OPTION VALUATION

Fall 2000

Essentially there are two models for pricing options

a. **Black Scholes Model**
b. **Binomial option Pricing Model**

→For equities, usual model is Black Scholes. For most bond options there are problems that eliminate the Black Scholes model from consideration.

1. **Recall that a bond's volatility is a function of Duration. The Volatility is generally considered to be a direct function of Duration. As time passes Duration declines. But B.S. assumes constant volatility.**

2. **Black Scholes assumes the evolution of stock prices is a stationary process. But bond prices must converge to par at maturity so process must change.**

3. **Black Scholes assumes a constant short rate. But assuming a constant risk-free rate while that long rate changes simultaneously does not make much sense.**

→Where are the Black Scholes assumptions not badly violated:

1. **A short-term option on a long instrument since duration will not change very much over the life of the option…**

2. **An option on a future. In this case the deliverable instrument is a "constant" maturity bond whose duration is fairly stable.**

In these two cases Black Scholes can be made to work well. For future ones use a variation in Black Scholes formula called Black Model. In other cases a binomial model is needed.

There are a lot of binomial models. There are a number of ways we can model changes in interest rates or discount functions. The basic characteristics that drive these models are:

1. **That they be arbitrage free or "no free lunch." However, we assume interest rates or discount function evolve, we can't find a strategy that always has a higher return no matter what.**

2. **No memory. An up movement followed by a down movement is the same as a down movement followed by an up movement, e.g., how we get somewhere is unimportant. Reason for this assumption is computational. It results in manageable problems.**

The Black Scholes Model

$$C = S_0 N(d_1) - Ee^{-rt} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / E) + (r + 1/2 s^2)t}{s \sqrt{t}}$$

$$d_2 = \frac{\ln(S_0 / E) + (r - 1/2 s^2)t}{s \sqrt{t}}$$

$S_0 =$ **Current price of security**
$E =$ **Exercise Price**
$r =$ **interest rate (continuously compounded)**
$t =$ **time to maturity in fraction of year**
$s =$ **Standard deviation of returns (continuously compounded)**
$N(\ ) =$ **Cumulative Normal**

(Black model is same as above except S_0 **is replaced by Futures price times** e^{-rt}**.)**

Some Definitions

1. **Hedge ratio** $d = \frac{\partial C}{\partial S} = N(d_1) > 0$

2. $g = \frac{\partial^2 c}{\partial^2 s} = \frac{\frac{1}{S}}{s\sqrt{t}} n(d_1)$

3. **Put is valued using Put Call Parity**

BINOMIAL OPTION PRICING

(SINGLE STATE MODELS)

EXAMPLE:

Assume

$R_{01} = 6\%$

$R_{02} = 7\%$

Assume one period rate can evolve to

1/2 $8.5 = S_u$

6

1/2 $5.5 = S_d$

ZEROS and their PRICES

One Period

Price	S_u	S_d
$\frac{1000}{1.06}$	1000	1000

0 1

1000

U

943.40

d

1000

Two Period

Price	S_u	S_d
$\frac{1000}{(1.07)^2}$	$\frac{1000}{1.085}$	$\frac{1000}{1.055}$

0	1	2
		1000
	921.66	
873.44		1000
	947.87	
		1000

NOTE

1. **Earlier mentioned Binomial option has to have "no memory."**
   This is why tree has three points at time 2.

2. **Check on "no free lunch."**

   873.44 x 1.06 = $925.85

   Thus, if investor buys two period bond, it can be better or worse than one period.

3. **Note [1/2(921.66)+1/2(947.87)]/1.06 (e.g., present value of bond prices at 1 does not equal $873.44). There is a risk premium.**

Pricing option using replication

Consider option to buy one year zero next year for $930

Next year price is $921.66 or $947.87

0

$17.87

Question: What is value?

Can buy at time zero a two-year pure discount and a one-year pure discount.

Let

$X_1 =$ **fraction of one-year pure discount**

$X_2 =$ **fraction of two year pure discount**

Construct portfolio with same payoff as option

At time one a then one-year zero will be worth either \$921.66 or \$947.87 and, thus, this will be the value of a time zero two period pure discount instrument

$$X_1(1000)+X_2(921.66)=0$$

$$X_1(1000)+X_2(947.87)=17.87$$

Solving

$$X_1=-.62839$$

$$X_2=+.68180$$

The price of this portfolio is

$$(-.62839)x(943.40)+(.68180)(873.44)=\$2.69$$

Since this is equivalent is should cost the same or have arbitrage.

NOTE

1. Probabilities were never used in valuing option.

2. Logically, if up movement was more likely option, would be worth less so probabilities must enter indirectly. If up movement is more likely bond would be worth less and working through analysis to obtain option value would result in a lower value.

3. Preceding was used to value call, but could be used to value any instrument whose value depended on interest rate movements.

Alternative way of pricing option
(risk neutral)

The basic idea of risk neutral price is as follows. We did not use characteristics of investors (accept prefer more to less) in deriving valuation. Therefore, must hold for all investors. One group that is convenient is risk neutral. The advantage of using this group is they only care about expected value.

Recall tree for two-year bond

		1000
	921.66	
873.44		**1000**
	947.87	
		1000

What probabilities will make expected value at one; give a price of $873.44 at zero

$$\frac{\$921.6\,P_1 + 947.87(1 - P_1)}{1.06} = \$873.44$$

$$P_1 = .84$$

Now consider option

0

.84

$2.69

.16

17.87

If risk neutral value is

$$\frac{(.84)0+.16(17.87)}{(1.06)}=\$2.69$$

Assume

$$r_{003} = 7.5\%$$

0	**1**	**2**
	Rates	
		9
	8.5	
6		**7**
	5.5	
		5

Prices

1000

917.43

P_u 1000

804.96 934.58

P_d 1000

952.38

1000

Solving for risk neutral probabilities

$$P_u = \frac{[P(917.43) + (1 - P)(934.58)]}{(1.085)}$$

$$P_d = \frac{[P(934.58) + (1 - P)(952.38)]}{(1.055)}$$

$$804.96 = \frac{.84P_u + .16P_d}{(1.06)}$$

				917.43
			.92	
		846.81		
	.84		**.08**	
804.96				**934.58**
	.16		**.92**	
		887.19		
			.08	
				952.38

<u>Note:</u>

1. **One factor model. Everything depends on evolution of six-month rates.**

2. **Can change frequency of up and down to month or day.**

Second Example of Getting Probabilities

Zero Prices

Zero Price	Maturity
943.40	1
881.66	2
816.30	3

0	1	2
		9
	7.5	
6		8
	5.5	
		7

One period zero

1000

943.40

1000

Examples

1. **Consider option to buy two-year zero in one-year a $860.**

0.0

.84

?

.16

27.19

$$\frac{27.19x.16}{1.06} = \$4.10$$

$$x_1 1000 + x_2 921.66 = 0$$

$$x_1 1000 + x_2 947.87 = 27.19$$

$$x_2 = 1.03739$$

$$x_1 = -.9561212$$

Getting risk neutral probabilities from one to two.

Two period

			1000
		930.23	
	P		
881.66			**1000**
	1-P		
		947.87	
			1000

930.23 P + 947.87 (1-P) = 881.66

P=.73

Three year zero.

				1000
			917.43	
		P_u		
	.73	(1-P)		1000
816.3			925.93	
	.27			
		P_d		
		(1-P)		1000
			934.58	
				1000

$$P_u = \frac{P(917.43)+(1-P)925.93}{(1.075)}$$

$$P_d = \frac{P(925.93)+(1-P)(934.58)}{(1.055)}$$

$$816.3 = \frac{.73(P_u)+.27P_d}{816.3}$$

$$P = .335$$

2. **Consider option to buy one-year zero in two years at $930.**

				0.00
			.92	
		.3377		
	.84		**.08**	
$1.13				**4.58**
	.16		**.92**	
		5.691		
			.08	
				22.38

$$\frac{4.58\text{x}.08}{1.085}=33.77$$

$$\frac{4.58 \text{ x } .92+22.38 \text{ x } .08}{1.055}=5.691$$

3. Consider a three-year bond callable at 1000 paying a coupon of 6% with call protection until time 2.

0		1		2
				0.0
			.92	
		0		
	.84		.08	
?				0.0
	.16		.92	
		.722		
			.08	
				9.524

value = .109

Non callable

$$P = \frac{60}{1.06} + \frac{60}{(1.07)^2} + \frac{1060}{(1.075)^3}$$

$$P = \$962.27$$

$$\text{Price} = 962.27 - .109 = 962.161$$

Caps, Floors and Collars

- **Issued with floating rate note or floating rate mortgages**

- **cap fixes a maximum interest rate, e.g., floating rate note can't exceed 8%**

- **cap is like a call option, e.g., an 8% cap on six-month LIBOR can be valued as if cap pays (actual LIBOR - 8%) x** $\frac{\text{(days in period)}}{360}$

- **floor fixes minimum rate, like a put option**

- **collar fixes both maximum and minimum rate**

Example

Consider three-period floating rate bond with a cap of 8.5%. The replicating portfolio is:

1. **a floating rate note**

2. **issuing a call option that pays off the rate minus 8% if interest exceeds 8%**

	0	**1**	**2**
			9 - pays 1% under this scenario
		8.5	
6			**7**
		5.5	
			5

0	1	2	3
			1
			1

value is: $$\frac{\frac{1 \text{ x } .92}{(1.09)\,1.085} \text{ x } .84}{1.06} = .61646$$

Value of floating rate bond with cap is:

$$100 - .61646$$

Assume the cap was a cap of 8%.

9

8.5

6 7

5.5

5

Loss due to cap

0 1 2 3

1

.5

Value $= \dfrac{\dfrac{2 \text{ x } .92}{1.085} + .5}{1.06} \text{ x } .84$

Value $= 1.3439$

<u>Need Process of Spot Changes</u>

- **Must be consistent with current rates**
- **Arbitrage free**

$u^2 r$

ur

r $\qquad\qquad$ udr

dr

$d^2 r$

one choice original Solomon Model

$$u = e^{m t + s\sqrt{t}}$$

$$d = e^{m t - s\sqrt{t}}$$

Note:

1. **Order of up and down does not matter**

$$ud = (e^{m1+s}) \text{ x } (e^{m2-s}) = e^{m1+m2}$$

$$du = (e^{m1-s}) \text{ x } (e^{m2+s}) = e^{m1+m2}$$

They fix:

1. **Probabilities = 1/2**
2. s **which is volatility of short term rate**

Note:

1. **Negative rates can't occur.**

2. **Rate changes proportional to level.**

3. **Does not account for term structure of volatility.**

Assume following rates

$$r_{oo1} = 10 \qquad\qquad P_1 = 909.09$$

$$r_{002} = 10\,1/2 \qquad\qquad P_2 = 818.98$$

$$r_{003} = 11 \qquad\qquad P_3 = 731.19$$

1. **Estimate volatility at .2 corresponds to 20% per year and up and down equally likely.**

2. **Imply m**

$$818.98 = \frac{\left[\frac{1/2(1000)}{1+ru} + \frac{1/2(1000)}{1+rd}\right]}{(1.10)}$$

Solving for m_1 **yields**

$$m_1 = .0817$$
$$ru = .1312$$
$$rd = .0897$$

$1/2ru + 1/2rd \cong$ **Forward**

1000

$$\frac{1000}{1+ru}$$

u **1000**

$$\frac{1000}{1+rd}$$

1000

$$up = \frac{\frac{.5(1000)}{(1+u^2r)} + \frac{.5(1000)}{(1+udr)}}{1+ur}$$

$$down = \frac{\frac{.5(1000)}{(1+udr)} + \frac{.5(1000)}{1+d^2r}}{(1+dr)}$$

$$731.19 = \frac{.5(up)+(.5)down}{(1.10)}$$

1000

$$\frac{1000}{1+r\bullet e^{m_2+2s}}$$

1/2

up

1/2 **1/2**

$$\frac{1000}{1+r\bullet e^{m_2}}$$

1000

731.19

1/2 **1000**

down

$$\frac{1000}{1+r\bullet e^{m_2-2s}}$$

1000

Unknown

m_2

up

down

$$M_2 = .1556$$

$$r_{uu} = 17.429$$

$$r_{ud} = 11.68$$

$$r_{dd} = 7.83$$

Black Derman Toy

(Goldman Sachs)

1. Fix term structure volatility, e.g., allow one-year rates to vary more than two-year rates etc. This is principle advantage of the model. Empirical evidence supports short rate varies more than long.

2. Fix probabilities

Assume Prior Example

$r_{oo1} = 10$ $\quad$ $P_1 = 909.09$

$r_{002} = 10.5$ $\quad$ $P_2 = 818.98$

$r_{003} = 11.0$ $\quad$ $P_3 = 731.19$

Estimate

1. $s_1 = .19$ $s_2 = .18$

2. **up + down = 1/2**

3. **expectations theory**

$$u_1 = e^{m_1 + s_1}$$

$$d_1 = e^{m_1 - s_1}$$

Calculating Second Period Spots

$$.19 = \frac{1}{2} \ln\left(\frac{u_1}{d_1}\right)$$

$$\frac{1}{(1+r_{002})^2} = \frac{\left(\frac{1}{(1+u_1 r)}\right) + \left(\frac{1}{(1+d_1 r)}\right)}{\frac{2}{1.10}}$$

13.12

10%

8.97

$$m_1 = .081$$

CALCULATES SPOT RATES IN PERIOD 3

$$\frac{1}{(1+r_{oo3})^{3}}=\frac{1}{(1+r_{001})}\frac{\left[\frac{1}{1+u2}\right]+\left[\frac{1}{1+d2}\right]}{2}$$

$$.5\ln\left(\frac{u2}{d2}\right)=.18$$

$$\frac{1}{1+u2}=\left[\frac{1}{[1+u]}\right]\left[\frac{\frac{1}{(1+uu)}+\frac{1}{(1+ud)}}{2}\right]$$

$$\frac{1}{(1+d2)}=\left[\frac{1}{(1+d)}\right]x\left[\frac{\frac{1}{(1+ud)}+\frac{1}{(1+dd)}}{2}\right]$$

$$uu.dd=(ud)^2$$