On the Maximization of the Geometric Mean with Lognormal Return Distribution

Edwin J. Elton; Martin J. Gruber

Note*

ON THE MAXIMIZATION OF THE GEOMETRIC MEAN WITH LOGNORMAL RETURN DISTRIBUTION†

EDWIN J. ELTON AND MARTIN J. GRUBER§

New York University

In this paper we discuss the relevancy of the geometric mean as a portfolio selection criteria. A procedure for finding that portfolio with the highest geometric mean when returns on portfolios are lognormally distributed is presented. The development of this algorithm involves a proof that the portfolio with maximum geometric mean lies on the efficient frontier in arithmetic mean variance space. This finding has major implications for the relevancy of much of portfolio and general equilibrium theory. These implications are explored.

One criterion for portfolio selection which has received a great deal of attention in the economic literature is the maximization of the geometric mean return (see Samuelson [22], Hakansson [11], Latane [15], and Latane and Tuttle [14]).[1] While this criterion has been advocated by a number of researchers, these same researchers have pointed out that no procedure currently exists for finding that portfolio which maximizes the geometric mean.[2]

The purpose of this paper is to prove the optimality of a procedure which can be used to construct that portfolio of assets which has the highest possible geometric mean return when the returns on portfolios of securities are lognormally distributed.[3] This case is of special interest because of the large body of theoretical and empirical literature which suggests that rates of return are in fact lognormally distributed.[4]

---

[1] Throughout this paper returns are used to refer to investment relatives (one plus the return). The geometric mean has been advocated by [2], [11], [15], and [16].

[2] For example see [11].

[3] Empirical evidence that the investment relatives are lognormally distributed for portfolios is provided by Hodges [13]. Kendall [14], Moore [19] and Osborne [21] provide further evidence by their examination of the distribution of price changes of indices. These latter authors have also provided evidence that investment relatives for securities are lognormally distributed. One set of theoretical conditions under which lognormally distributed portfolio returns are consistent with lognormally distributed security returns have been set forth by Merton [18].

[4] A disagreement exists today about whether returns are in fact lognormally distributed [14], [19], [21] or follow a stable Paretain distribution [10], [17]. While the resolution of this debate is beyond the scope of this paper, there is sufficient evidence to suggest that deviations from the lognormal distribution are nonexistent or of such a small order of magnitude to make the lognormal case worth further study.

In the first section of this paper, we will discuss circumstances under which the geometric mean is the appropriate decision criterion. In the second section of this paper, we shall show that if returns are lognormally distributed, the portfolio with the highest geometric mean must lie on the efficient frontier in (arithmetic) mean variance space. This not only is important for computational considerations, it also has implications for economic theory.[5] For example, it means that Markowitz-Tobin mean variance analysis is relevant for an investor maximizing the geometric mean, and much of the efficient market literature which depends on investors being on the efficient frontier is relevant. Finally, in the third section, we will show that the portfolio on the efficient frontier which has the highest geometric mean return can be readily defined in terms of the slope of the efficient frontier.

1. Conditions for the Optimality of the Geometric Mean

The maximizaton of geometric mean return has been justified as a criteria for portfolio selection in two ways. First because it can lead to the selection of portfolios with the highest expected utility of terminal wealth, if utility functions are of the log form and second because it leads to the selection of portfolios with intuitively appealing characteristics. We shall examine each of these justifications in turn.

In order to examine the relationship between the geometric mean and the expected utility of terminal wealth we must define some terms.

Let

1. W'_T = the invester's wealth as of a horizon year T, a stochastic variable,
2. W_t = the investor's initial wealth,
3. R_{it} = a random variable equal to one plus return, (t is a period index and i denotes a particular return),
4. P_{it} = the probability of R_{it} occurring.

Investor's terminal, wealth is equal to $W_T = W_t(R_{it} \cdot R_{it+1} \cdots R_{iT-1}$. If the investor's utility function is of the log form then the utility of terminal wealth is given by

$$\ln W_T = \ln W_t + \ln R_{it} + \ln R_{it+1} + \cdots + \ln R_{iT-1}$$

and the expected utility of terminal wealth is given by

$$E[\ln W_T] = \ln W_t + E[\ln R_{it}] + E[\ln R_{it+1}] + \cdots + E[\ln R_{iT}]_{-1}$$

or

$$(1) \quad E(\ln W_T) = \ln W_t + \sum_i \ln R_{it}^{P_{it}} + \sum_i \ln R_{it+1}^{P_{it+1}} + \cdots + \sum_i \ln R_{iT-1}^{P_{iT-1}}.$$

Since utility functions are unique up to a linear transform we can subtract $\ln W_t$

---

[5] It is possible that this problem could be solved more efficiently as a nonlinear programming problem. This could only be determined by direct tests. Two cautions are in order. First, promising techniques such as the gradient method do not reach an optimum in a finite number of steps where our procedure guarantees an optimum in a finite number of iterations. Second, our experience with problems of 80 constraints is that we could produce the full efficient frontier in 2 minutes on a 360-50. If we modify the program for just this problem, time could be somewhat reduced. While no direct comparison is possible, results for nonlinear techniques as reported by Colville [3] would not suggest a computational saving. Actually a combination of an initial solution produced by gradient techniques as a starting solution to a quadratic programming algorithm might be most efficient.

MAXIMIZATION OF THE GEOMETRIC MEAN WITH LOGNORMAL RETURN DISTRIBUTION 485

(a constant) from both sides of the equation and state that the portfolio which maximizes

$$E(\ln (W_T/W_t)) = \sum_i \ln R_{it}^{P_j t} + \sum_i \ln R_{it+1}^{P_j t+1} + \cdots + \sum_i \ln R_{iT-1}^{P_j T-1}$$

maximizes the expected utility of terminal wealth.

The formula for the single period geometric mean is $\prod_i R_{it}^{P_j i}$. The single period geometric mean criteria calls for us to select a portfolio in each period (j) so as to maximize this expression. But from expression (1) the expected utility of investor wealth is maximized if

$$(2) \qquad \sum_{j=t}^{T-1} \sum_i \ln R_{ij}^{P_j j}$$

is maximized. If new portfolios can be selected each period, this is equivalent to maximizing $\sum_i \ln R_{ij}^{P_j j}$ for all j between t and T.

We can take the antilog of this expression without affecting the maximum, since the antilog is a monotonic transform. Thus, an equivalent criteria is Maximize $\prod_i R_{ij}^{P_j j}$ or simply maximize the geometric mean return in every period.[6]

The use of the geometric mean as a decision criteria has also been defended because it produces a series of decisions with intuitively appealing characteristics. To see this let X' = the growth rate (assuming continuous compounding) in capital over the investor's horizon. Then by definition

$$(6) \qquad W'_T = W_t \exp (X')$$

$$(7) \quad \text{or} \qquad X' = \ln (W'_T/w_t).$$

We saw earlier that the one period geometric mean maximizes the expected value of (7). Thus for a multi-period horizon selecting that portfolio with maximum geometric mean each period, produces the maximum expected growth rate in wealth. Furthermore, as the number of periods over which this decision rule is applied becomes very large, the probability approaches one that the growth rate in wealth produced by the geometric mean decision rule will be higher than the growth rate produced by any other strategy. Several authors have asserted that since maximizing the long term growth rate in wealth should be almost a universal goal, the geometric mean is often (if not always) a proper decision rule for selecting portfolios.[7]

2. Maximizing the Geometric Mean and Mean-Variance Efficiency

As discussed earlier, there is substantial empirical evidence supporting an assumption that one plus the return is log normally distributed. Let R_i be one plus return, a variable which is log normally distributed and let:

1. μ be the mean of R_i,
2. σ be the standard deviation of R_i,
3. m be the mean of log R_i,
4. s be the standard deviation of log R_i.

[6] An alternative of the geometric mean criterion is to select that portfolio which has the highest expected geometric mean over the multiperiod time horizon under consideration. The conditions under which this leads to the same selection criterion as maximizing a log utility function are derived in Elton and Gruber [4].

[7] This line of reasoning has been followed by Latane and Tuttle [15], Latane [16], and Hakansson [11]. The reader should be careful to note that counterarguments have been offered by some authors, e.g., Samuelson [21].

It is well known that there exists a one-to-one mapping between a m, s^2 pair and the corresponding moments of R_i :[3]

(8) $$\mu = \exp\,(m + \tfrac{1}{2}s^2)$$

(9) $$\sigma^2 = \exp\,(m + \tfrac{1}{2}s^2)^2[(\exp\,s^2) - 1].$$

Utilizing equations (8) and (9) to solve for m in terms of σ and μ yields:

(10) $$m = \ln\,\mu - \tfrac{1}{2}\ln\,[(\sigma^2/\mu^2) + 1]$$

Maximizing m, the mean log return is equivalent to maximizing the geometric mean return since, as shown in §1:

(a) when the geometric mean is the appropriate decision criterion, it is sufficient to find that portfolio which maximizes the geometric mean of one period returns even in the context of a multi-period portfolio problem;

(b) the geometric mean is defined as $\prod_i R_i^{P_i}$;

(c) taking the log of the geometric mean does not affect the ranking of portfolios since the log function is a monotonic transformation;

(d) the log of the geometric mean is $\sum_i P_i \ln R_i$ which is the mean log return.

To show that the portfolio that maximizes m is mean-variance efficient, we must show that (for a constant μ), m is increased as σ is decreased and (for a constant σ), m is increased as μ is increased or:

$$\frac{\partial m}{\partial \mu} > 0, \qquad \frac{\partial m}{\partial \sigma} < 0.$$

Utilizing equation (10) we have:

$$\frac{\partial m}{\partial \mu} = \frac{2\sigma^2 + \mu^2}{\mu(\sigma^2 + \mu^2)}\frac{+}{+} = + \quad and \quad \frac{\partial m}{\partial \sigma} = \frac{-\sigma}{\sigma^2 + \mu^2} = \frac{-}{+} = -$$

where μ is the mean of R_i a lognormally distributed variable. Since μ is the price relative or one plus the rate of return from holding a portfolio, and since the return from a portfolio can never be less than minus one hundred per cent, μ cannot be negative. The variance and standard deviation are of course also positive and thus the derivations have the appropriate signs. In short, if returns are lognormally distributed, the portfolio that maximizes the geometric mean lies on the efficient frontier in (arithmetic) mean return standard deviation space.[9]

3. The Portfolio that Maximizes the Geometric Mean

Having proved that the portfolio with the highest geometric mean is on the efficient frontier, we shall now identify that point on the frontier which provides the highest geometric mean. Letting $\sigma = f(\mu)$ and taking the derivative of equation (10) with respect to μ to obtain the condition for a maximum yields:

(11) $$f'(\mu) = \frac{2f(\mu)}{\mu} + \frac{\mu}{f(\mu)} \,.$$

---

[3] In standard portfolio analysis, μ and σ^2 refer to the mean and variance of returns. Here of course they refer to the mean and variance of 1 plus the returns. See [1] for a derivation of (1) and (2).

[9] Hakansson [11] has proved that in general the portfolio which has the largest geometric mean is not necessarily mean variance efficient. While this is true in general as just demonstrated in the text, it is not true when returns are lognormally distributed.

Substituting σ for $f(\mu)$, $\partial\sigma/\partial\mu$ for $f'(\mu)$ and rearranging yields

$$(12) \qquad\qquad \frac{\partial\mu}{\partial\sigma} = \frac{\sigma\mu}{2\sigma^2 + \mu^2}$$

as a condition for an optimum.

The efficient frontier is concave from below.[10] Therefore, the left hand side of (12) $\partial\mu/\partial\sigma$ is monotonically decreasing. The right hand side of (12) $\sigma\mu/(2\sigma^2 + \mu^2)$ first increases and then decreases.[11] This implies that (12) can hold for up to two portfolios. If both portfolios exist, then a comparison between them in terms of the original equation (10) is necessary. If (12) never holds, then one of the end points (minimum variance or maximum return) is optimum.[12]

4. Conclusion

In this paper we have presented a method for determining that portfolio which has the largest geometric mean return when price relatives are lognormally distributed. The determination of this portfolio has major implications for economic theory. Hakansson [11], has shown that in general, maximizing the geometric mean will lead to the selection of portfolios which are not on the efficient frontier in (arithmetic) mean variance space. This implies that Markowitz-Tobin type analysis is not only irrelevant, but misleading for an investor who either (1) has a logarithmic utility function and is facing a single or multi-period portfolio selection problem, or (2) wants to maximize the expected long run growth rate of wealth. Since most tests of efficient markets, evaluation of mutual funds, and descriptive models of investor behavior depend on mean variance efficiency, this is disturbing.

We have shown that when price relatives are log normally distributed, the portfolio which maximizes the geometric mean lies on the efficient frontier in (arithmetic) mean variance space. Thus single period Markowitz-Tobin analysis is consistent with certain types of multi-period problems and with problems involving the maximization of long run growth.

References

1. AITCHISON, JOHN AND BROWN, JAMES A. C., *The Lognormal Distribution with Special Reference to its Use in Economics*; (Cambridge University), 1957.
2. BROIMAN, LEO, "Optimal Gambling Systems for Favorable Games," Fourth Berkeley Symposium on Mathematical Statistics and Probability, (Berkeley, University of California Press, 1961).
3. COLVILLE, A. R., "A Comparative Study of Nonlinear Programming Codes," (Kuhn, editor), *Proceedings of the Princeton Symposium on Mathematical Programming*, (Princeton University Press; Princeton, New Jersey), 1970.
4. ELTON, EDWIN J. AND GRUBER, MARTIN J., "On the Optimality of Some Multi-period Portfolio Selection Criteria," Forthcoming in *Journal of Business.*
5. —— AND ——, "Estimating the Dependence Structure of Share Prices—Implications for Portfolio Selection," Vol. 28, No. 5 (Dec. 1973), pp. 1203–1232. *Journal of Finance.*

---

[10] Proofs that the efficient frontier is concave in arithmetic mean-variance space are numerous and do not depend on the distribution of returns. Rather they depend on the assumption of less than perfect correlation between assets.

[11] Implicitly differentiating $\sigma\mu/{-2}\,\sigma^2 + \mu^2$-yields $1 - (\sigma^2/\mu^2) - 2(\sigma^4/\mu^4)$ which can be positive or negative depending upon whether σ/μ is less than or greater than $(1/2)^{1/2}$.

[12] With unlimited borrowing at the riskless rate of interest, the end point can be an infinitely levered portfolio.

6. —— AND ——, "Portfolio Theory When Investment Relatives are Lognormally Distributed," Forthcoming in *Journal of Finance*.

7. —— AND ——, "The Multi-period Portfolio Problem—Implications for the Optimality of Single Period Decision Rules," Forthcoming in *Oxford Economic Papers*.

8. FAMA, EUGENE F., AND MILLER, MERTON H., *The Theory of Finance*, (Holt, Rinehart and Winston), 1972.

9. ——, "Portfolio Analysis in a Stable Paretain Market," *Management Science*, Vol. 11, No. 3, (1965), pp. 404–419.

10. ——, "The Behavior of Stock Prices," *Journal of Business*, Vol. 38, No. 1, (1965).

11. HAKANSSON, NILS H., "Capital Growth and the Mean-Variance Approach to Portfolio Selection," *Journal of Financial and Quantitative Analysis*, Vol. VI, No. 1, (1971), pp. 517–558.

12. ——, "An Induced Theory of the Firm Under Risk: The Pure Mutual Fund," *Journal of Financial and Quantitative Analysis*, (June, 1970).

13. HODGES, STEWART, "On Operational Model for Portfolio Selection," Ph.D. Thesis, University of London, 1972.

14. KENDALL, M. G., "The Analysis of Economic Time Series," *Journal of the Royal Statistical Society*, (Ser. A) XCVI, (1953), pp. 11–25.

15. LATANE, HENRY AND TUTTLE, DONALD, *Security Analysis and Portfolio Management*, (The Ronald Press), 1970.

16. LATANE, HENRY, "Criteria for Choice Among Risky Ventures," *Journal of Political Economy*, (April 1959).

17. MANDELBROT, BENOIT, "The Variation of Certain Speculative Prices," *Journal of Business*, Vol. 36, No. 4, (1963), pp. 394–419.

18. MERTON, ROBERT, "Optimal Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, Vol. 3, No. 4, (December 1971), pp. 373–413.

19. MOORE, ARNOLD, "A Statistical Analysis of Common Stock Prices," unpublished doctoral dissertation (Graduate School of Business, University of Chicago, 1962).

20. MOSSIN, JAN, "Optimal Multi-period Portfolio Policies," *Journal of Business*, Vol. 41, (April 1968), pp. 215–219.

21. OSBORNE, M. F. M., "Brownian Motion in the Stock Market," *Operations Research*, Vol. VII, (1959), pp. 145–173.

22. SAMUELSON, PAUL, "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, Vol. LI, (August, 1969), pp. 139–146.