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SIMPLE CRITERIA FOR OPTIMAL PORTFOLIO SELECTION

EDWIN J. ELTON, MARTIN J. GRUBER AND MANFRED W. PADBERG*

MODERN PORTFOLIO THEORY DATES from Markowitz's [9, 10] pioneering article published in 1952 and subsequent book. Markowitz's suggestions were intended to be practical and implementable. It is ironic that the primary outgrowth has been normative and theoretical and that modern portfolio theory has rarely been implemented.

There are three major reasons why portfolio theory has not been implemented. These are:

1. the difficulty in estimating the type of input data necessary (particularly correlation matrices);
2. the time and cost necessary to generate efficient portfolios (solve a quadratic programming problem); and
3. the difficulty of educating portfolio managers to relate to risk return tradeoffs expressed in terms of covariances as well as returns and standard deviations.

There have been two approaches in the literature to solving the first of these problems. One has been to use a single index model to generate variance-covariance structures. The second is to assume a simple structure for the variance-covariance matrix. In particular the assumption that all pairwise correlations are the same has been shown to do an excellent job of forecasting future correlation structures.¹ In this paper we shall employ these two approaches, which were formulated to solve the first problem, in a manner which should go a long way towards eliminating the second and third problem.

Specifically, we will show that if one is willing to accept the existence of a risk free asset and is willing to either (1) assume that the single index model adequately describes the variance-covariance structure or (2) assume that a good estimate of all pairwise correlation coefficients is a single number, then a simple decision criterion (which does not involve mathematical programming) can be used to reach an optimal solution to the portfolio problem. Furthermore, this simple decision criterion has an intuitive interpretation and its basis is easily understood. This simple method not only allows one to determine which securities are included in an optimal portfolio but also how much to invest in each. Furthermore, the technique allows the definition of a cut-off rate defined solely in terms of the characteristics of the individual security, such that the impact on the optimal portfolio of the introduction of any new security into the manager's decision set can quickly and easily be seen. Finally, the technique makes clear to the manager what characteristics of a security are desirable.²

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1. See Elton and Gruber [1].

2. The results contained in this paper have been announced previously (without proofs) in [3].

This paper is divided into two sections corresponding to the two approximations to the variance-covariance structure. In the first section we assume that the only source of joint movement of two securities comes about because of a common response to market movements. In the second section we will derive a simple numerical solution technique when the correlation coefficient between all pairs of securities can be assumed to be the same. Throughout, we will assume the existence of a riskless asset. This implies that the separation theorem holds and that the investor should maximize the ratio of excess return on a portfolio divided by the standard deviation of the portfolio. Also, throughout the paper we will make the blanket assumption that there is at least one security in the set of all investment opportunities whose expected return is strictly greater than the return on the riskless asset. In each section we will consider both the case where short sales are allowed and the case where they are forbidden.

I. THE SINGLE INDEX MODEL AND THE CONSTRUCTION OF OPTIMAL PORTFOLIOS

In this section we shall assume that the standard single index model is an accurate description of reality. That is

1. $R_i = \alpha_i + \beta_i I + \epsilon_i$
2. $I = A_{N+1} + \epsilon_{N+1}$
3. $E(\epsilon_{N+1} \epsilon_i) = 0, \quad i = 1, \dots, N$
4. $E(\epsilon_i \epsilon_j) = 0, \quad i = 1, \dots, N; j = 1, \dots, N; i \neq j$

where R_i = the return on security i (a random variable)

I = a market index (a random variable)

β_i = a measure of the responsiveness of security i to changes in the market index

α_i = the return on security i that is independent of changes in the market index

ϵ_i = a variable with a mean of zero and variance $\sigma_{\epsilon_i}^2$

σ_m^2 = the variance of the market index

The last two equations characterize the approximation of the standard single index model to the variance-covariance structure. The assumption implied by these equations is that the only joint movement between securities comes about because of a common response to a market index.

We shall show that under these assumptions one can solve for optimal portfolios with simple decision criteria without resorting to mathematical programming. The methods we shall derive for finding optimal portfolios are more accurate than the linear programming approximations which have been put forth and in fact reach the same solution to the portfolio problem as the exact quadratic programming method.³ In addition they are so simple that once the β for each stock has been derived the optimal portfolio can be found, without the use of a computer, in a few minutes time.

3. Sharpe [11] presents a linear programming approximation to the quadratic programming problem.

We shall study two cases involving different degrees of complexity. In the first case we shall assume that short selling is allowed while in the second case we shall not allow short selling. In both these cases lending and borrowing can take place at the riskless rate of interest.

1. *Optimum Portfolios with Short Selling*

In this section we shall derive the expression for that portfolio which has the highest excess return to standard deviation (θ) when short selling is allowed.

We shall first find an unconstrained vector of the relative weights for each security so that the θ for the portfolio is maximized. Then we shall scale these weights to insure that we are fully invested.

In addition to the symbols already defined

Let R_f = the riskless lending-borrowing rate

X_i = the relative weights we place on each security ($X_i > 0$ for a long position, $X_i < 0$ for a short position)

R_p = return on the portfolio (a random variable)

σ_p = the standard deviation of the return on the portfolio

Then the problem is given the assumption of the single index model to find a set of X_i 's to maximize

$$\theta = \frac{\bar{R}_p - R_f}{\sigma_p} \tag{1}$$

where the bar over a variable denotes its expected value.

Now⁴

$$\bar{R}_p - R_f = \sum_{i=1}^N X_i (\bar{R}_i - R_f)$$

and

$$\sigma_p^2 = E \left(\sum_{i=1}^N X_i R_i - \sum_{i=1}^N X_i \bar{R}_i \right)^2$$

Employing the single index assumptions outlined above⁵

$$\sigma_p = \left[\sum_{i=1}^N X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^N X_i^2 \sigma_{\epsilon_i}^2 \right]^{1/2}$$

4. X_i can be negative for short sales. We are following Lintner's [6] suggestion in treating short sales. That is the short seller pays any dividends which accrue to the person who lends him the stock and gets a capital gain (or loss) which is the negative of any price appreciation. In addition the short seller is assumed to receive interest at the riskless rate on both the money loaned to the owner of the borrowed stock and the money placed in escrow when the short sale is made. See Lintner [6] for a full discussion of these assumptions.

5. The expected return is unchanged when the single index model is employed.

Then

$$\theta = \frac{\sum_{i=1}^N X_i (\bar{R}_i - R_f)}{\left[\sum_{i=1}^N X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^N X_i^2 \sigma_\epsilon^2 \right]^{1/2}}$$

Note that θ is homogeneous of degree zero. Thus maximizing θ without the constraint on the sum of the X_i 's yields the same optimum as maximizing θ with the constraint.

To find that set of X_i 's which maximize θ we take the derivative of θ with respect to each X_i and set it equal to zero.⁶

$$\frac{d\theta}{dX_i} = (\bar{R}_i - R_f) - \frac{\sum_{i=1}^N X_i (\bar{R}_i - R_f)}{\sigma_p^2} \left[X_i \beta_i^2 \sigma_m^2 + \beta_i \sum_{\substack{j=1 \\ j \neq i}}^N X_j \beta_j \sigma_m^2 + X_i \sigma_\epsilon^2 \right] = 0$$

for $i = 1, 2, \dots, N$

Defining $Z_i = (\bar{R}_i - R_f / \sigma_p^2) X_i$ and solving this expression for any Z_i yields⁷

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_\epsilon^2} - \frac{\beta_i \sigma_m^2 \sum_{j=1}^N Z_j \beta_j}{\sigma_\epsilon^2} \quad (2)$$

6. Since the denominator appearing in θ is defined with respect to a positive-definite quadratic form, θ is continuously differentiable everywhere except for the point X with all coordinates $X_i = 0$ for $i = 1, \dots, N$. It is not difficult, however, to verify that θ is bounded by zero. For $X \neq 0$, it follows from the Cauchy-Schwartz inequality that θ is bounded from above and that the maximum is unique up to a multiplicative factor. Consequently, the calculation outlined above produces a maximum. See Appendix C and Lintner [6]. Note furthermore, that the maximum value of θ and thus the transformation of X to Z below involves a positive factor. This follows since the standard deviation and excess return of the optimal portfolio are both positive since otherwise the investor holds the riskless asset.

7. A complication could occur if residual risk is zero. If only one security (i) has a zero residual risk then equation (2) is

$$\sigma_m^2 \sum_{j=1}^N \beta_j X_j = (\bar{R}_i - R_f) / \beta_i$$

If more than one security has zero residual risk then a riskless portfolio can be found since $\sigma_p^2 = (X_1^2 \beta_1^2 \sigma_m^2 + X_2^2 \beta_2^2 \sigma_m^2 + 2X_1 X_2 \beta_1 \beta_2 \sigma_m^2) = (X_1 \beta_1 + X_2 \beta_2)^2 \sigma_m^2$. This equals zero if $X_1 \beta_1 = -X_2 \beta_2$. This riskless portfolio is an alternative to the riskless asset. The analysis proceeds after any adjustments in R_f . In practice one would not expect zero residual risk.

Multiplying both sides of the equation by β_i

$$Z_i \beta_i = \frac{\bar{R}_i - R_f}{\sigma_\epsilon^2} \beta_i - \frac{\beta_i^2 \sigma_m^2 \sum_{j=1}^N Z_j \beta_j}{\sigma_\epsilon^2}$$

Adding together the N equations of this form yields

$$\sum_{j=1}^N Z_j \beta_j = \sum_{j=1}^N \left[\frac{\bar{R}_j - R_f}{\sigma_\epsilon^2} \beta_j \right] - \sigma_m^2 \sum_{j=1}^N Z_j \beta_j \left[\sum_{j=1}^N \frac{\beta_j^2}{\sigma_\epsilon^2} \right]$$

or

$$\sum_{j=1}^N Z_j \beta_j = \frac{\sum_{j=1}^N \left[\frac{\bar{R}_j - R_f}{\sigma_\epsilon^2} \beta_j \right]}{1 + \sigma_m^2 \sum_{j=1}^N \frac{\beta_j^2}{\sigma_\epsilon^2}} \quad (3)$$

Substituting equation (3) into equation (2) yields

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_\epsilon^2} - \left[\frac{\sum_{j=1}^N \left[\frac{\bar{R}_j - R_f}{\sigma_\epsilon^2} \beta_j \right]}{1 + \sigma_m^2 \sum_{j=1}^N \frac{\beta_j^2}{\sigma_\epsilon^2}} \right] \frac{\beta_i}{\sigma_\epsilon^2}$$

Note that the term in brackets depends only on the population of the stocks being considered and is independent of the composition of the optimal portfolio. This term can be calculated before the search for the optimal portfolio begins. Let us call this term C_0 . Then

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_\epsilon^2} - C_0 \frac{\beta_i}{\sigma_\epsilon^2} = \frac{\beta_i}{\sigma_\epsilon^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - C_0 \right]$$

The value of Z_i for all stocks can easily be calculated. Having done so we must scale the Z_i so that we have invested 100% of our funds.⁸ That is we must insure that

$$\sum_{j=1}^N |Z_j| = 1.$$

8. We take the absolute values because of the possibility of short sales as explained above. Short sales involve an outlay equal to the price of the stocks involved.

Thus the fraction of our portfolio which we should invest in any stock (X_i^0) is equal to⁹

$$X_i^0 = \frac{(\bar{R}_i - R_f) - C_0 \beta_i}{\sigma_\epsilon^2} \quad (4)$$

$$\sum_{j=1}^N \left| \frac{(\bar{R}_j - R_f) - C_0 \beta_j}{\sigma_\epsilon^2} \right|$$

The advantage of a formula like (4) is that it can easily be calculated and an optimal portfolio arrived at for any population of stocks.

Let us illustrate this with an example. Consider the four securities whose characteristics are shown in Table 1. Further assume that $R_f = 2$ and $\sigma_m^2 = 1$. Then

$$\sum_{j=1}^N \frac{\beta_j^2}{\sigma_\epsilon^2} = 4 \quad \text{and} \quad \sum_{j=1}^N (\bar{R}_j - R_f) \beta_j / \sigma_\epsilon^2 = [(10/50)(20/\sqrt{8}) + (8/32)(2\sqrt{8})$$

$$+ (6/8)\sqrt{8} + (4/2)(1/2)(\sqrt{8})] = 22/\sqrt{8}.$$

Thus we have

$$Z_1 = \frac{10}{50} - \frac{20}{\sqrt{8} \cdot 50} \left(\frac{22}{5\sqrt{8}} \right) = \frac{10-11}{50} = -\frac{1}{50}$$

$$Z_2 = \frac{8}{32} - \frac{2\sqrt{8}}{32} \left(\frac{22}{5\sqrt{8}} \right) = \frac{40-44}{32 \cdot 5} = -\frac{1}{40}$$

$$Z_3 = \frac{6}{8} - \frac{\sqrt{8}}{8} \left(\frac{22}{5\sqrt{8}} \right) = \frac{30-22}{8 \cdot 5} = \frac{1}{5}$$

$$Z_4 = \frac{4}{2} - \frac{\sqrt{8}}{4} \left(\frac{22}{5\sqrt{8}} \right) = \frac{40-22}{20} = \frac{9}{10}$$

$$\sum_{i=1}^4 |Z_i| = 229/200$$

Thus

$$X_1^0 = -4/229 \quad X_2^0 = -5/229 \quad X_3^0 = 40/229 \quad X_4^0 = 180/229$$

9. Treynor and Black [14] have constructed a model for the first of the four cases dealt with in this paper, namely the single index representation of the variance-covariance matrix under the short sales case. However, our results for this case differ from theirs. The differences arise because Treynor and Black [14] assume both that the single index model holds and that a market security exists which while defined as the weighted sum of individual securities has zero residual risk. Fama [4] has shown that these two assumptions are inconsistent and can't be made simultaneously.

TABLE I

Security No.	Mean Return	Beta	Residual Risk	Stan. Dev.
1	12	$20/\sqrt{8}$	50	10
2	10	$2\sqrt{8}$	32	8
3	8	$\sqrt{8}$	8	4
4	6	$\frac{1}{2}\sqrt{8}$	2	2

2. *Optimal Portfolios When Short Sales Are Not Allowed*

If we restrict management prerogatives by disallowing short sales we must modify the solution presented in the last section. In particular, if short selling is disallowed then we must introduce the constraints that all $X_i \geq 0$. This requires employing the Kuhn-Tucker conditions. Since the variance-covariance matrix is positive definite the Kuhn-Tucker conditions are both necessary and sufficient for an optimum. (See Appendix C.) The equivalent of equation (2) using the Kuhn-Tucker conditions is ¹⁰

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_e^2} - \frac{\beta_i}{\sigma_e^2} \sigma_m^2 \sum_{j=1}^N \beta_j Z_j + \mu_i \tag{5}$$

where

$$Z_i \geq 0, \mu_i \geq 0, \text{ and } \mu_i Z_i = 0 \quad \text{for all } i.$$

Now let us assume for a moment that we can find all stocks which would be in an optimal portfolio (call the set of such stocks k) and arrange these stocks as $i = 1, 2, \dots, k$. For the sub-population of stocks that make up the optimal portfolio

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_e^2} - \frac{\beta_i}{\sigma_e^2} \sigma_m^2 \sum_{j=1}^k Z_j \beta_j \quad \text{and} \quad \mu_i = 0$$

Multiplying both sides by β_i summing over all stocks in k and rearranging yields

$$\sum_{j=1}^k Z_j \beta_j = \frac{\sum_{j=1}^k \left[\frac{\bar{R}_j - R_f}{\sigma_e^2} \beta_j \right]}{1 + \sigma_m^2 \sum_{j=1}^k \frac{\beta_j^2}{\sigma_e^2}} \tag{6}$$

10. A complication could occur if residual risk were zero. In this case $\sum_{i=1}^k \beta_i Z_i$ equals the maximum $(\bar{R}_i - R_f)/\beta_i$ with zero residual risk. The rest of the analysis follows. It should be noted that the Kuhn-Tucker multipliers μ_i appearing in (5) as well as elsewhere in this paper are the usual multipliers up to multiplication by the constant σ_p for the optimal portfolio.

Notice since the set k contains all stocks with positive Z_i 's

$$\sum_{j=1}^N Z_j \beta_j = \sum_{j=1}^k Z_j \beta_j \quad (7)$$

and let

$$\phi_k = \sigma_m^2 \frac{\sum_{j=1}^k \frac{\bar{R}_j - R_f}{\sigma_\epsilon^2} \beta_j}{1 + \sigma_m^2 \sum_{j=1}^k \frac{\beta_j^2}{\sigma_\epsilon^2}} \quad (8)$$

Using (8) we obtain after substitution and rearranging from equation (5) the following expression for Z_i :

$$Z_i = \frac{\beta_i}{\sigma_\epsilon^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \phi_k \right] + \mu_i \quad (9)$$

Since $\mu_i \geq 0$, including μ_i can only increase the value of Z_i . Thus if Z_i is positive with $\mu_i = 0$, the inclusion of μ_i can never make it zero. Hence if Z_i is positive when $\mu_i = 0$ the security should be included. If $Z_i < 0$ when $\mu_i = 0$ positive values of μ_i can increase Z_i . However, since the product of μ_i and Z_i must equal zero, positive values of μ_i imply $Z_i = 0$. Hence any security with $Z_i < 0$ when $\mu_i = 0$ must be rejected.

In order to determine if a security should be included it is necessary to deal with three types of securities; those with positive β_i , those with negative β_i , and those with $\beta_i = 0$. Let us start by assuming that all securities have positive β_i 's.

Then we show in Appendix A that if a security with a particular $(\bar{R}_i - R_f)/\beta_i$ is included in the optimal portfolio all securities with higher values of $(\bar{R}_i - R_f)/\beta_i$ must be included in the optimal portfolio. This holds because with $\beta_i > 0$ the sign of Z_i depends on the sign of the term in the brackets. The term in the brackets is $(\bar{R}_i - R_f)/\beta_i$ minus a constant. Thus if a stock with a particular $(\bar{R}_i - R_f)/\beta_i$ has a positive Z_i all stocks with higher excess return to β ratios will also have a positive Z_i . Thus all we have to do to find the securities included in the optimal portfolio is rank from 1 to N all securities by $(\bar{R}_i - R_f)/\beta_i$. Then compute a value for equation (8) as if the set k only contained the first security. This will always be positive. Next we calculate equation (8) setting $i=2$ and letting the set k contain the first two securities. We proceed for $i=3, 4, \dots$ until Z_i computed from equation (9) with $\mu_i = 0$ turns negative. If it turns negative for the $j+1$ st security then the set k contains the first j securities ($i=j$). Appendix B presents a proof that once Z_i from equation (9) turns negative it can never be made positive for any security not in the set k by adding more securities to the portfolio. Hence we have found a simple and fast way to define all securities in the set k . Once these securities are found the Z_i value for all securities in the set can be found simply by calculating the Z_i for each

security from equation (9) recognizing that the μ_i for each of these securities equals zero.

The fraction of our funds which should be placed in each security can be found by recognizing that the sum of the fractions must equal 1 or by dividing the Z_i found for each security in the set k by the sum of the Z_i for the set k . Once again, a simple and very quick procedure has been found for designing an optimal portfolio.

Let us illustrate this with an example. Consider the numbers shown in Table 1. Ranking by excess return to β shows that the securities in decreasing order of desirability are 4,3,2,1. The term in the brackets for security 4 is¹¹

$$\left[\frac{4}{\frac{1}{2}\sqrt{8}} - \left(\frac{4 \cdot \frac{1}{2}\sqrt{8}}{2} \right) \left(\frac{1}{1+2/2} \right) \right] > 0$$

The term in the brackets for security 3 when the portfolio consists of 3 and 4 is

$$\left[\frac{6}{\sqrt{8}} - \left(\frac{4 \cdot \frac{1}{2}\sqrt{8}}{2} + \frac{6 \cdot \sqrt{8}}{8} \right) \left(\frac{1}{1+2} \right) \right] > 0$$

The term in the brackets for security 2 when the portfolio consists of 2, 3 and 4 is

$$\left[\frac{8}{2\sqrt{8}} - \left(\frac{4 \cdot \frac{1}{2}\sqrt{8}}{2} + \frac{6 \cdot \sqrt{8}}{8} + \frac{8 \cdot 2\sqrt{8}}{32} \right) \left(\frac{1}{1+3} \right) \right] < 0$$

Thus the optimum portfolio consists of a portfolio of security 3 and 4 with the weights of¹²

$$X_3^0 = Z_3 = \frac{\sqrt{8}}{8} \left[\frac{6}{\sqrt{8}} - \left(\frac{14\sqrt{8}}{8} \right) \left(\frac{1}{3} \right) \right] = \frac{1}{6}$$

$$X_4^0 = Z_4 = \frac{\frac{1}{2}\sqrt{8}}{2} \left[\frac{8}{\sqrt{8}} - \left(\frac{14\sqrt{8}}{8} \right) \left(\frac{1}{3} \right) \right] = \frac{5}{6}$$

We have not as yet dealt with the case of securities with negative β 's. If all securities had negative β 's an argument and set of proofs analogous to that described above would hold except that stocks would be ranked in ascending order by $(\bar{R}_i - R_f)/\beta_i$.

If stocks with both negative and positive β 's are present then one should follow the procedure outlined above to see which of the positive β stocks should be included in the portfolio. When no more positive β stocks are included, stocks with negative β 's should be tried (starting with the one with the smallest $(\bar{R}_i - R_f)/\beta_i$,

11. Inspection shows that the first security is always included. Thus this step is not necessary.

12. By chance the Z_i add to one.

until no more enter. If any negative β stocks enter, it must decrease the size of the term in brackets in (8) and so the highest excess return to positive β stocks previously rejected should be checked to see if it now enters. If more positive β stocks enter, then the negative β stock list should be checked and the procedure repeated iteratively until no more stocks enter. In actual practice this iterative procedure will converge almost instantaneously because of the very small number of stocks with negative β 's.

The last problem left to deal with is the problem of stocks with zero β . From equation (9) if any stocks exist with zero β then they should be included in the optimal portfolio if their expected return exceeds the riskless rate of interest. Hence when the first positive stocks (and all other stocks) are checked to see if they belong in the optimal portfolio, all zero β stocks with an expected return above the riskless rate should be included in the portfolio since they do not alter the value of ϕ_k given by (8).

Before leaving this section, it is worthwhile considering the implications of our model for the revision of portfolios when a new stock is introduced into the population of stocks under consideration. The present framework drastically simplifies the revision problem.

Let us take the case of a new stock with a positive β . If the excess return to β ratio for the new stock is below the excess return to β ratio of the highest numbered stock excluded from the portfolio (that is the excluded stock with the highest excess return to β ratio), then the new stock will not enter our optimum portfolio. If the new stock has an excess return to β ratio above the stock with the lowest excess return to β ratio which was previously included in the portfolio, it will enter. Furthermore, stocks previously in the portfolio with low excess return to β ratios will have to be rechecked using equations (8) and (9) to see if they remain. Finally, if the stock has an excess return to β ratio between the lowest in the portfolio and the highest not in the portfolio, equation (9) will have to be used to see if it enters or not. If it enters, all stocks which were previously in the portfolio remain in the portfolio. Negative β stocks can be handled in an analogous manner.

II CONSTANT CORRELATION COEFFICIENTS AND THE CONSTRUCTION OF OPTIMAL PORTFOLIOS

In this section, we will assume that all pairwise correlation coefficients are equal. While this probably does not represent the true pattern one finds in the economy, it is very difficult to obtain a better estimate. Elsewhere [1] we have shown that this assumption produces better estimates of future correlation coefficients than do historical correlation coefficients or those produced from the single index approximation discussed in section one. In fact, the assumption of a constant correlation coefficient produced forecasts which were about as accurate as any of nine techniques we tried. As discussed earlier, the optimum portfolio is that which maximizes the ratio of excess return on the portfolio to its standard deviation of returns. Letting

1. σ_{ij} = covariance between security i and security j
2. σ_i^2 = the variance of security i

3. ρ = the correlation coefficient between any two securities
4. all other terms as before

then excess returns on a N security portfolio is

$$\bar{R}_p - R_f = \sum_{i=1}^N X_i (\bar{R}_i - R_f)$$

and the standard deviation is

$$\sigma_p = \left[\sum_{i=1}^N X_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N X_i X_j \sigma_{ij} \right]^{1/2}$$

A. Optimal Policies When Short Sales Are Allowed

If we allow short sales then we maximize $\theta = (\bar{R}_p - R_f) / \sigma_p$ without restricting the sign of X_i . The first order conditions necessary for a maximum were presented by Lintner [4] and are;

$$Z_i \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^N Z_j \sigma_{ij} = \bar{R}_i - R_f \quad i = 1, \dots, N$$

where

$$Z_i = X_i (\bar{R}_p - R_f) / \sigma_p^2$$

This is equal to

$$Z_i \sigma_i^2 (1 - \rho) + \rho \sigma_i \sum_{j=1}^N Z_j \sigma_j = \bar{R}_i - R_f$$

Solving for Z_i yields¹³

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_i^2 (1 - \rho)} - \frac{\rho}{1 - \rho} \frac{1}{\sigma_i} \sum_{j=1}^N Z_j \sigma_j \quad (10)$$

The term $\sum_{j=1}^N Z_j \sigma_j$ can be eliminated by multiplying through by σ_i and summing

$$\sum_{j=1}^N Z_j \sigma_j = \sum_{j=1}^N \frac{\bar{R}_j - R_f}{\sigma_j (1 - \rho)} - \frac{\rho}{1 - \rho} N \sum_{j=1}^N Z_j \sigma_j$$

13. Any security with $\sigma_i^2 = 0$ is a riskless asset. The one with the highest return is the preferred one and R_f is the return on this security. Hence at this step $\sigma_i^2 > 0$.

Solving for $\sum_{j=1}^N Z_j \sigma_j$ yields

$$\sum_{j=1}^N Z_j \sigma_j = \frac{1}{1-\rho+N\rho} \sum_{j=1}^N \frac{\bar{R}_j - R_f}{\sigma_j}$$

Substituting this into equation (10) yields

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_i^2(1-\rho)} - \frac{\rho}{1-\rho} \frac{1}{1-\rho+N\rho} \frac{1}{\sigma_i} \sum_{j=1}^N \frac{\bar{R}_j - R_f}{\sigma_j} \quad (11)$$

Thus the amount to invest in each security is

$$X_i^0 = \frac{Z_i}{\sum_{j=1}^N |Z_j|} = \frac{1}{\sum_{j=1}^N |Z_j|} \frac{1}{\sigma_i} \frac{1}{1-\rho} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \frac{\rho}{1-\rho+N\rho} \sum_{j=1}^N \frac{\bar{R}_j - R_f}{\sigma_j} \right] \quad (12)$$

This equation can be used to determine whether a stock is sold short or purchased long and the amount invested in each stock. Several conclusions can be drawn. First, the above is an extremely simple equation. For even a medium sized portfolio X_i could be determined from the estimates quickly using only a pencil and paper. Second, if a stock with a particular excess return to standard deviation is purchased long all stocks with higher ratios are also purchased long. This is true since the terms in the bracket of equation (12) determines the sign of X_i and the second term is the same constant for all securities. The contrary is of course also true if a security with a particular excess return to standard deviation ratio is purchased short all securities with a smaller ratio are also purchased short. This can be illustrated with a simple example. Consider the example shown in Table 1. Calculating excess returns to standard deviation and other useful quantities yield the numbers shown in Table 2.

With a correlation coefficient assumed to be .5 we have from equation (11)

$$Z_1 = \frac{1}{1-.5} \frac{1}{10} \left[1 - \frac{.5}{1-.5+2} (5.5) \right] = 2(1/10)[1-1.1] = -2/100$$

$$Z_2 = 2(1/8)[1-1.1] = -1/40$$

$$Z_3 = 2(1/4)[1.5-1.1] = 1/5$$

$$Z_4 = 2(1/2)[2-1.1] = .9$$

Scaling the above so that the sum of the absolute values equals 1.0 yields the same results as shown in Section I.

B. Optimal Policies When Short Sales Are Not Allowed

If short selling is not allowed then we have to rely on the Kuhn-Tucker conditions. As detailed in Appendix C they are both necessary and sufficient. The

TABLE 2

Security	$\bar{R}_i - R_f$	$1/\sigma_i$	$(\bar{R}_i - R_f)/\sigma_i$
1	10	1/10	1
2	8	1/8	1
3	6	1/4	1.5
4	4	1/2	2.0
			5.5

Kuhn-Tucker conditions which maximize θ are

1. $\bar{R}_i - R_f - Z_i \sigma_i^2 - \sum_{\substack{j=1 \\ j \neq i}}^N Z_j \sigma_j + \mu_i = 0$
2. $Z_i \geq 0 \quad \mu_i \geq 0$
3. $Z_i \mu_i = 0$

Solving for Z_i we have

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_i^2(1 - \rho)} - \frac{1}{\sigma_i} \frac{\rho}{1 - \rho} \sum_{j=1}^N Z_j \sigma_j + \frac{\mu_i}{\sigma_i^2(1 - \rho)}$$

Without loss of generality, we can assume that the first k securities are included. For an excluded security, $Z_i = 0$ so that the summation in the equation can be written as summing to k rather than N . For an included security, μ_i is zero. Thus for these securities

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_i^2(1 - \rho)} - \frac{1}{\sigma_i} \frac{\rho}{1 - \rho} \sum_{j=1}^k Z_j \sigma_j$$

Multiplying both sides by σ_i and summing allows us to solve for $\sum_{j=1}^k Z_j \sigma_j$.

$$\sum_{j=1}^k Z_j \sigma_j = \sum_{j=1}^k \frac{\bar{R}_j - R_f}{\sigma_j} \frac{1}{1 - \rho + k\rho}$$

Substituting this expression into the equation for Z_i given the above yields

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_i^2(1 - \rho)} - \frac{1}{\sigma_i} \frac{\rho}{1 - \rho} \frac{1}{1 - \rho + k\rho} \sum_{j=1}^k \frac{\bar{R}_j - R_f}{\sigma_j} + \frac{\mu_i}{\sigma_i^2(1 - \rho)} \tag{13}$$

The term containing the μ_i can only increase Z_i . Hence if Z_i is positive with μ_i zero a positive value of μ_i cannot make $Z_i = 0$. Thus any security with positive Z_i when $\mu_i = 0$ must be included. Correspondingly, any security with negative Z_i when $\mu_i = 0$ must be excluded.

Rearranging the above for included securities yields

$$Z_i = \frac{1}{1-\rho} \frac{1}{\sigma_i} \left[\frac{\bar{R}_i - R_f}{\sigma_i} - \frac{\rho}{1-\rho+k\rho} \sum_{j=1}^k \frac{\bar{R}_j - R_f}{\sigma_j} \right]$$

The sign of Z_i depends on the terms in the brackets. Since the last term in the brackets is a constant for any k if a security with a particular rate $(\bar{R}_i - R_f)/\sigma_i$ has a positive Z_i , then all securities with a higher ratio must also be included. It can be shown in a manner analogous to Appendix B that if a stock has a negative Z_i all lower ranking stocks will also have a negative Z_i if they are added. These two characteristics can be used to determine decision rules for solving the portfolio problem. Rank stocks in decreasing order of excess return to standard deviation, add securities until the term in the brackets becomes negative. Once the term in the brackets is negative for the security added last, it will be negative for any additional securities that are added.

The optimum amount to invest in each security is given by the above divided by the sum of the Z_i . Thus

$$X_i^0 = \frac{Z_i}{\sum_{i=1}^k Z_i}$$

Let us illustrate this with an example. Consider the example shown in Table 1 with $\rho = .5$ and the intermediate calculations shown in Table 2. The ranking of the securities in decreasing order of desirability is 4,3,2,1. Including only 4 the expression in the brackets is¹⁴

$$\left[2 - \frac{.5}{1-.5+.5} 2 \right] > 0$$

Including 3 and 4 the expression in the brackets for security 3 is

$$\left[1.5 - \frac{.5}{1-.5+1} 3.5 \right] > 0$$

Including 2,3 and 4 the expression in the brackets becomes

$$\left[1 - \frac{.5}{1-.5+1.5} 4.5 \right] < 0$$

Thus 3 and 4 are included. The amount of each is

$$X_4^0 = 2 \cdot \frac{1}{2} \left[2 - \frac{.5}{1-.5+1} 3.5 \right] = 5/6$$

$$X_3^0 = 2 \cdot \frac{1}{4} \left[1.5 - \frac{.5}{1-.5+1} 3.5 \right] = 1/6$$

These are the same as the proportions calculated in Section One.

14. Inspection shows that the highest ranking security is always included so that this step could have been skipped.

Before closing this section a couple of observations are in order. First, if a new stock is under consideration it will be included if its excess return to standard deviation ratio exceeds that of the lowest ranking stock included in the portfolio, and will be excluded if its excess return to standard deviation ratio is below that of the highest ranking excluded stock.¹⁵ This makes very explicit the characteristics that will make a stock enter and means that the proportions in the optimum portfolio will only have to be recalculated occasionally. Second, the introduction of a new stock is unlikely to cause a radical change in the stocks included in the optimal portfolio. At most, the entry of a new stock should cause a marginal change in the optimal excess return to standard deviation cutoff rate. Thus, using the old cutoff rate as a starting point in determining the new cutoff rate should lead to a quick solution to the problem.

III. CONCLUSION

In this paper we have developed decision rules that allow one to reach optimal solutions to realistic portfolio problems without ever solving a mathematical programming problem. Furthermore, the characteristics of the stock that make it desirable are readily understood and calculated. In a forthcoming paper, we will show that similar (but more complex) results can be reached when we introduce multiple indices and more complicated correlation structures.

APPENDIX A

Proof that if any stock with a positive β belongs in an optimal portfolio, all stocks with positive β 's which have a higher ratio of excess return to β belong in that portfolio.

Given that an optimal portfolio exists containing some set of securities k , then by equation (9) it follows from $\mu_i = 0$ and $Z_i > 0$ that

$$\frac{\bar{R}_i - R_f}{\beta_i} > \phi_k$$

if stock i is contained in the portfolio. Consequently, for any stock j with

$$\frac{\bar{R}_j - R_f}{\beta_j} > \frac{\bar{R}_i - R_f}{\beta_i}$$

it follows that the expression for Z_j given by (9) is positive with $\mu_j = 0$. Consequently, by the Kuhn-Tucker conditions (5), stock j belongs to the optimal portfolio.

APPENDIX B

Proof that if any stock with a positive β is not in an optimal portfolio no stock with a lower excess return to β ratio and a positive β can be in the optimal portfolio.

15. If its excess return to standard deviation ratio falls between these two limits, equation (13) will have to be used to see if the stock should be included or excluded.

Given that an optimal portfolio exists containing some set of securities k , then by equation (9) it follows from $Z_i = 0$ and $\mu_i \geq 0$ that

$$\frac{\bar{R}_i - R_f}{\beta_i} \leq \phi_k$$

if stock i is not contained in the portfolio. Consequently, for any stock j with

$$\frac{\bar{R}_j - R_f}{\beta_j} < \frac{\bar{R}_i - R_f}{\beta_i}$$

it follows that the expression for Z_j given by (9) is non-positive with $\mu_j = 0$. Consequently, by the Kuhn-Tucker conditions (5), stock j does not belong to the optimal portfolio.

APPENDIX C

In order to prove that the methods employed in the paper yield optima in both the cases where short sales are allowed and disallowed, respectively, we need to show that the function θ is pseudo-concave on the domain of positivity of θ . More precisely, let $\Gamma = \{X \in \mathbf{R}^n \mid \sum_{i=1}^N (\bar{R}_i - R_f) X_i > 0\}$. By our blanket assumption mentioned in the introduction, it follows that Γ is nonempty and furthermore, that Γ is an open convex subset of \mathbf{R}^n . To prove that $\theta = \theta(X)$ is pseudo-concave on Γ we have to show (see Mangasarian [7, p. 141]) that for every $\bar{X} \in \Gamma$ and all $X \in \Gamma$ such that $\nabla \theta(\bar{X})(X - \bar{X}) \leq 0$ it follows that $\theta(X) \leq \theta(\bar{X})$, where $\nabla \theta(\bar{X})$ is the gradient of θ evaluated at \bar{X} . Since $\bar{X} \in \Gamma$, it follows that $\bar{X} \neq 0$ and consequently, by the positive-definiteness of the variance-covariance matrix, that the numerator of θ does not vanish. Denoting by Q the variance-covariance matrix and by R the vector with components $\bar{R}_i - R_f, i = 1, \dots, N$, we can write $\theta(X) = (RX)(XQX)^{-\frac{1}{2}}$. Consequently, $\nabla \theta(\bar{X})(X - \bar{X}) = (RX)(\bar{X}Q\bar{X})^{-\frac{1}{2}} - (R\bar{X})(\bar{X}Q\bar{X})^{-\frac{1}{2}}(\bar{X}QX)$. From the Cauchy-Schwartz inequality (see Mangasarian [7, p. 7], it follows that $\bar{X}QX \leq (\bar{X}Q\bar{X})^{\frac{1}{2}}(XQX)^{\frac{1}{2}}$ and hence that $\theta(X)$ is pseudo-concave. (For an alternative proof of the pseudo-concavity of $\theta(X)$ see also Mangasarian [7, p. 148, Problem 6.1].) Having established the pseudo-concavity of $\theta(X)$, it now follows from Theorem 9.3.3 of Mangasarian [7, p. 141] that in the case where short-sales are allowed the calculations of Sections I.1 and II.1 produce an optimum. Similarly, from Théorem 10.1.2 of [7] sufficiency of the Kuhn-Tucker conditions ensues, when short-sales are disallowed. Since the constraints of our optimization problem are linear, it follows from our blanket assumption mentioned in the introduction, that Slater's constraint qualification [7, p. 155] is satisfied and hence by Theorem 10.2.7 [7, p. 156], necessity of the Kuhn-Tucker conditions follows.

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