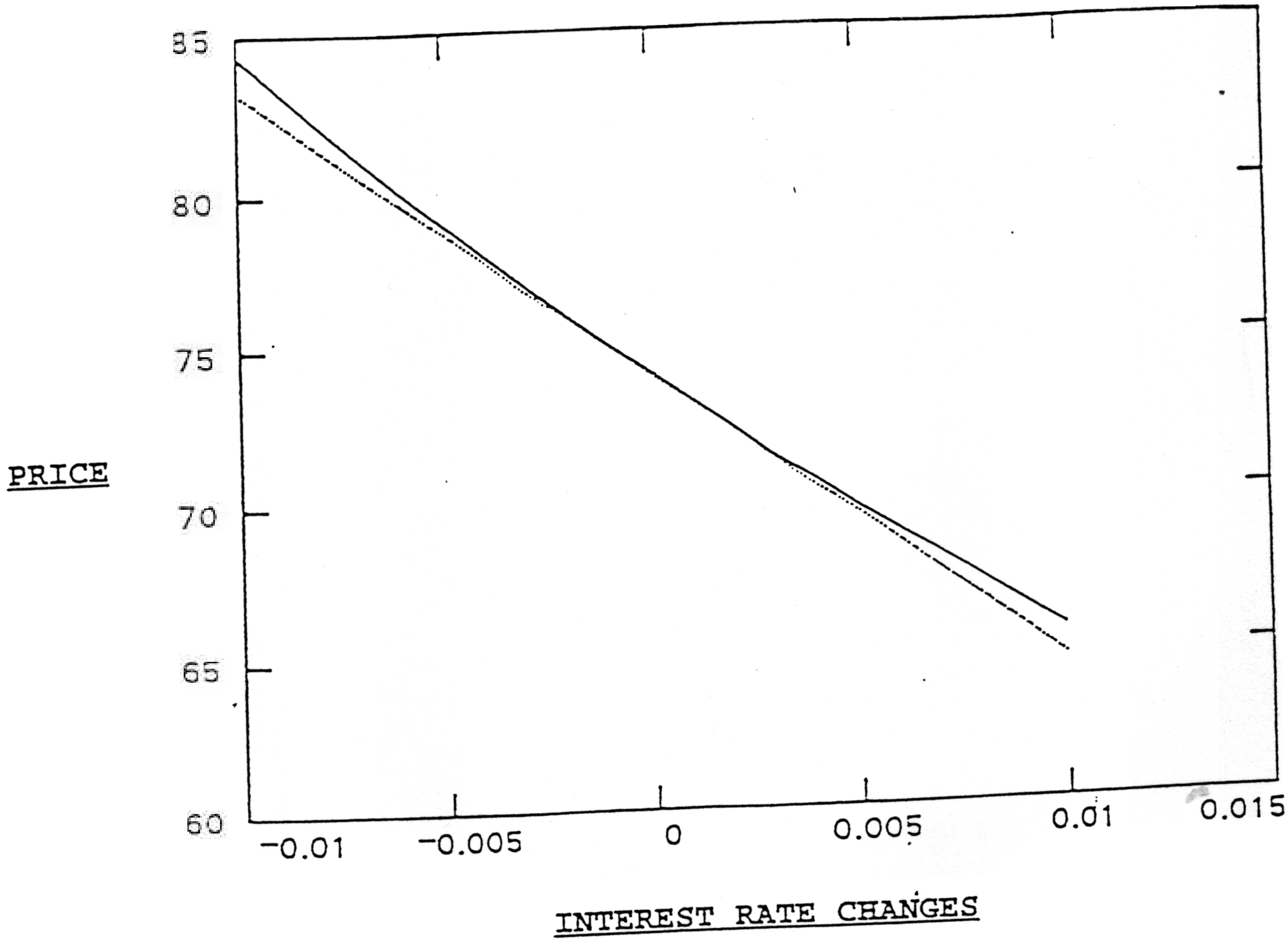
$$\frac{dP_0}{P_0} = -D_2 \frac{d(1+r_{01})}{(1+r_{01})}$$

% CHANGE
IN PRICE = [MINUS DURATION] X[% CHANGE
IN (1 + r_{01})]

EXAMPLE 3

Coupon	**10% (semi-annual)**
Principal	**$100**
Maturity	**15 years**
Duration	**6.527**
Initial Price	**$740.27**
Yield Curve (see attached)	


YIELD
CURVE
0.1
0.08
0.06
0.04
0.02
0
10
20
30
MATURITY



Actual Price ———

Implied Price

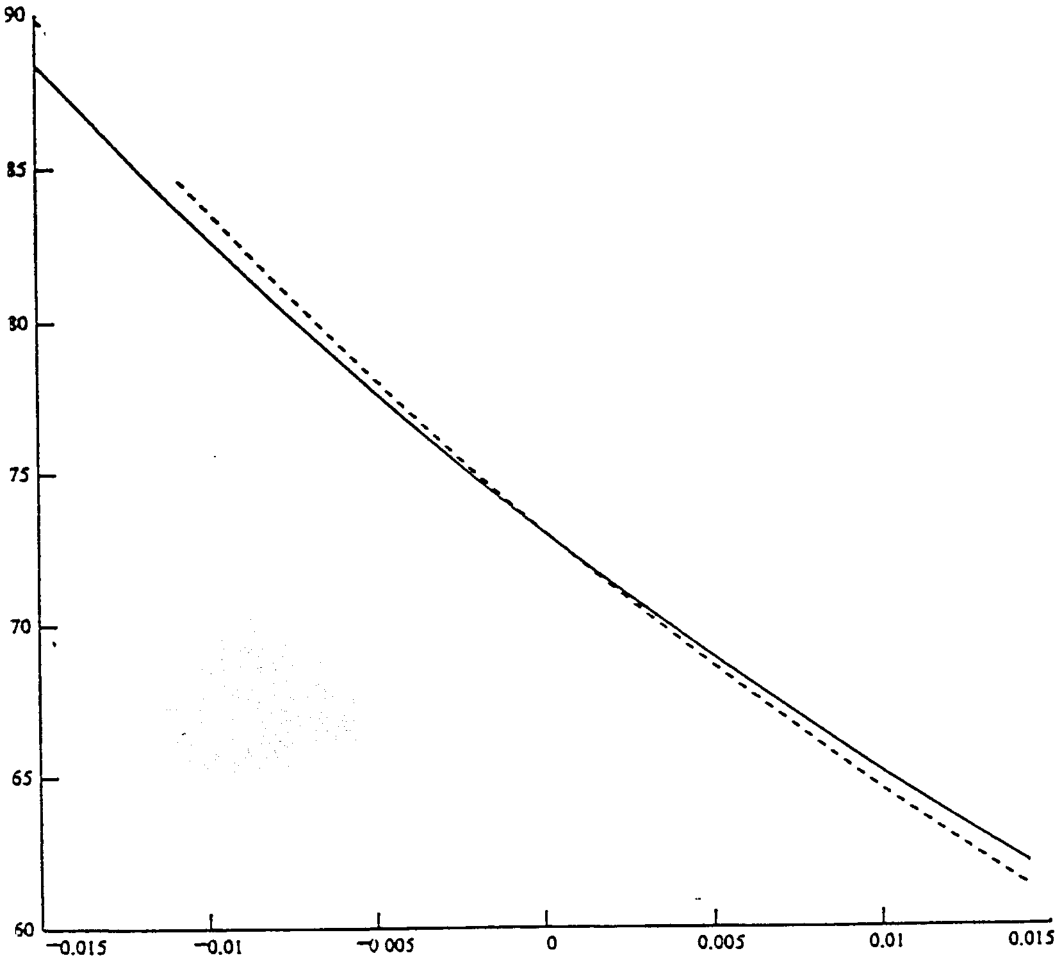
INTEREST RATE CHANGE	TRUE	DURATION
x	p(x)	newprice (x)
-0.01	84.381	83.28
-0.009	83.244	82.351
-0.008	82.131	81.421
-0.007	81.041	80.491
-0.006	79.975	79.561
-0.005	78.931	78.632
-0.004	77.909	77.702
-0.003	76.908	76.772
-0.002	75.928	75.842
-0.001	74.968	74.912
0	74.027	73.983
0.991	73.106	73.053
0.002	72.204	72.123
0.003	71.32	71.193
0.004	70.453	70.264
0.005	69.604	69.334
0.006	68.772	68.404
0.007	67.957	67.474
0.008	67.157	66.544
0.009	66.373	65.615
0.01	65.605	64.685

EXAMPLE 4

Coupon	**5%**
Principal	**$100**
Maturity	**30 years**
Annual Interest	
Duration	**13.054**

Price Change for Different Assumed Shifts in the Yield Curve

Price



Interest Rate Change

Flat Yield Curve ----------

Upward Sloping Yield Curve ——————

Adding additional terms (convexity)

The formula for the first three terms in a MacLaurin series expansion of a function f(X + h) in the region of h as h approaches zero is

$$f(X+h) = f(X) + \frac{df(X)(h)}{1} + \frac{df^{2}(X)(h)^{2}}{2:1} + \ldots$$

Where the prime denotes derivatives. Define P(r) as the price of a bond at an interest rate r. Then, writing the price of the bond at a new interest rate (r + h) using the series expansion results in

$$P(r+h) = P(r) + P'(r)h + 1/2P''(r)h^{2}$$

The price of the bond is

$$P_0 = \sum_{t=1}^{T} \frac{cf(t)}{(1+r)^{t}}$$

Then the first derivative with respect to (1 + r) is

$$\frac{dp_0}{d(1+r)} = \sum_{t=1}^{T} \frac{-tcf(t)}{(1+r)^t} \quad \frac{1}{1+r}$$

and the second derivative is

$$\frac{d^2 P_0}{d(1+r)^2} = \sum_{t=1}^{T} \frac{t(t+1)cf(t)}{(1+r)^t} \quad \frac{1}{(1+r)^2}$$

This second derivative is called convexity.

$$\text{return} = \frac{P(r+h)-P(r)}{P(r)} = -D_i\Delta_i + 1/2C_i(\Delta_i)^2$$

$$D = \left[\sum_{t=1}^{T} \frac{tC(t)}{(1+r)^t}\right] P_0$$

$$C_i = \left[\sum_{t=1}^{T} \frac{t(t+1)C(t)}{(1+r)^t} \right] \div P_0$$

$$\Delta_i = \frac{d(1+r)}{(1+r)}$$

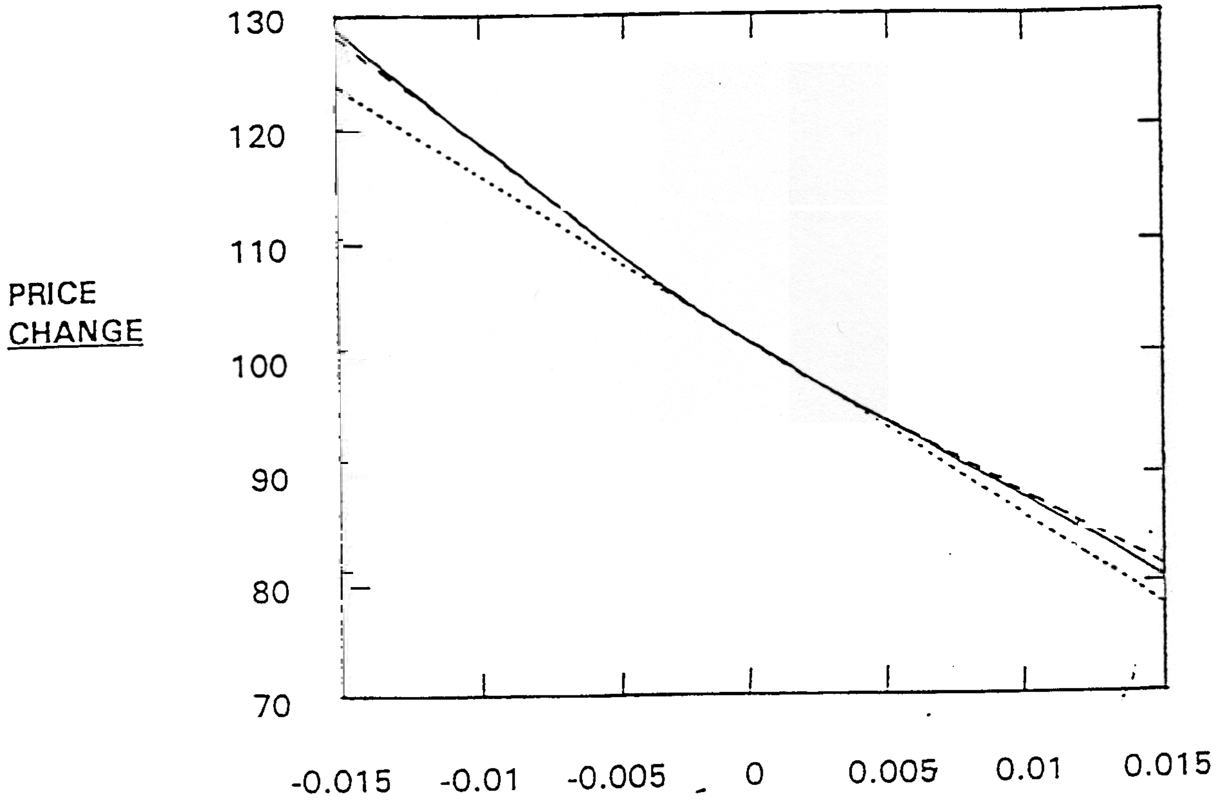
→ **Note convexity is positive gives higher return under all changes.**

→ **Consider two bonds with same duration and different convexity assume $C_A > C_B$. Does A give higher return?**

→ **Answer: If it did, we would have dominance. Thus return without yield curve shift must be higher for B.**

EXAMPLE 5

Coupon	**10%**
Maturity	**15 years**
Principal	**$100**
Price	**$100**
Yield curve flat	**10%**
Duration	**8.07**
Convexity	**96.59**



Actual Price ————————

Duration Only ....

Duration and Convexity - - - - - - - - - - -

R (x)	**p (x)**	**new price (x)**	**newprice2 (x)**
0.035	**127.588**	**122.059**	**127.001**
0.036	**125.429**	**121.521**	**124.956**
0.037	**123.322**	**119.984**	**122.946**
0.038	**121.264**	**118.447**	**120.97**
0.039	**119.254**	**116.91**	**119.03**
0.04	**117.292**	**115.372**	**117.125**
0.041	**115.376**	**113.835**	**115.255**
0.042	**113.504**	**112.298**	**113.419**
0.043	**111.674**	**110.761**	**111.619**
0.044	**109.889**	**109.223**	**109.854**
0.045	**108.144**	**107.686**	**108.124**
0.046	**106.44**	**106.149**	**106.429**
0.047	**104.774**	**104.612**	**104.769**
0.048	**103.146**	**103.074**	**103.145**
0.049	**101.555**	**101.537**	**101.555**
0.05	**100**	**100**	**100**
0.051	**98.48**	**98.463**	**98.48**
0.052	**96.994**	**96.926**	**96.996**
0.053	**95.542**	**95.388**	**95.546**
0.054	**94.122**	**93.851**	**94.131**
0.055	**92.733**	**92.314**	**92.752**
0.056	**91.375**	**90.777**	**91.407**
0.057	**90.047**	**89.239**	**90.098**
0.058	**88.748**	**97.702**	**88.824**
0.059	**87.478**	**86.165**	**87.584**
0.06	**86.235**	**84.628**	**86.38**
0.061	**85.019**	**83.09**	**85.211**
0.062	**83.83**	**81.553**	**84.076**
0.063	**82.666**	**80.016**	**92.977**
0.064	**81.527**	**78.479**	**91.913**
0.065	**80.412**	**76.941**	**80.884**

MEASURING

SENSITIVITY

NUMERICALLY

Numerical Estimation of Duration

t	r_{00t}	r'_{00t}
1	**10**	**11**
2	**11**	**12**
3	**12**	**13**
4	**13**	**14**
5	**14**	**15**

$$\frac{\Delta P}{P} = -D \;\; dr$$

$$P = \frac{5}{(1.05)} + \frac{5}{(1.055)^2} + \frac{5}{(1.06)^3} + \frac{5}{(1.065)^4} + \frac{105}{(1.07)^5}$$

$$P' = \frac{5}{(1.055)} + \frac{5}{(1.06)^2} + \frac{5}{(1.065)^3} + \frac{5}{(1.07)^4} + \frac{105}{(1.075)^5}$$

By definition:

$$\frac{P' - P}{P} = \textit{fraction change in price}$$

Calculation Duration

$P = 92.202$

$P' = 90.2817$

$$\frac{P' - P}{P} = -.021$$

$dr = .01$

Duration = 2.1

Two Factor Example

Assume two key rates

a. six-month rate

b. ten-year rate

Estimate Functional Relationship

$$\Delta r_{001} = 1\,\Delta r_{001} + 0\,\Delta r_{020}$$

$$\Delta r_{002} = .8\,\Delta r_{001} + .05\,\Delta r_{020}$$

$$\Delta r_{003} = .4\,\Delta r_{001} + .1\,\Delta r_{020}$$

$$\Delta r_{004} = .1\,\Delta r_{001} + .15\,\Delta r_{020}$$

$$\Delta r_{005} = .05\,\Delta r_{001} + .20\,\Delta r_{020}$$

t	r_{00t}	r_{00t}'	r_{00t}''
1	**10**	**11**	**10**
2	**11**	**11.8**	**11.05**
3	**12**	**12.4**	**12.10**
4	**13**	**13.1**	**13.15**
5	**14**	**14.05**	**14.20**

$$\frac{\Delta P}{P} = -D_1\, dr_1 - D_2\, dr_2$$

$$P = \frac{5}{(1.05)} + \frac{5}{(1.055)^2} + \frac{5}{(1.06)^3} + \frac{5}{(1.065)^4} + \frac{105}{(1.07)^5}$$

$$P' = \frac{5}{(1.055)} + \frac{5}{(1.059)^2} + \frac{5}{(1.062)^3} + \frac{5}{(1.0655)^4} + \frac{105}{(1.07025)^5}$$

$$P'' = \frac{5}{(1.05)} + \frac{5}{(1.05525)^2} + \frac{5}{(1.0605)^3} + \frac{5}{(1.06575)^4} + \frac{105}{(1.071)^5}$$

$\rightarrow \quad P = 92.202$

$P' = 92.028$

$P'' = 91.835$

$$\frac{P' - P}{P} = -0.0019$$

$$\frac{P'' - P}{P} = -.0040$$

$D_1 = .19$

$D_2 = .40$

TABLE 2

Estimated Sensitivities For The One Factor Model

The factor proxy is the four year spot rate.

$$d(r_{it}) = a_i + b_i d(r_{4t}) + e_{it}$$

Time to Maturity (in yrs)	MODEL 1[a] a_i	MODEL 1[a] b_i	MODEL 2[b] a_i	MODEL 2[b] b_i
0.16	-0.0043	0.798	-0.2193	0.886
	-0.0064	0.878	-0.1137	1.071
1	-0.0081	1.084	-0.0637	1.134
	-0.0074	1.117	-0.0366	1.161
2	-0.0059	1.125	-0.0211	1.138
3	-0.0023	1.072	-0.0590	1.055
4	0.0000	1.000	0.0000	1.000
5	0.0023	0.930	0.0046	0.942
6	0.0042	0.863	0.0081	0.886
7	0.0058	0.803	0.0180	0.836
8	0.0072	0.753	0.0128	0.790
9	0.0084	0.710	0.0142	0.748
10	0.0093	0.671	0.0151	0.707
11	0.0100	0.636	0.0157	0.669
	0.0105	0.604	0.0164	0.619
13	0.0109	0.573	0.0181	0.571

a. Model 1 assumes that the innovation in the spot rate follows a random walk. All changes are perceived as unexpected.

b. Model 2 derives the innovation in the spot rate under the pure expectation theory. The unexpected change is the difference between the actual spot rate and the forward rate from previous period.

TABLE 4

Estimated Sensitivities For The Two Factor Model

The factor proxies are the six year and the difference between the six year and the eight months rate.

$$d(r_{it}) = a_i + b_{i1} d(r_{6,t}) - b_{i2} d(r_{6-.6,t}) + e_{it}$$

Time to Maturity (in yrs)	MODEL 1[a] a_i	MODEL 1[a] b_{i1}	MODEL 1[a] b_{i2}	MODEL 2[b] a_i	MODEL 2[b] b_{i1}	MODEL 2[b] b_{i2}
0.16	0.0074	0.700	-1.000	-0.1176	0.734	-1.049
0.5	0.0025	0.915	-1.038	-0.0209	0.949	-1.039
0.66	0.0000	1.000	-1.000	0.0000	1.000	-1.000
1	-0.0036	1.079	-0.847	0.0066	1.071	-0.856
1.5	-0.0061	1.145	-0.640	0.0108	1.145	-0.524
2	-0.0055	1.162	-0.502	0.0113	1.141	-0.459
3	-0.0035	1.131	-0.305	0.0056	1.099	-0.235
4	-0.0032	1.091	-0.169	0.0022	1.072	-0.129
5	-0.0019	1.049	-0.069	0.0004	1.039	-0.051
6	0.0000	1.000	0.000	0.0000	1.000	0.000
7	0.0017	0.949	0.041	0.0065	0.958	0.029
8	0.0031	0.900	0.062	0.0019	0.915	0.043
9	0.0045	0.852	0.069	0.0034	0.870	0.046
10	0.0056	0.806	0.068	0.0051	0.824	0.042
11	0.0067	0.761	0.060	0.0070	0.778	0.033
12	0.0077	0.717	0.050	0.0087	0.716	0.026
13	0.0086	0.674	0.036	0.0110	0.652	0.022

a. Model 1 assumes that the innovation in the spot rate follows a random walk. All changes are perceived as unexpected.

b. Model 2 derives the innovation in the spot rate under the pure expectation theory. The unexpected change is the difference between the actual spot rate and the forward rate from previous period.

COMMENTS

1) **The advantage of Numerical calculation of duration lies in determining the duration for bonds with option features. One can numerically value the options at both sets of spot rates and then determine the price changes including the change in the value of the option.**

2) **The other consideration is that charges in spot rates are linked. The above assumption of a constant 1% change for all spots is unrealistic.**

3) **If one believed two factors one would obtain a D_1 and D_2 for each factor**

TIME SERIES ESTIMATION

Total Return =	Expected Return + Due to Passage of Time	Sensitivity to Unex- pected Term Structure Shifts	Unexpected Term + Structure Shifts	Random Error

$$r_{it} = \bar{r}_i - d_i \left(\frac{dr}{1+r} \right) + \varepsilon_i$$

Note:

(1) In stock area estimate directly

(2) In bonds D changes over time but can do for spots(unchanged duration), and build up to coupon bonds. Above is a "factor model."

Duration on portfolio is weighted average of duration bonds comprising it.

empirical duration = $\Sigma X_i D_i$

where D_i for each pure discount bond

Maturity	Empirical / Theoretical
329	1.294
.732	1.439
1.438	1.393
2.493	1.237
3.562	1.123
4.466	1.028
6.055	1.028
12.877	.755

Relation of Empirical to Theoretical Duration

Immunization

(1) Protecting against shifts in spot rates

a. $D_A = D_L$
b. $D_A = D_L$ $\qquad$ $C_A = C_L$
c. $D_{A1} = D_{L1}$ $\qquad$ $D_{A2} = D_{L2}$

(2) If PV (assets) > PV (Liabilities) immunization involves scaling

$$D_A = \frac{L}{A} D_L$$

Note:

Exact match portfolios are of course immunized.

Sensitivity Matching		Cash flow Matching
	vs	

1. **If all bonds fairly priced always cash flow match.**

2. **Must have sufficient mispricing to justify sensitivity.**

3. **Often optimum to do both.**

I. Terms

- **a. Duration**
- **b. Cash flow matching**
- **c. Sensitivity matching**

II. Concepts

- **a. Risks of cash flow matching**
- **b. Risks of sensitivity matching**
- **c. Analytical duration measures**
- **d. Numerical duration measures**
- **e. What affects duration**
- **f. Convexity**

III. Calculations

- **a. Various duration and convexity measures**

Problem

1. **Assume a bond has been semi-annual payments of $5 and a life of four years. What is its duration with a flat yield curve of 8%?**

   Answer:

   Price of bond is $106.73, which is the present value of cash flows.

$$Duration = \frac{\frac{1/2 \times 5}{(1.04)} + \frac{1 \times 5}{(1.04)^2} + \frac{3/2 \times 5}{(1.04)^3} + \dots + \frac{4 \times 105}{(1.04)^8}}{106.73} = 3.42$$

calculates duration with flat yield curve and examines change in price for shift in that curve and that predicted by duration

DURFLAT

$r := .05$ $c := 5$ $prin := 100$

$t := 1 .. 30$ $T := 30$ $x := -.015, -.014 .. .015$

$R(x) := r + x$

$$d(t) := \frac{1}{(1 + r + x)^t} \qquad d1(t) := \frac{1}{(1 + r)^t}$$

$t := 1 .. 29$

$$p(x) := \sum_t c \cdot d(t) + (c + prin) \cdot d(T) \qquad po := \sum_t c \cdot d1(t) + (c + prin) \cdot d1(T)$$

calculates duration

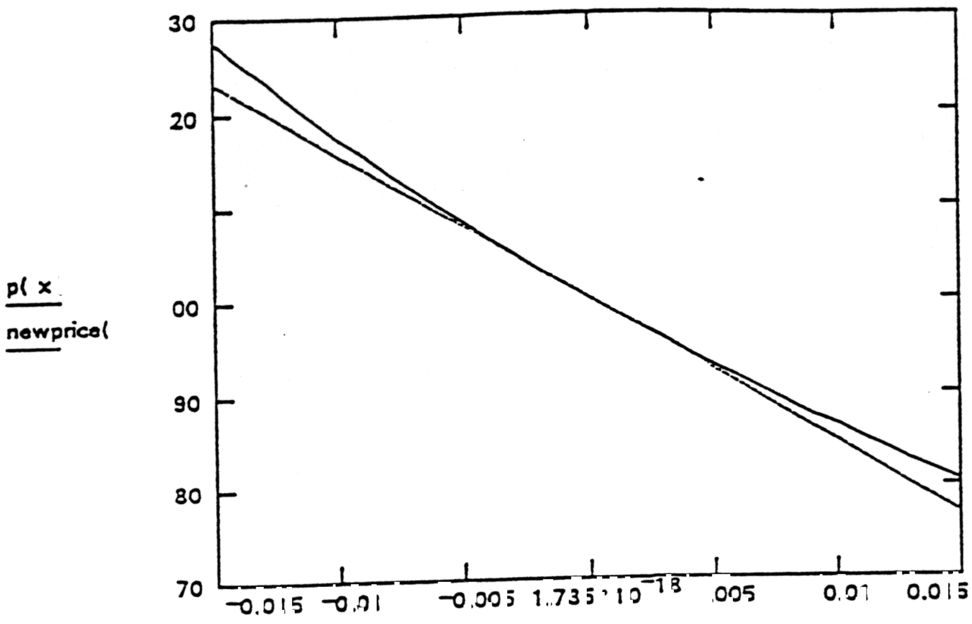
$$duration := \frac{\sum_t c \cdot t \cdot \left[\frac{1}{(1 + r)^t}\right] + (c + prin) \cdot \left[\frac{1}{(1 + r)}\right]^T \cdot T}{po}$$

$duration = 16.141$

calculates estimated price using duration

$$newprice(x) := -duration \cdot \frac{(x)}{(1 + r)} \cdot po + po$$

x	R(x)	p(x)	newprice(x)
− 0.015	0.035	127.588	123.059
− 0.014	0.036	125.429	121.521
− 0.013	0.037	123.322	119.984
− 0.012	0.038	121.264	118.447
− 0.011	0.039	119.254	116.91
− 0.01	0.04	117.292	115.372
− 0.009	0.041	115.376	113.835
− 0.008	0.042	113.504	112.298
− 0.007	0.043	111.675	110.761
− 0.006	0.044	109.889	109.223
− 0.005	0.045	108.144	107.686
− 0.004	0.046	106.44	106.149
− 0.003	0.047	104.774	104.612
− 0.002	0.048	103.146	103.074
− 0.001	0.049	101.555	101.537
0	0.05	100	100
$1 \cdot 10^{-3}$	0.051	98.48	98.463
0.002	0.052	96.994	96.926
0.003	0.053	95.542	95.388
0.004	0.054	94.122	93.851
0.005	0.055	92.733	92.314
0.006	0.056	91.375	90.777
0.007	0.057	90.047	89.239
0.008	0.058	88.748	87.702
0.009	0.059	87.478	86.165
0.01	0.06	86.235	84.628
0.011	0.061	85.019	83.09
0.012	0.062	83.83	81.553
0.013	0.063	82.666	80.016
0.014	0.064	81.527	78.479
0.015	0.065	80.412	76.941



30
20
00
90
80
70
p(x
newprice(
-0.015
-0.01
-0.005
.005
0.01
0.015

compares price response of pure discount to coupon paying bond

DURPURED

$r := .04, .05 .. .16$ $t := 1, 2 .. 10$ $T := 11$

$cf(t) := 10$ $cfT := 110$

price of coupon and pure discount bond

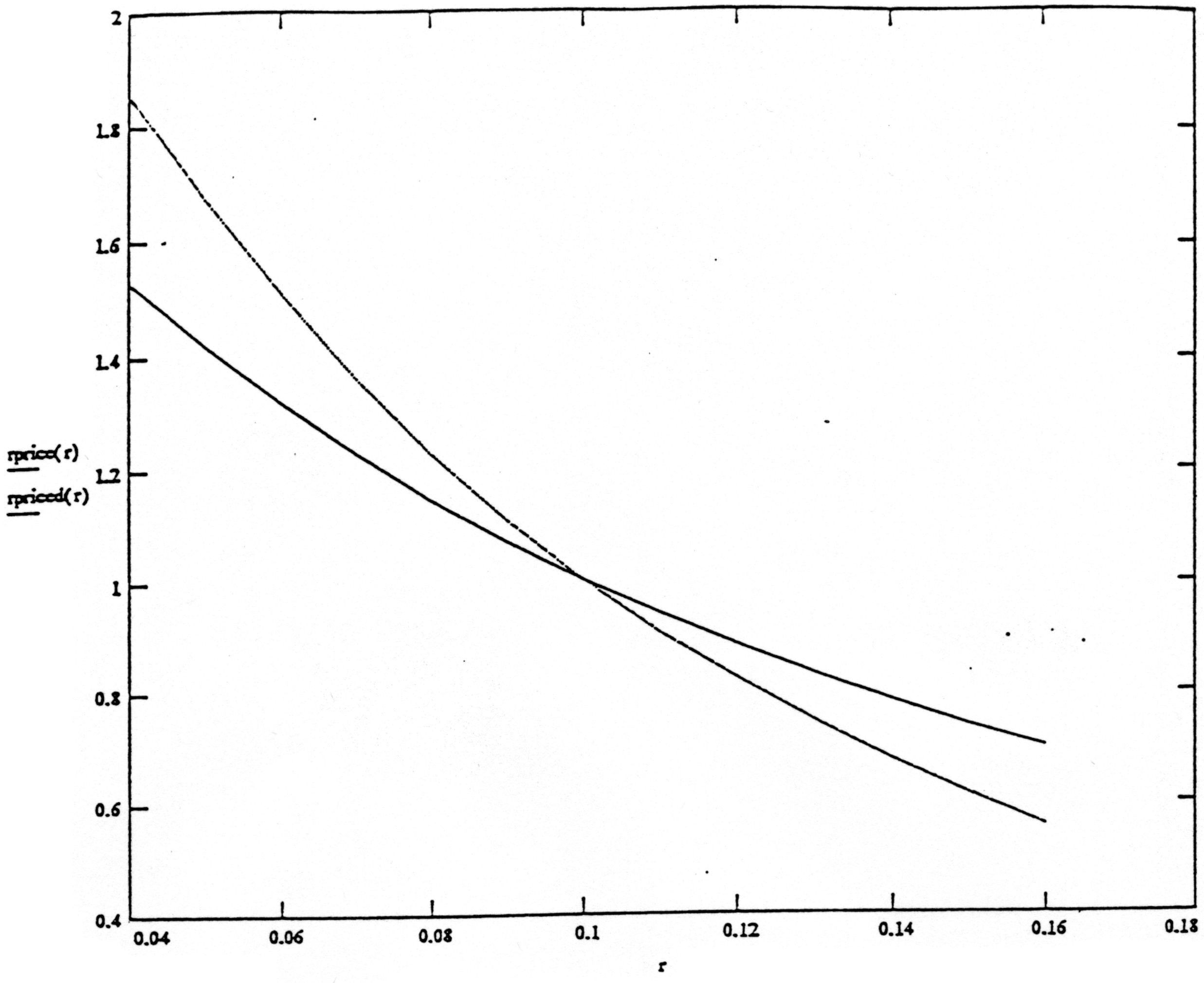
$$price(r) := \sum_t \frac{cf(t)}{(1+r)^t} + \frac{cfT}{(1+r)^T}$$

$$priced(r) := \frac{cfT}{(1+r)^T}$$

calculates relative price

$$rprice(r) := \frac{price(r)}{price(.10)}$$

$$rpriced(r) := \frac{priced(r)}{priced(.10)}$$



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
rprice(r)
rpriced(r)
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
r

price versus interest for coupon paying bond with upword sloping yield curve and changing yield curve

DURERROR

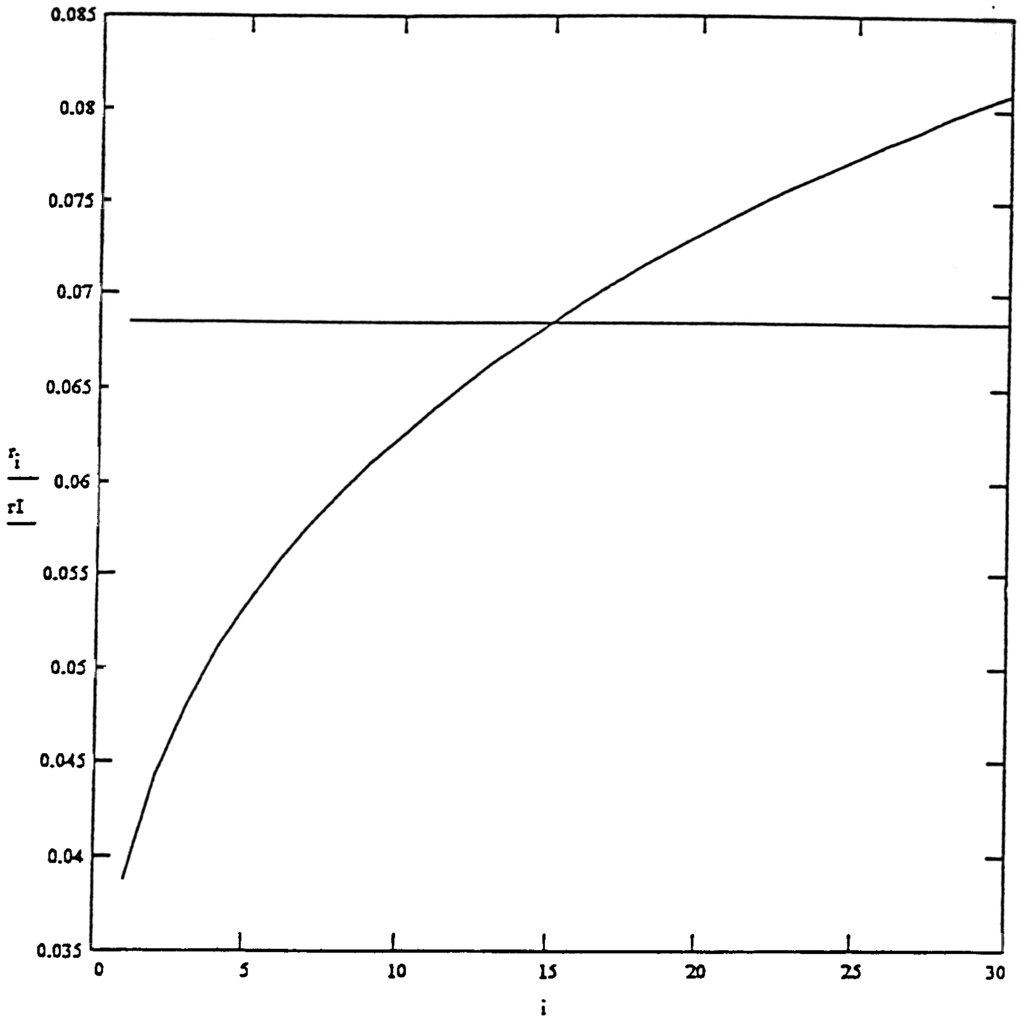
$t := 1 .. 29$ $\quad T := 30$ $\quad i := 1 .. 30$

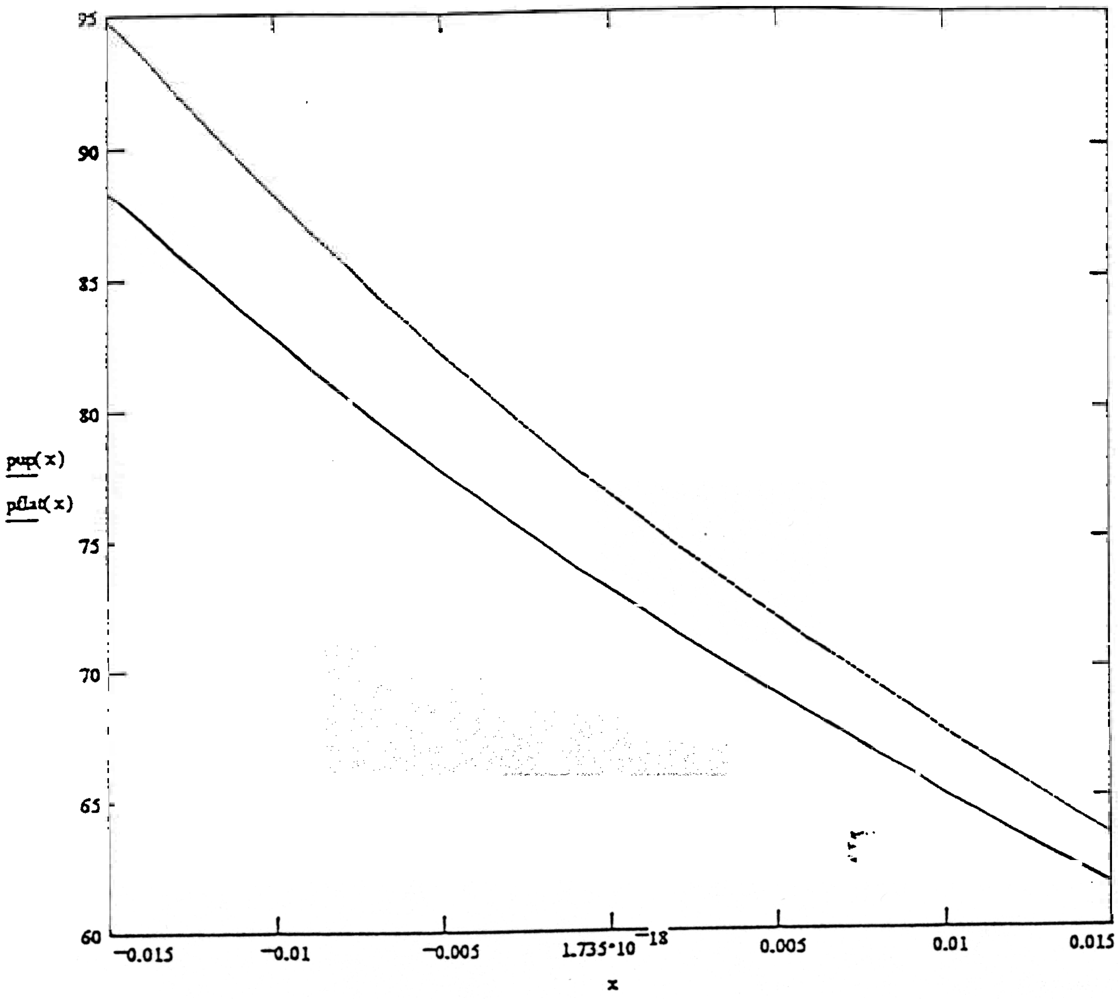
$c := 5$ $\quad prin := 100$ $\quad x := -.015, -.014 .. .015$

$$r_i := \left[.015 + .05 \cdot \left(\frac{i}{12} \right)^{.3} \right] \qquad r1 := .015 + .05 \cdot \left(\frac{15}{12} \right)^{.3}$$

$$d(t) := \frac{1}{(1 + r_t + x)^t} \qquad d1(t) := \frac{1}{(1 + r1 + x)^t}$$

$$pup(x) := \sum_t c \cdot d(t) + (c + prin) \cdot d(T) \qquad pflat(x) := \sum_t c \cdot d1(t) + (c + prin) \cdot d1(T)$$





95
90
85
80
75
70
65
60
pup(x)
pflat(x)
−0.015
−0.01
−0.005
1.735·10^−18
0.005
0.01
0.015
x

calculates duration for a upword sloping yield curve and the actual price after a shift in the curve as well as that predicted by duration

DURUP

starting values

$c := 5$ $\quad$ $prin := 100$ $\quad$ $r_0 := 1$ $\quad$ $T := 30$

calculates spots and forwards

$i := 1 .. 31$

spots

$$r_i := \left[.015 + .05 \cdot \left(\frac{i}{12} \right)^{.3} \right]$$

forwards

$$f(i) := \frac{(1 + r_i)^i}{(1 + r_{i-1})^{i-1}}$$

calculates discount functions

$x := -.01, -.009 .. .01$

$t := 1 .. 31$

$k := 1 .. 30$

$$d(t, x) := \frac{1}{\prod_k \text{if}\left[t<k, 1, f(k) + x \cdot \left(\frac{f(k)}{f(1)} \right) \right]} \qquad d1(t) := \frac{1}{(1 + r_t)^t}$$

calculate starting price and price with changing yield curve

$$po := \sum_t c \cdot d1(t) + (c + prin) \cdot d1(T)$$

$$p(x) := \sum_t c \cdot d(t, x) + (c + prin) \cdot d(T, x)$$

calculates duration in half years

$$duration := \frac{\left[\sum_t c\cdot t\cdot\left[\frac{1}{(1+r_t)^t}\right]\right] + (c + prin)\cdot\left[\frac{1}{(1+r_T)}\right]^T\cdot T}{po}$$

$duration = 13.054$

calculates price using duration alone

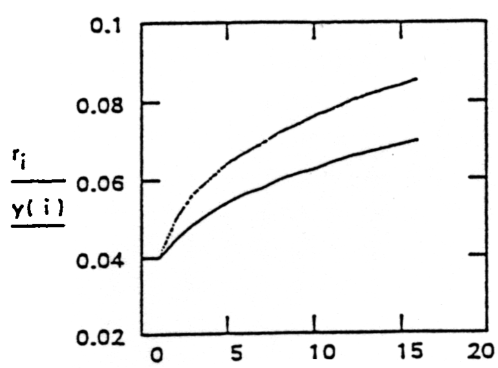
$$-\frac{(x)}{(1+r_1)}\cdot po + po$$

$y(i) := f(i) - 1$

$i := 1 .. 16$

plots spots and forwards

i	r_i	$y(i)$
1	0.039	0.039
2	0.044	0.05
3	0.048	0.056
4	0.051	0.06
5	0.053	0.063
6	0.056	0.066
7	0.058	0.069
8	0.059	0.072
9	0.061	0.074
10	0.062	0.076
11	0.064	0.078
12	0.065	0.079
13	0.066	0.081
14	0.067	0.082
15	0.068	0.084
16	0.07	0.085

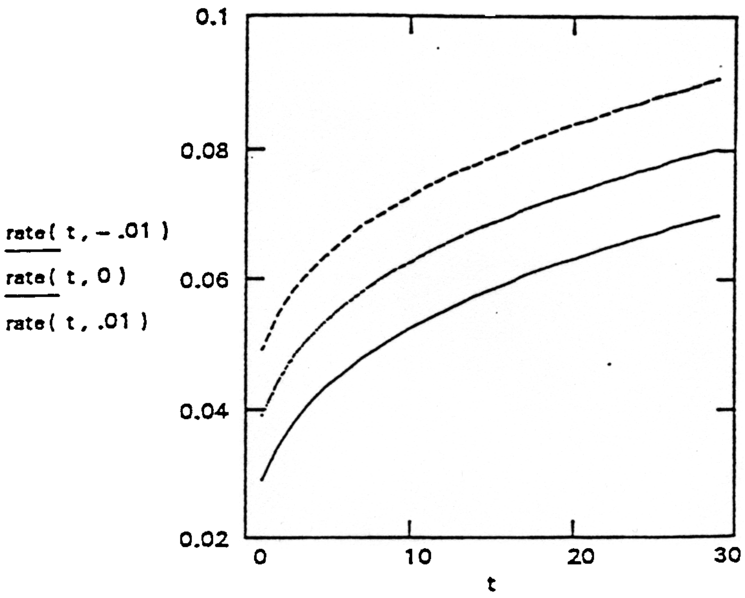


EXAMINES YIELD CURVE SHIFTS

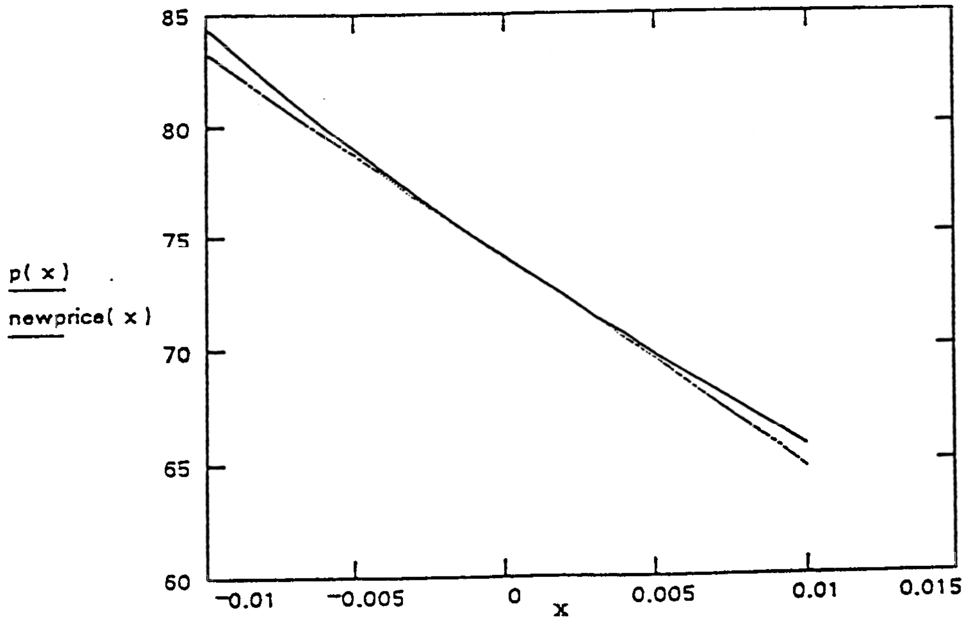
$$\text{rate}(t,x) := \left(\frac{1}{d(t,x)}\right)^{\left(\frac{1}{t}\right)} - 1$$

$$t := 1 .. 29$$

rate(t, –	rate(t,	(rate(t, .01)
0.029	0.039	0.049
0.034	0.044	0.054
0.038	0.048	0.058
0.041	0.051	0.061
0.043	0.053	0.064
0.045	0.056	0.066
0.047	0.058	0.068
0.049	0.059	0.069
0.051	0.061	0.071
0.052	0.062	0.073
0.053	0.064	0.074
0.055	0.065	0.075
0.056	0.066	0.076
0.057	0.067	0.078
0.058	0.068	0.079
0.059	0.07	0.08
0.06	0.071	0.081
0.061	0.071	0.082
0.062	0.072	0.083
0.063	0.073	0.084
0.064	0.074	0.084
0.065	0.075	0.085
0.065	0.076	0.086
0.066	0.077	0.087
0.067	0.077	0.088
0.068	0.078	0.088
0.068	0.079	0.089
0.069	0.079	0.09
0.07	0.08	0.091



x	p(x)	newprice (x)
− 0.01	84.381	83.28
− 0.009	83.244	82.351
− 0.008	82.131	81.421
− 0.007	81.041	80.491
− 0.006	79.975	79.561
− 0.005	78.931	78.632
− 0.004	77.909	77.702
− 0.003	76.908	76.772
− 0.002	75.928	75.842
− 0.001	74.968	74.912
0	74.027	73.983
0.001	73.106	73.053
0.002	72.204	72.123
0.003	71.32	71.193
0.004	70.453	70.264
0.005	69.604	69.334
0.006	68.772	68.404
0.007	67.957	67.474
0.008	67.157	66.544
0.009	66.373	65.615
0.01	65.605	64.685



Shows the effect of changes in the price of a bond do to a shift in the yield curve as well as the predicted price using duration and convexity

$r := .05$ $c := 5$ $prin := 100$

CONVEX

$x := -.015, -.014 .. .015$

$t := 1 .. 29$ $T := 30$ $R(x) := r + x$

$$d(t := \frac{1}{(1 + r + x)^t} \qquad d1(t := \frac{1}{(1 + r)^t}$$

$$p(x := \sum_t c \cdot d(t) + (c + prin) \cdot d(T \quad po := \sum_t c \cdot d1(t) + \ c + prin) \cdot d1(T)$$

calculates duration and convexity

$$duration := \frac{\left[\sum_t c \cdot t \cdot \left[\frac{1}{(1 + r)^t}\right]\right] + (c + prin) \cdot \left[\frac{1}{(1 + r)}\right]^T \cdot T}{po}$$

$$convexity := \frac{\left[\sum_t c \cdot t \cdot (t + 1) \cdot \left[\frac{1}{(1 + r)^t}\right] + (c + prin) \cdot \left[\frac{1}{(1 + r)}\right]^T \cdot T \cdot (T + 1)\right]}{po}$$

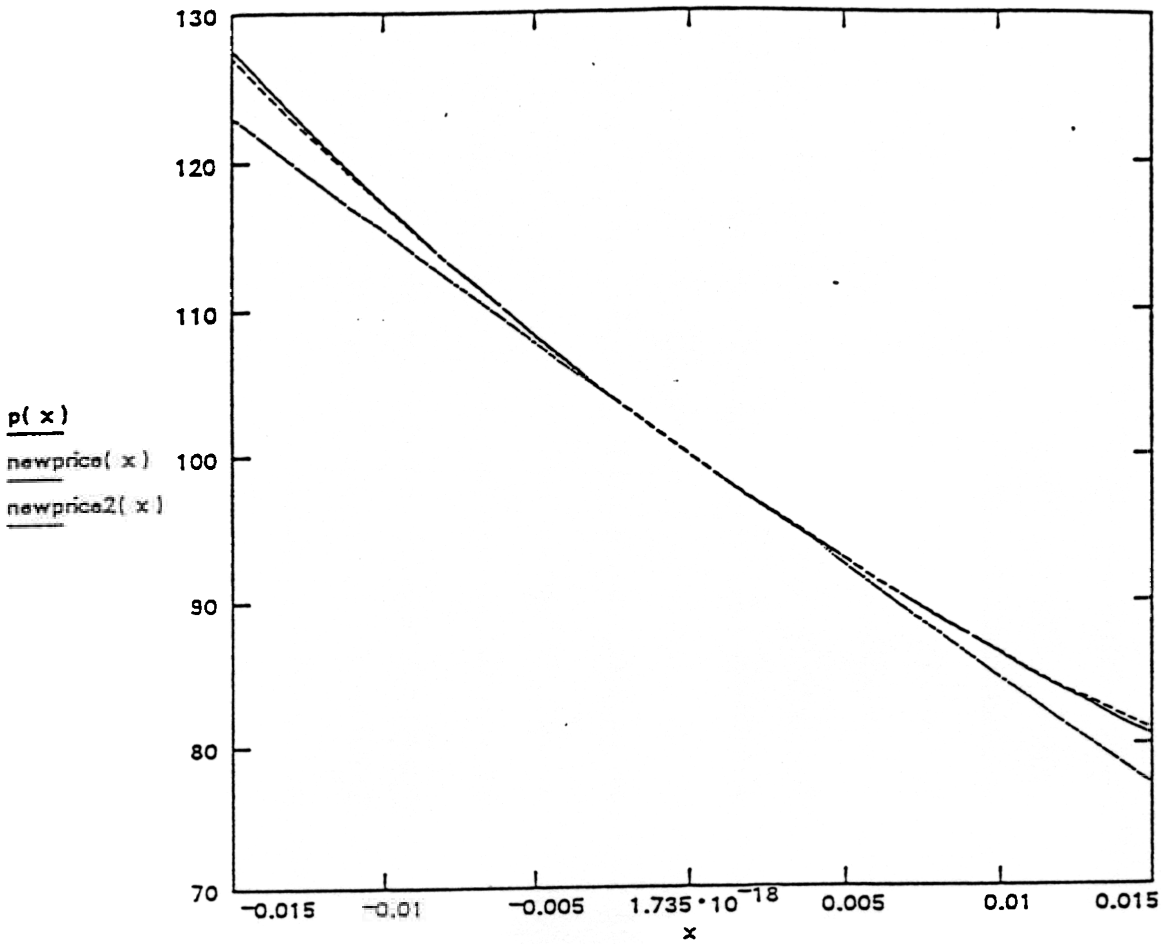
$duration = 16.141$ $convexity = 386.39$

calculates estimated price using duration alone and using duration and convexity

$$newprice(x) := po - duration \cdot \frac{x}{(1 + r)} \cdot po$$

$$newprice2(x) := po - duration \cdot \frac{x}{(1 + r)} \cdot po + .5 \cdot convexity \cdot \frac{x^2}{(1 + r)^2} \cdot po$$

R(x)	p(x)	newprice(x)	newprice2(x)
0.035	127.588	123.059	127.001
0.036	125.429	121.521	124.956
0.037	123.322	119.984	122.946
0.038	121.264	118.447	120.97
0.039	119.254	116.91	119.03
0.04	117.292	115.372	117.125
0.041	115.376	113.835	115.255
0.042	113.504	112.298	113.419
0.043	111.675	110.761	111.619
0.044	109.889	109.223	109.854
0.045	108.144	107.686	108.124
0.046	106.44	106.149	106.429
0.047	104.774	104.612	104.769
0.048	103.146	103.074	103.145
0.049	101.555	101.537	101.555
0.05	100	100	100
0.051	98.48	98.463	98.48
0.052	96.994	96.926	96.996
0.053	95.542	95.388	95.546
0.054	94.122	93.851	94.131
0.055	92.733	92.314	92.752
0.056	91.375	90.777	91.407
0.057	90.047	89.239	90.098
0.058	88.748	87.702	88.824
0.059	87.478	86.165	87.584
0.06	86.235	84.628	86.38
0.061	85.019	83.09	85.211
0.062	83.83	81.553	84.076
0.063	82.666	80.016	82.977
0.064	81.527	78.479	81.913
0.065	80.412	76.941	80.884



p(x)
newprice(x)
newprice2(x)
130
120
110
100
90
80
70
−0.015
−0.01
−0.005
1.735 · 10^−18
0.005
0.01
0.015
x