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INTER-TEMPORAL PORTFOLIO ANALYSIS BASED ON SIMULATION OF JOINT RETURNS*[1]

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An inter-temporal quadratic programming model for selecting portfolios of risky assets is formulated. The model's application to capital budgeting is discussed in considerable detail. This is followed by briefer discussions of its application in other areas. The paper develops a new and more efficient way of using simulation to calculate the variance-covariance elements required as input to Markowitz-type models; this new procedure makes no further demands on the decision maker than do other simulation models that have been suggested. The paper also shows how such models can be made inter-temporal. Finally, the paper analyzes the potentials for extending such models to include other theoretical considerations that have been proposed in the literature.

This paper develops an inter-temporal model for the selection of risky assets. The most relevant antecedents of this model are the works of Weingartner [28] in capital budgeting, Hertz [9] and Bossons [3] in simulation, and Markowitz [17], Chambers and Charnes [5], and Cohen and Hammer [6] in portfolio selection.[2] In particular, we present a new and more efficient way of calculating the variance-covariance terms in asset selection models that have quadratic objective functions; discuss how to make these models inter-temporal; consider how the current portfolio of the investor or firm should be incorporated; and indicate directions in which the model might be extended. Throughout this paper we indicate the range of assumptions under which our method of analysis is valid (although the results of the analysis may sometimes change when assumptions are altered).

So that what we are doing is made more explicit, we shall go into a considerable amount of detail in our treatment of capital budgeting. This will be followed by a short discussion of the application of the model in selecting securities, and a still briefer consideration of its application in other areas. Before developing the model and its application to the firm's asset selection decision, we shall discuss the appropriateness of the standard deviation as a measure of risk.

Standard Deviation as a Measure of Risk

It is necessary for our analysis to assume the existence of a well-defined and operational measure of risk. Risk can mean a number of things: variability of

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[2] Quirin [20], Van Horne [27], and Weingartner [29] in recent articles have suggested applying Markowitz portfolio analysis to capital budgeting. This paper goes well beyond their analyses in specifying how and when such a model should be applied, as is indicated in the remaining sentences of the first paragraph in the text.

earnings, probability of loss greater than a certain amount, and probability of bankruptcy, among others. At least in many cases, the main source of risk is variability in earnings. In these cases, if the variability in earnings of the firm approximates a two parameter distribution, if investors have quadratic utility functions,[3] or if the shape of the frequency function of the firm's earnings remains relatively constant over time,[4] then the standard deviation is a good measure of risk. Even in cases where none of these conditions is exactly met, the standard deviation may still be the most reasonable operational measure available. Thus, in the rest of this paper we shall adopt the heuristic of using the standard deviation to represent risk. This naturally leads to the use of an "expected return, standard deviation of return" (E, σ) type of efficient frontier.

In generating the efficient frontier, we have used the present value of cash flows as the random variable whose expected value is maximized and whose standard deviation is a measure of risk. Our analysis would still be appropriate with many other objective functions or random variables.[5] Finally, it should also be noted that in order to solve the models we propose, it is necessary to assume that the discount rate between certain and uncertain assets of a given type (i.e., the risk premium) remains constant throughout time.

The Capital Budgeting Model Without Budget Constraints

With this background, we are now ready to discuss the discounting of a single cash flow before considering the model itself. In order to find the present value of a cash flow that occurs in period t, we must discount this flow for risk and time. From our previous discussion, risk is some function of the standard deviation of the cash flow discounted for time.

Therefore, the present value of a cash flow received in period t is:

$$PV(f_{it}) = f_{it}/\{(1 + p_i)(1 + r_t)^t\}$$

where

1. $PV(f_{it})$ = present value of cash flow i in period t.
2. p_i = the discount rate for returns of the same risk class as $f_{it}/\{(1 + r_t)^t\}$

[3] For a justification, see Tobin [25]. Note also that the quadratic utility function we are talking about is a quadratic rate of return function. For a discussion of the properties of this and other possible utility functions, see Freimer and Gordon [8].

[4] In this case when one compares a firm to others, as will be done later, it is necesary to find firms with a similar risk structure. Operationally this could mean industry groupings; see King [14].

[5] It is not necessary to use present value as the criterion. *E.g.*, the analysis developed in this paper would be equally applicable (although the solutions generated by the model may not be the same) if the internal rate of return or the horizon value of the firm were used as the firm's objective function.

Note that in the context of this paper, "present value" is being used as a random variable to represent the discounted returns to a firm or an investor if a particular risky asset is added to the portfolio. "Present value" is not being used here to represent the maximum amount that a firm or an investor would pay for the particular risky asset; given well-defined preferences, this maximum amount is, of course, a constant, not a random variable.

that changes $f_{it}/(1 + r_t)^t$ into its certainty equivalent. p_i depends on the standard deviation of $f_{it}/(1 + r_t)^t$.

3. r_t = the (riskless) time discount rate that converts certain returns in period t to returns in period $t - 1$.
4. t = time index.
5. i = project index.

Since we assume that r_t is a known constant, the above can be simplified to

$$PV(f_{it}) = E_{it}/(1 + p_i)$$

where

$$E_{it} = f_{it}/(1 + r_t)^t$$

If we have a second cash flow in t, however, then the discount rate of the two projects taken together need have no relationship to the discount rate of the projects taken separately unless they are perfectly correlated. The appropriate measure of risk is the standard deviation of the present value of the combined flows. If the cash flows occur in different periods the same conclusion follows, since these also may have a positive covariance.

This discussion suggests the use of a model adapted from Markowitz but utilizing present value rather than rate of return.[6] The simplest form of the model is:[7]

$$\text{(1)}\qquad \begin{aligned} &\text{Minimize } x'\,\Sigma\, x\\ &\text{Subject to:}\\ &\quad 1.\ \mu x = c\\ &\quad 2.\ x_0 = 1\\ &\quad 3.\ x_i \leqq 1 \quad i = 1 \cdots n\\ &\quad 4.\ x_i \geqq 0 \quad 1 = 1 \cdots n + m \end{aligned} \qquad \mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{n+m} \end{bmatrix} \quad x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

where

1. μ is a $1 \times (n + m + 1)$ vector of the expected present values of the various investment projects. In terms of previous notation μ_i is $\Sigma_t E_{it}$.
2. The subscript 0 refers to current investments. For example, μ_0 is the expected present value of the cash flows associated with current investments. The subscripts 1 to n refer to physical investments, and $n + 1$ to $n + m$ refer to investments in securities.
3. x is a $(n + m + 1) \times 1$ vector giving the amount invested in a project. It is assumed to be 0 or 1 for physical investments and unbounded from above for investments in securities.[8] x_i is an element of that vector.[9]

[6] For discussion of the reasons why present value is preferred to rate of return, see Alchian [1], Lorie and Savage [16], Hirscheifer [11], and Weingartner [28].

[7] For computational purposes one might want to reformulate the model so that the objective function becomes maximize the expected return (μx) and the variance-covariance matrix is put into the constraints ($x'\Sigma x \leqq C$). See, e.g., van de Panne and Whinston [26].

[8] For both economic and computational reasons, upper bound constraints on investments in securities may often be required.

[9] The projects need not all be started in the same period. We can formulate the model to

4. $x_0 = 1$ forces us to continue with current investments.
5. c is a parameter whose value changes in generating the efficient frontier.
6. Σ is the variance-covariance matrix between projects' present values; its derivation is described below.

The formulation of the problem in (1) is the most simplified model. Variations will be discussed later.

We shall now indicate how to generate the variance-covariance matrix (Σ) using a simulation of joint returns approach. First, we must specify for each current or anticipated investment and for the firm as a whole a structural model indicating what factors determine the cash flows in each period.[10] For example, we may decide that cash flows associated with an equipment replacement project depend on the sales of the company, the cost of labor, the cost of material, etc. For this investment, we would then determine the relationship between the cash flows and these underlying factors.[11]

Next we specify the joint probability distribution of these underlying factors; frequently this can be done most conveniently by specifying separate distributions for each underlying factor and the inter-relationships among the factors. The functional form and/or parameter values of the joint distribution may change over time, and the change may depend on the values assumed by specific random variables (including, perhaps, the values of these underlying factors themselves) in the same or previous periods. The specification of the structural models and the distributions of the factors must be detailed enough so that all inter-relationships are explicitly stated. In the above example, the joint probability distribution may reasonably be specified in such a way that (a) the cost of labor depends on sales, and (b) the mean of the distribution of sales is a constant times the previous period's sales.

After having established these specifications, we then simulate period-by-period the values of each of the underlying factors. Initially, we use the values of the factors that were generated for the first period to calculate the cash flow during the first period for *each* investment project (current or anticipated) and for the firm as a whole. The cash flow of each of these investment projects is stored separately. We then use the values of the underlying factors that were generated for the second period to calculate the cash flow for each investment project during the second period. The cash flows generated in the second period are then discounted (using the riskless discount rate r_2) and added to the cash

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consider the possibility of starting a project in any of several periods by regarding this project as the equivalent of several related projects, one to be started in each period, and then imposing a constraint that permits at most one of these related projects to be undertaken. This type of formulation can be adopted for both physical investments and investments in securities; its advantages will be discussed in more detail in a later section.

[10] Some simplifying assumptions may be needed to reflect the effects of future investments on future cash flows.

[11] Some of the underlying factors may, of course, be the probabilities that new investments come along.

TABLE 1

Present Values of Investment Projects Generated by Each Simulation Run

	Run 1	Run 2	. . .	Run w
Investment 0	V_{01}	V_{02}	. . .	V_{0w}
Investment 1	V_{11}	V_{12}	. . .	V_{1w}
⋮	⋮	⋮		⋮
Investment $n + m$	$V_{n+m,1}$	$V_{n+m,2}$		$V_{n+m,w}$

flows that were calculated in the first period. This process of calculating the cash flows for the projects, discounting them to the present, and then adding the newly generated discounted cash flows to the cumulative discounted cash flows from previous periods is continued until the last period of the simulation.[12] At this point we have calculated the present value of the cash flows for each investment project for the first simulation run. Note that the cash flows have been discounted for time but not for risk. Further simulation runs are then made, and the sequence of present values for each of the projects is stored. After a sufficient number of runs, the simulation is terminated.

We now have the information shown in Table 1. Each entry in the table is the present value of one investment project for a given multi-period trial. For example, V_{i1} is the present value of project i (i.e., $\Sigma_t E_{it}$) that was calculated during the first simulation run. Since we stored the sequence of present values, we can easily calculate the covariances, variances, and means of the present values.[13] After we have calculated the above present values, we apply a quadratic programming algorithm to the model formulated in (1) to generate the efficient frontier.[14] An example of the efficient frontier is presented in Figure 1.

[12] This may be the point where the future returns for all projects still generating cash are constant, or the point where future periods in the simulations become irrelevant either because they do not affect the optimal first-period decision, make negligible contributions to the value of the objective function, or require computing costs which exceed possible further gains. See Modigliani and Cohen [18] for further discussion of this point.

[13] This is analogous to the security analysis input used in the Markowitz security portfolio selection model. What is required by our procedure, however, is the specification of underlying distributions on which the returns are based, rather than the joint distribution of the returns themselves.

Note that in order to compute the covariances of the present values of cash flows, it is necessary to use present values that were generated during the same simulation run. This is the only reason for storing the present values in sequence as they are generated. In practice, one might instead wish to calculate and accumulate the relevant sums of present values and their squares and cross-products, rather than storing all of the present values.

[14] This is really an integer quadratic programming problem. One algorithm which can be used to solve a limited class of such problems has been developed by Burford and Lang [4].

As yet, we do not have enough experience to be able to generalize on how important is the problem of fractional projects. Note that fractional projects may provide information useful for the design of new projects (see Weingartner [28]). Also, if the projects are related

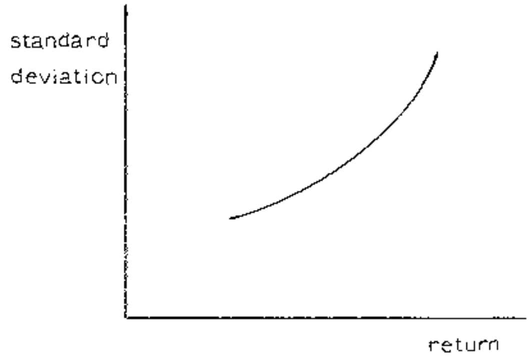

FIG. 1. Possible risk-return combination for the firm

In the capital budgeting context the efficient frontier indicates the achievable risk-return trade-offs, given the firm's available investment alternatives. At this stage, it may be desirable for the firm's managers to exercise their judgment in selecting one particular point on the efficient frontier. If one wished to formalize this selection process, however, then from the viewpoint of the stockholders, the optimum point on the efficient frontier is that set of investments which maximizes the market value of the firm's stock. This point depends upon the market's trade-off between risk and return. For a particular firm this would require estimating the trade-off for the various combinations of projects it is considering. In practice this trade-off between risk and return can best be estimated by studying the market valuation of other firms. Consequently, managers trying to decide which is the optimum point on the efficient frontier would compare their firm to other firms that react to similar underlying variables and have either the same risk-return combination as their firm or else are in a risk-return combination into which they are considering shifting their firm. It is beyond the scope of this paper, however, to develop the detailed procedure for selecting a single point on the efficient frontier.

One final characteristic of our simulation of joint returns approach should be noted. We have been very careful to separate the time and risk discounts.[15]

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to uncorrelated indexes, Weingartner's [29] algorithm would be useful. If budget constraints are not present, then Reiter's [21] algorithm is appropriate. Finally, we know from existing techniques that the movement from one extreme point to another involves changes in only one asset. This has two consequences. First, the maximum number of fractional projects in general is one plus the maximum number of fractional projects at extreme points. Second, we could theoretically use integer linear programming to calculate extreme points, and then ignore or separately generate the points between.

[15] For a detailed discussion of why the risk and time discounts should be separated, see Robichek and Myers [22] and [23]. Their formulation is more general than ours, however, in that they do not recommend any specific weighting scheme, whereas we are recommending the use of a particular weighted average of the standard deviations of yearly cash flows in which the weights used are $1/(1 + r_t)^t$. If one wished to use a more general formulation, the simulation program would increase in size and a different efficient frontier would result, but the rest of the discussion would remain unchanged.

During the simulation each cash flow is discounted for time, but not for risk. The discounting for risk is implicitly done later when one decides on the most appropriate risk-return combination for the firm, i.e., when a specific point on the efficient frontier is selected.

The Capital Budgeting Model With Budget Constraints

The basic model formulated in (1) can be augmented by adding budget limitations and other interdependencies besides the covariances. A model incorporating a sample of these features based on the Weingartner [29] model is:

(2) Minimize $x'\Sigma x - x'\theta x$
Subject to

1. $\mu x = c,$
2. $a_{10}x_0 + a_1 x^* + s_0 - s_1 = 0,$
3. $a_{t0}x_0 + a_t x^* + s_{t-1} - s_t = 0 \qquad t = 2 \cdots T,$
4. $x_i \leqq 1 \qquad i = 1 \cdots n,$
5. $s_t \geqq 0 \qquad t = 1 \cdots T,$
   $x_i \geqq 0 \qquad i = 1 \cdots n + m,$ and
6. $x_0 = 1.$

$$a'_t = \begin{bmatrix} a_{t1} \\ a_{t2} \\ \vdots \\ a_{t,n+m} \end{bmatrix},$$

where

1. μ, x, and c are defined as before.
2. a_t is a $1 \times (n + m)$ vector representing the cash flow associated with the new investment projects in period t. A positive (negative) element of a_t represents a net cash inflow (outflow). a_{ti} is the cash flow associated with a particular investment i in period t. a_{t0} is a scalar equal to the cash flow from the current operations of the firm in period t.
3. x^* is the same as x except that x_0 is not included.
4. s_t is the amount held in cash in period t (s_0, of course, is the opening cash balance). This formulation of the model allows the forward transfer of money between periods.[16] One might want to hold cash either because transactions costs make the returns from securities negative or because of adverse effects of security holdings on the portfolio variance.
5. T is the final period for which a budget constraint is considered.

[16] We do not include borrowing possibilities in the model because the cost of funds would often depend on the projects selected. If this were not true, e.g., because of market imperfections, borrowing possibilities could be included. An iterative procedure might also allow borrowing possibilities to be included.

6. θ is a matrix expressing mutual exclusion between projects. This will be explained in more detail below.

The budget limitations are stated above only in an expected value sense. We could express these limitations probabilistically, but to do so operationally would require (given the current state of chance-constrained programming) that the sum of the cash flows in a period be approximately normal—an assumption that would be hard to justify.[17]

The question arises as to the appropriateness of the use of present value when we have constraints. If the discount rate is, in fact, a function of the cash constraints, then the model is inappropriate, since we cannot determine the discount rate prior to project selection. In most cases, however, this will not be true. The typical constraints that arise will be:

1. Non-cash constraints on managerial talent, floor space, etc.
2. Cash constraints that do not affect the way the market views new investments by the firm. These could be cash constraints imposed by other lenders, by transactions costs limiting external issues of securities, or by management policy.

Mutual exclusion between projects can be handled by suitable choice of values in the second matrix (θ). For example, mutual exclusion between projects i and j can be handled by inserting a large negative number in position ij of θ. In this manner, project interdependency can be incorporated into the model without increasing the number of fractional projects and with no change in the number of explicit constraints.

The Dual

The dual to the model formulated in (2) for an included physical project x_i would be[18]

$$\gamma_i = \mu_i - 2\Sigma_i x - \Sigma a_{ti} \rho_t ,$$

where

1. γ_i is the dual associated with the equation that at most one unit of physical investment i be undertaken.
2. ρ_t is the dual associated with the budget constraint in period t. It gives the value of an additional dollar available for investment in period t.
3. Σ_i is the i^{th} row of the matrix Σ.
4. a_{ti}, x are defined as before.

For an accepted project γ_i is positive and represents the sum of the expected present value, a term representing the effect of the project on the risk of the firm, and a term representing the cash flows associated with the project during periods with budget constraints evaluated at the opportunity cost of funds for that period. It is interesting to note that a project could be accepted (or rejected) due to the expected present value of its return, because it caused a reduction (in-

[17] See, for example, Näslund [19] and Hillier [10].

[18] All elements of the θ matrix that are in the dual are zero, and therefore the θ matrix is excluded.

crease) in the risk of the firm, or because it generated (used) funds at an important time.[19]

Portfolios of Securities

Our previous discussion indicated how we can extend Markowitz's portfolio selection model to include inter-temporal considerations. Markowitz assumes as input data the expected return, variance, and covariances of a number of securities, and he then calculates an efficient frontier of portfolios. He does not, however, explicitly consider an investor's present security holdings; nor does he include possible sales and purchases of securities after the current decision.

To include all possible future sales and purchases may make the resulting quadratic program too large for computational feasibility. However, a subset that on *a priori* grounds seems promising or a subset that seems promising after some initial calculations can easily be included.

We are again suggesting a simulation of joint returns from investment in various assets analogous to what was described above. The possibility of selling an asset in a specific future period can be included by formally dealing with two different assets, one which is purchased to be held indefinitely and the other which is purchased to be held only until the relevant future period. This requires only that the simulation results for that asset be recorded twice. For example, in order to consider selling asset A in period 3, the results of the simulations of asset A for periods 1, 2, and 3 are recorded both in the record of an asset A which is assumed to be held indefinitely and in the record of the new asset A_{1-3}. In order to consider purchasing an asset in a particular future period, one can similarly record that asset's returns from the relevant future period to the horizon and then discount the returns to the current date using the riskless time discount rate. Transactions costs (including taxes) are handled by regarding them as negative cash flows in the relevant periods.

Budget constraints in future periods are incorporated by expressions of the following type:

$$\Sigma_i C_{it} x_i \leqq B_t \qquad t = 1 \cdots T$$

where

1. B_t is the amount available for investment in period t.
2. x_i is the number of shares of security i held in the portfolio.[20]
3. C_{it} is the price per share of security i in period t if it is being considered for purchase, or the negative of the expected selling price net of transactions costs (including taxes) if it is being considered for sale, or else it is the negative of the dividend per share.

[19] Working from less general assumptions, Lintner [15] has derived conclusions similar to those stated above in connection with both the acceptance or rejection of projects and the selection of a point on the efficient frontier.

[20] With this formulation, the upper bound constraints, $x_i \leqq 1$, no longer are relevant. Realistic upper bounds on the numbers of shares of particular securities may still be pertinent, however, especially for institutional investors.

Many of the techniques developed previously in connection with capital budgeting are also applicable to the analysis of portfolios of securities. First, one can include the transfer of money between periods through lending opportunities.[21] Next, the θ matrix can be used to express mutual exclusion between assets. Finally, the effect of the current portfolio is also readily handled. If x_a shares of an asset A are currently held and x_a^* shares are held after the next portfolio is selected, then the present value of the return on the investment of x_a^* in asset A is:

either

$$Y = Rx_a^* - (x_a - x_a^*)B_s = (R + B_s)x_a^* - x_aB_s \quad \text{if} \quad x_a^* \leq x_a$$

or

$$Y = Rx_a^* - (x_a^* - x_a)B_b = (R - B_b)x_a^* + x_aB_b \quad \text{if} \quad x_a^* > x_a$$

where

1. Y is the return from investing in x_a^* shares of asset A.
2. R is the present value of a share of asset A.
3. B_s and B_b are the transactions costs (including taxes) per share of asset A sold and bought, respectively.

This shows that effectively one receives a per share return of $(R + B_s)$ from investing in asset A up to x_a shares, and a return of $(R - B_b)$ thereafter. This discontinuity, and therefore the effect of the current portfolio, can be reflected by:

(a) having two assets with the same variance and covariance but with unit returns of $R + B_s$ and $R - B_b$,

(b) restricting the amount of the asset with return $R + B_s$ to be no greater than x_a ,

and

(c) subtracting an amount B_sx_a from μx in the parametric constraint $\mu x = c$.

The effect of current holdings can be handled in most programs for calculating the optimum portfolio; it does not depend on our suggestion for calculating variances and covariances through the simulation of joint returns. However, the practicability of including inter-temporal considerations is very highly dependent on our suggestions for generating variances and covariances. If one calculates them in the manner we suggest, then including inter-temporal considerations does not increase the information burden on the person supplying the input. Instead, the full burden rests on the computer which performs increased calculations.

Other Applications

The approach to inter-temporal portfolio analysis based on the simulation of joint returns that has been developed for physical investments and securities can also be applied in other areas. Two brief examples will indicate how this

[21] Note that by including an asset with zero risk and zero transactions costs such as bank savings accounts, the budget constraint can be written as an equality, since such an investment dominates holding cash.

approach can be used to strengthen other types of models. In general, this approach can add inter-temporal dimensions, risk considerations, or operational procedures for generating required input data to the existing relevant models

Chambers and Charnes [5] have presented a linear programming model for bank asset management. This model maximizes the horizon value of a bank subject to an availability of funds constraint and a constraint requiring that the Federal Reserve Board of Governors' bank examiners' standards be met. Although the latter constraint can be interpreted as limiting a bank's risk of solvency, other dimensions of risk (e.g., inability to meet loan demand, fluctuations in market prices, etc.) still are relevant. It is thus appropriate to derive the "Markowitz" efficient frontier for bank portfolios in the context of the Chambers-Charnes inter-temporal programming model. The variance-covariance matrix, derived through the simulation of joint returns, can be appended to the criterion function in exactly the manner shown above.[22]

Weingartner [29] has suggested a method for selecting portfolios of *R* and *D* projects. The several alternative models that he presents all require as input mean values of the project returns and probabilities of success, and some also require variances and covariances. The simulation of joint returns is the most appropriate operationally feasible method for generating the input data required by these models, for two reasons. First, returns from R and D are subject to a great deal of variability, so that it is important for a model to include an explicit treatment of risk. Second, since this variability is often related to several underlying factors, the simulation of joint returns is often the only feasible procedure for generating the required variance-covariance matrix.

Further Theoretical Considerations in the Analysis of Risky Assets

It is usually assumed that for a given expected return, investors prefer that firm whose portfolio of assets has minimum risk. This assumption was used in the preceeding sections, and it is the standard assumption of other operational methods for selecting risky assets. There are two reasons why this assumption might need to be modified for theoretical completeness. First, a high-risk firm having low covariance with other firms may sell at a more favorable price than a less risky firm having high covariance with other firms because the "more risky" firm decreases the risk of the investor's portfolio whereas the "less risky" firm does not.[23] This is the effect of market inter-relationships. Second, two firms which have the same mean and variance of return may not sell at the same price if one of the firms is more likely to provide high returns when investors need them most. For example, a gold mining firm may sell at a premium over a construction supply firm even though both offer the same average return and variance of return to the investor because the gold mining firm serves as a hedge against infla-

[22] The suggestions made in this paragraph are also relevant for the Cohen and Hammer [6] model.

[23] See Markowitz [17] and Sharpe [24] for another discussion of this point.

tion whereas the construction supply firm does not. This is the effect of different "states of the world" on investors' need for income.[24] Each of these two theoretical considerations will now be discussed in more detail

Under the following circumstances, market inter-relationships should not affect the firm's asset selection decision:

(a) new projects do not significantly change the inter-relationship of the firm with other firms; or

(b) the only investments that change the market inter-relationships of the firm are investments in which a stockholder could also invest and there are no special economies to a single investor holding both assets.[25]

Under most other circumstances, the previous analysis should theoretically be extended to include market inter-relationships. Such an extension, unfortunately, must await research on the underlying factors which affect a firm's cash flow (and are common between firms) and development of a more detailed general equilibrium theory for capital markets

The other theoretical extension of the previous analysis would include the effect of the state of the world on investors' need for money. Two types of events may generate a change in an individual's need for income: Personal events and economy-wide events. In general, one would not expect a firm's cash flows to be affected by the personal events of its stockholders such as changes in health and family obligations.[26] Consequently, prospective investors will not be able to buy securities whose returns are affected by the investors' cash needs. If these were the only kinds of events that affect investors' portfolio desires, the state of the world could be ignored entirely. Unfortunately, this is not always the case. Economy-wide events, such as war, depression, prosperity and inflation, affect both investors' needs and companies' returns. For example, investors may have a high need for income during depression, and hence bid up those securities whose returns are high under economic adversity.[27] Even so, the effect of the state of the world on investors' need for money could still be ignored, if:

(a) potential projects react to the state of the world like past projects;

(b) the state of the world does not differentially affect the relative amount of risk of various companies; or

[24] The relevance of the state of the world to the choice among commodities was formulated by Arrow [2] and Debreu [7]. A recent application of this framework to the problem of capital budgeting is contained in Hirshleifer [13]. Note that in this paper we are using the 'state of the world" concept in a slightly different way than the authors just cited. We are using "state of the world" to mean a general category of occurrence, such as sickness or depression, rather than a specific value of either of these categories, such as smallpox or GNP for the USA of $400 billion.

[25] Under these circumstances, the prices of the relevant securities have already been bid up to compensate for the low covariance between these securities. See, e.g., Sharpe [24].

[26] A sole proprietorship, some partnerships, and key personnel in corporations are obvious exceptions.

[27] That most of these securities are also low-risk securities may account for part of the difficulty in measuring a risk premium in the market.

(c) the market is such that the state of the world does not change the investors' risk-return preferences.

If none of these conditions holds, our proposed analysis can theoretically be extended by calculating the mean and variance of return for each category of the state of the world that influences the investors' utility for money and by developing an explicit theory of how the state of the world affects the investors' utility for money.

Summary

In this paper we have set up a quadratic programming model which utilizes information about the risk and return of interdependent assets to generate the set of efficient portfolios. We have further shown how to develop the input data required by the model from a simulation which makes no further demands on the decision maker than do standard simulation models in the area. Our model is inter-temporal, so that consideration is given to the effect of present decisions on future actions. Although the model has been formulated primarily in the capital budgeting context, its application to other types of asset selection has been discussed. The paper concludes with some suggestions for extending the model to include additional theoretical considerations.

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