# A THEORY OF FINANCING CONSTRAINTS AND FIRM DYNAMICS [*]

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## Abstract

There is widespread evidence supporting the conjecture that borrowing constraints have important implications for firm growth and survival. In this paper we model a multi-period borrowing/lending relationship with asymmetric information. We show that borrowing constraints emerge as a feature of the optimal long-term lending contract, and that such constraints relax as the value of the borrower's claim to future cash-flows increases. We also show that the optimal contract has interesting implications for firm dynamics. In agreement with the empirical evidence, as age and size increase, mean and variance of growth decrease and firm survival increases.

# I. Introduction

There is considerable evidence suggesting that financing constraints may be important determinants of firm dynamics.[1] Such constraints may arise in connection to the financing of investment opportunities or temporary liquidity needs, such as those required to survive a recession. This paper develops a theory of endogenous financing constraints and studies its implications for firm growth and survival. In our model, borrowing constraints arise as part of the optimal design of a lending contract under asymmetric information.

The model is as follows. At time zero the borrower (entrepreneur) has a project that requires a fixed initial investment. Once in operation, the project yields revenues that are subject to i.i.d. shocks and increase with the amount of working capital advanced in the period. The project can be discontinued at anytime, providing a fixed liquidation value. A lender finances the initial investment and provides for working capital. Both the borrower and the lender are risk neutral and discount future cash-flows at the same rate. Informational asymmetries arise as the lender cannot monitor either the use of funds or the outcome of the project. We study the optimal dynamic contract subject to the incentive compatibility and limited liability constraints.

The environment is chosen so that in the absence of asymmetric information, the outcome is trivial: there exists an efficient level of working capital which is advanced every period. There is neither growth nor exit. In contrast, with asymmetric information the optimal contract determines non-trivial stochastic processes for firm size (working capital invested), equity (the entrepreneur's share of total firm value), and debt (the lender's share). These processes have two absorbing states: either the firm is liquidated, or a point is reached where borrowing constraints cease to bind and the firm attains its efficient size.

Revenue shocks affect the financial structure of the firm and thus have persistent effects on firm size, growth, and probability of survival. As in all models of moral hazard, rewards and punishments serve to discipline the conduct of the entrepreneur. This explains why the value of equity (i.e. the expected discounted value of the cash flows accruing to the entrepreneur) increases with high revenue shocks and decreases with low ones. Moreover, the spread between future contingent equity values increases with the amount of working capital advanced. This mimics the standard moral hazard problem of unobserved effort, where higher sensitivity of a

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1. Fazzari, Hubbard and Petersen [1988], Gilchrist and Himmelberg [1994], and Whited [1992] among others, find that at the firm level capital expenditures respond positively to innovations in the cash flow process, even after controlling for measures of the expected marginal return on investment. Gertler and Gilchrist [1994] argue that liquidity constraints may explain why small manufacturing firms respond more to a tightening of monetary policy than larger manufacturing firms do. Perez-Quiros and Timmermann [2000] show that in recessions smaller firms are more sensitive to the worsening of credit market conditions as measured by higher interest rates and default premia. For surveys see Hubbard [1998] and Stein [2003].

worker's compensation to output is needed if higher effort is required. Due to the interaction of limited liability and a concave profit function, in our model the total value of the firm is a concave function of the entrepreneur's equity. As a result, a spread in future equity is costly. This accounts for the borrowing constraints. A sequence of good shocks results in an increasing path for equity. As its value approaches a threshold, the working capital eventually increases to the unconstrained efficient level. On the other hand, a sufficiently long sequence of bad shocks leads the value of equity to a region where the project is liquidated.

In spite of its simplicity, the model seems to match most of the qualitative properties of firm dynamics that have been recently documented. In average, investment (change in the working capital invested in the project) is sensitive to innovations in the cash flow process. This sensitivity decreases with age and size. Firm size increases with age; survival increases with firm size; hazard rates for exit first increase and then decrease with the age of the firm; mean and variance of the growth rate decrease with size and age.[2]

The model also yields prescriptions for optimal security design. The incentive compatibility and limited liability constraints imply a set of feasible combinations of debt and equity claims. The Pareto frontier of this set defines implicitly the value of equity as a decreasing function of debt. Values on the frontier can be implemented by one-period contingent contracts and debt rollover. Interestingly, this arrangement provides for some debt forgiveness when low shocks occur; the lender is compensated for the implied loss when high shocks are realized. There is also a maximum sustainable debt, which corresponds to the lowest value of equity on the Pareto frontier. Two possible cases can arise, depending on whether this equity value is zero (the smallest feasible value) or greater than zero. In the first case, the contract is renegotiation-proof and the maximum level of debt is equal to collateral of the project (i.e. its liquidation value). In the second case, the maximum level of debt exceeds the value of the collateral. The first case occurs for high levels of liquidation value while the second occurs for low ones.

Our work builds on recent contributions by Gertler [1992] and Bolton and Scharfstein [1990]. Gertler [1992] studies the optimal contract between a lender and a borrower in a three-period production economy with asymmetric information. Similarly to ours, in Gertler's model the tradeoff between current employment and "ex-post financial position" induces borrowing constraints and investment-cash flow sensitivity. A shortcoming of this model, as pointed out by Gertler, "is that it lies well short of a fully dynamic framework to be matched to the data." Our work is a move in that direction and can be viewed as a dynamic extension of Gertler's

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2. All of these facts have been widely documented. Caves [1998] provides a comprehensive survey of the literature.

- with a minor difference in the timing of the allocation of capital. In addition, our model introduces a positive liquidation value and derives implications for firm survival. Bolton and Scharfstein [1990] consider a two-period model with asymmetric information similar to ours, but without a choice of scale. In their scenario, either the project is funded or not. The threat of not providing funds in the second period (i.e. the threat of liquidation) provides incentives for truthful reporting in the first one. As in our model, moral hazard can lead to inefficient liquidation.

The two studies that are most closely related to ours are due to DeMarzo and Fishman [2001] and Quadrini [2003], respectively. Both consider optimal long-term lending contracts in environments characterized by asymmetric information. DeMarzo and Fishman [2001] focus on the implementation of the long-term arrangement by means of simple contracts, while Quadrini [2003] characterizes renegotiation-proof contracts.

The effects of moral hazard on investment when long-term lending contracts are allowed, have also been investigated by Atkeson [1991] and Marcet and Marimon [1992]. Atkeson shows that asymmetric information and limited enforcement can explain why developing countries experience capital outflows when hit by bad idiosyncratic shocks. Marcet and Marimon study the effects of the same imperfections on capital accumulation.

In recent work, Albuquerque and Hopenhayn [2004] study lending and firm dynamics in a model with limited enforcement. In spite of the similarities between our approach and theirs, some of the implications are radically different. To illustrate this point, Section VIII considers a variant of our model where moral hazard is replaced by incomplete enforcement. In contrast to the moral hazard case, the entrepreneur's equity and firm size never decrease and there is no exit.

Our model is one of repeated moral hazard. The recursive representation that we use was developed by Green [1987] and Spear and Srivastava [1987]. The pioneering work in the area goes back to Radner [1985] and Rogerson [1985].

The remainder of this paper is organized as follows. The model is introduced in Section II. In Section III we characterize the main properties of the optimal contract. In particular, we describe the optimal capital advancement and repayment policies, and the evolution of equity over time that they imply. The implications for firm growth and survival are described in Section IV. In Section V we consider the implementation of the optimal lending contract by means of short-term loans. Section VI discusses the role of collateral. In Section VII we consider the financial feasibility of the optimal contract. In Section VIII we contrast moral hazard with incomplete enforcement. Section IX concludes.

# II. The Model

Time is discrete and the time horizon is infinite. At time zero the entrepreneur has a project which requires a fixed initial investment $I_0 > 0$ and a per-period investment of working capital. Let $k_t$ be the amount of working capital invested in the project - its scale - in period $t$ . The project is successful with probability $p$, in which case the entrepreneur collects revenues $R(k_t)$. If the project fails, revenues are zero. We assume that the function $R$ is continuous, uniformly bounded from above, and strictly concave. At the beginning of every period the project can be liquidated. The liquidation generates a scrap value $S \geq 0$.

We assume that the lender cannot observe the revenue outcome. In other words, such outcome is <u>private information</u> for the entrepreneur.[3]

The entrepreneur's net worth is given by $M < I_0$. Therefore, to undertake the project, he requires a lender (bank) to finance part of the initial setup cost and the investments of working capital. We assume that in every period the entrepreneur is liable for payments to the lender only to the extent of current revenues. That is, the firm is restricted at all times to a nonnegative cash flow.[4]

Both the borrower and the lender are risk neutral, discount flows using the same discount factor $\delta \in (0, 1)$, and are able to commit to a long term contract. This means that, once they have accepted the terms of a contract, both individuals commit to abide by its terms in every possible future contingency, no matter the stream of cash-flows it guarantees to them.

At time $t = 0$ the bank makes a take-it-or-leave-it offer to the entrepreneur. The offer consists of a contract whose terms, i.e. cash-flow and liquidation policies, can be contingent on all information provided by the agent. This information consists of a sequence of reports on revenue realizations. More formally, let $\theta$ be a Bernoulli random variable, with $\theta \in \Theta \equiv \{H, L\}$ and $\text{prob}\{\theta = H\} = p$. Revenues are positive (and equal to $R(k)$) when $\theta = H$, and identically zero when $\theta = L$. A reporting strategy for the entrepreneur is given by $\hat{\boldsymbol{\theta}} = \{\hat{\theta}_t(\theta^t)\}_{t=1}^{\infty}$, where $\theta^t = (\theta_1, ..., \theta_t)$ denotes the actual history of realizations of $\theta$.[5] Letting $h^t = (\hat{\theta}_1, ..., \hat{\theta}_t)$ denote the history of reports of the agent, the contract $\boldsymbol{\sigma} = \{\alpha_t(h^{t-1}), Q_t(h^{t-1}), k_t(h^{t-1}), \tau_t(h^t)\}$ specifies contingent liquidation probabilities[6] $\alpha_t$, transfers $Q_t$ from the lender to the entrepreneur (in case of liquidation), capital advancements $k_t$, and payments $\tau_t$ from the entrepreneur to the lender (in case of no liquidation).

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3. An alternative formulation where revenue is observable but the use of funds (or investment) cannot be monitored is considered in Appendix A. The two formulations turn out to be equivalent.

4. This assumption can be easily relaxed to a lower bound.

5. In principle the entrepreneur's communications can be arbitrarily complicated. However, the Revelation Principle allows us to reduce the message space to the set $\Theta$ without loss of generality.

6. As it will become clear later in this section, randomizations over the liquidation decision will be optimal in some states of nature.

The timing is the same after every history $h^t$ and is described in Figure I. At the beginning of the period, the lender has the chance of liquidating the project. The contract dictates that he will do so with probability $\alpha_t(h^{t-1})$. In case liquidation occurs, the entrepreneur is compensated with a value $Q_t(h^{t-1})$, while the lender receives $S - Q_t(h^{t-1})$. If the project is not liquidated, the lender provides the entrepreneur with capital $k_t(h^{t-1})$. Thereafter, the entrepreneur observes the revenue realization and makes a report to the lender. The lender will require a transfer $\tau_t(h^t)$, where $h^t = (h^{t-1}, \hat{\theta}_t)$.

[ Figure I here ]

The entrepreneur's net cash-flow will be $R(k_t(h^{t-1})) - \tau_t(h^{t-1}, \hat{\theta}_t)$. Without loss of generality, we assume that these resources are fully consumed by the entrepreneur (i.e. not reinvested in the business venture).[7]

A contract is feasible if the probabilities of liquidation are well defined, and if the entrepreneur's cash-flow is nonnegative after every possible history of reports.

**Definition** A contract $\boldsymbol{\sigma}$ is feasible if $\forall\, t \geq 1$ and $\forall\, h^{t-1} \in \Theta^{t-1}$

(i) $\alpha_t(h^{t-1}) \in [0, 1]$,

(ii) $Q_t(h^{t-1}) \geq 0$,

(iii) $\tau_t(h^{t-1}, H) \leq R(k_t(h^{t-1}))$,

(iv) $\tau_t(h^{t-1}, L) \leq 0$.

Notice that after every history $h^{t-1}$, the contract and reporting strategies $(\boldsymbol{\sigma}, \hat{\boldsymbol{\theta}})$ imply expected discounted cash flows for the entrepreneur and the lender. Denote such values as $V_t(\boldsymbol{\sigma}, \hat{\boldsymbol{\theta}}, h^{t-1})$ and $B_t(\boldsymbol{\sigma}, \hat{\boldsymbol{\theta}}, h^{t-1})$, respectively. We will refer to them as equity and debt.

A contract is said to be incentive compatible if it implies that misreporting the revenues' realization is never in the entrepreneur's interest.

**Definition** A contract $\boldsymbol{\sigma}$ is incentive compatible if $\forall\, \hat{\boldsymbol{\theta}}, \ V_1(\boldsymbol{\theta}, \boldsymbol{\sigma}, h^0) \geq V_1(\hat{\boldsymbol{\theta}}, \boldsymbol{\sigma}, h^0)$.

Now define the set $\mathcal{V} \equiv \left\{ V \mid \exists\, \boldsymbol{\sigma} \ s.t. \ (feas), \ (ic) \text{ and } V_1\left[\boldsymbol{\sigma}, \boldsymbol{\theta}, h^0\right] = V \right\}$. $\mathcal{V}$ is the set of equity values that can be generated by feasible and incentive compatible contracts.[8] For any $V \in \mathcal{V}$, an optimal contract maximizes the value obtained by the lender among all incentive

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7. Alternatively, we might allow the entrepreneur to save and assume that the bank can observe the return on his wealth and monitor the size of the project.

8. As shown below, there is an incentive compatible and feasible contract for any $V \geq 0$.

compatible and feasible contracts that deliver an initial value $V$ to the entrepreneur. This defines a frontier of values

$$B(V) = \sup\{B \mid \exists\, \boldsymbol{\sigma}\ \ s.t.\ V_1\left[\boldsymbol{\sigma},\boldsymbol{\theta},h^0\right] = V \text{ and } B_1\left[\boldsymbol{\sigma},\boldsymbol{\theta},h^0\right] = B\}.$$

Every point on the frontier corresponds to a capital structure $(V, B(V))$ and implies a value for the firm $W(V) = V + B(V)$. Notice that in general the Modigliani-Miller Theorem will not hold: firm value will not be invariant to changes in the capital structure. In the remainder of this paper we characterize the set of optimal contracts, i.e. the set of contracts yielding values on the frontier.

## II.A.  A Benchmark: Contracts under Symmetric Information

As a first step, it is useful to consider the case of symmetric information, where the lender observes the revenue realizations and can write contracts contingent on them. Since the two agents discount cash flows at the same rate, the optimal contract will maximize the total expected discounted profits for the match. This result is achieved by having the lender provide the entrepreneur with the unconstrained efficient amount of capital in every period.

The unconstrained efficient capital advancement is given by $k^* \equiv \arg\max_{k} pR(k) - k$ and implies a per-period total surplus $\pi^* \equiv \max_{k} pR(k) - k$. Then, the total surplus is given by $\widetilde{W} \equiv \pi^*/(1-\delta)$. Undertaking the project is efficient as long as $\widetilde{W} > I_0$. Once started, the firm will never grow, shrink, or exit. Any division of the surplus $\widetilde{W}$ among the lender and the entrepreneur is feasible, provided that the latter obtains a nonnegative value.

## II.B.  The Contract with Private Information

For any lending contract, upon continuation the evolution of equity over time satisfies the following accounting identity:[9]

(1) $$V_t = p\left(R(k_t) - \tau_t\right) + \delta\left[pV_{t+1}^H + (1-p)V_{t+1}^L\right],$$

where the first term on the right-hand side corresponds to expected dividends and the second to the expected discounted value of the entrepreneur's claim at the beginning of next period. The quantities $V_{t+1}^H$ and $V_{t+1}^L$ are the continuation values (next period's equity values as implied by the contract) contingent on high and low report, respectively. Equation (1) suggests a tradeoff in the assignment of values over time (the current period versus the future) and across states

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9. The limited liability constraint implies that, conditional on a low report, the payment demanded to the entrepreneur will always be zero. Therefore we will adopt the convention of using $\tau_t$ to denote the payment conditional on a high report.

(low versus high revenue realization). These will be the margins considered in the optimal contract design.

The above observations also suggest a recursive representation of the optimum problem defined in the previous section. It can be shown that after any history of reports, the value of equity effectively summarizes all the information provided by the history itself.[10] Therefore, equity can be used as state variable for the problem.

We start by defining the total value of the firm upon continuation. The flow equation (1) becomes now a constraint, as the initial value $V$ must be delivered to the entrepreneur by the continuation contract. Suppressing time subscripts, this gives:

$$(2) \qquad V = p\left(R(k) - \tau\right) + \delta\left[pV^H + (1-p)V^L\right].$$

As usual, the only relevant incentive constraint is the one imposing truthful reporting in the high state:

$$R\left(k\right) - \tau + \delta V^H \geq R\left(k\right) + \delta V^L.$$

By reporting truthfully, the entrepreneur obtains dividends $R(k) - \tau$ and a continuation value $V^H$. By misreporting, the agent avoids making any repayment, thereby obtaining a value $R(k) + \delta V^L$. The incentive compatibility condition can be rewritten as

$$(3) \qquad \tau \leq \delta\left(V^H - V^L\right).$$

Finally, the limited liability constraint requires that

$$(4) \qquad \tau \leq R(k).$$

Any nonnegative continuation value $V^i$ $(i = H, L)$ is feasible, as it can be implemented by giving the entrepreneur no capital and a transfer equal to $V^i$, and then liquidating the project. Any continuation value $V^i < 0$ is not feasible, for it would violate the limited liability constraint (4) in some future period. Recall that $V$ is the expected discounted value of the entrepreneur's future cash flows. For it to be negative, at least one of such cash flows must be negative. Hence, a value $V$ can be supported by a feasible contract if and only if $V \geq 0$.

We denote firm value contingent upon continuation as $\widehat{W}$. This is different from the value of the firm prior to the liquidation decision, which is denoted as $W$. The choice variables are

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10. Green [1987] and Spear and Srivastava [1987] were the first to show that, under mild boundedness conditions, there exists a recursive formulation for the maximization problem faced by the principal in models of repeated moral hazard. Such conditions hold in our case. We omit the proof of equivalence between the sequence problem and the recursive problem, because it consists of the mere application of the techniques used by Atkeson and Lucas [1992], among others.

the capital advancement for the period $k$, the repayment $\tau$, and the contingent continuation values $V^H$ and $V^L$. The value $\widehat{W}(V)$ is given by

$$\widehat{W}(V) = \max_{k,\tau,V^H,V^L} [pR(k) - k] + \delta \left[ pW(V^H) + (1-p)W(V^L) \right]$$

(P1)
$$\text{subject to (2), (3), (4)}$$

$$\text{and } V^H, V^L \geq 0.$$

The first term of the maximand corresponds to the current period's expected profits. The second term indicates the expected discounted value of the firm at the beginning of the following period, prior to the liquidation decision.

We now turn to the liquidation decision. As made clear by Figure II, a positive liquidation value $S$ implies that a stochastic liquidation decision is optimal in some states. Allowing for randomizations over the liquidation decision is equivalent to assuming that at the beginning of every period the lender offers a lottery to the borrower. The firm is liquidated with probability $\alpha$, in which case the borrower receives $Q$, and it is kept in operation with probability $1 - \alpha$. In the latter case, the borrower receives $V_c$, where $c$ is mnemonic for *continuation*. Then, the function $W(V)$ solves the following functional equation:

$$W(V) = \max_{\alpha \in [0,1], Q, V_c} \alpha S + (1-\alpha)\widehat{W}(V_c)$$

(P2)
$$\text{subject to } \alpha Q + (1-\alpha) V_c = V,$$

$$\text{and } Q, V_c \geq 0.$$

Figure II illustrates the basic features of the firm value function $W(V)$. Following standard dynamic programming arguments, it is easy to show that the $W(V)$ is concave and increasing. The domain of $V$ can be partitioned in three regions: (i) a region where liquidation is possible, $0 \leq V \leq V_r$; (ii) a region where there is no liquidation in the current period and firm value increases with the value of equity, $V_r \leq V \leq \widetilde{V}$; (iii) a region where the value of the firm equals its value under symmetric information, $V \geq \widetilde{V}$. We examine each region in turn.

[ Figure II here ]

First note that if $V = 0$ at the liquidation stage, the project is scrapped for sure, providing a total value $W(0) = S$. If the firm is not liquidated, $V = 0$ implies that no capital can be advanced in the current period and the firm will be liquidated for sure the following period, so $\widehat{W}(0) = \delta S$. Further, under the assumption that $\widetilde{W} > S$, there exists $V_r > 0$ such that for values $0 < V \leq V_r$, it is optimal to give the entrepreneur a lottery with prizes of $Q = 0$ in case of liquidation and $V_c = V_r = V/(1-\delta)$ in case of continuation. The value of the

firm in this region is given by a linear combination of $S$ and $\widehat{W}(V_r)$, with weights $\alpha(V)$ and $1 - \alpha(V)$, respectively. The probability of liquidation is $\alpha(V) = (V_r - V)/V_r$. Simple inspection of Figure II reveals that such policy yields a value for the firm which is higher than $\max[S, \widehat{W}(V)]$, the value implied by the optimal deterministic termination rule. Notice also that for $V \le V_r$ the probability of liquidation is decreasing in the equity: the larger the claim of the entrepreneur, the lower the chance of liquidation.

Consider now region (iii). The threshold $\widetilde{V}$ is the minimum equity such that the policy of providing every period the efficient level of capital $k^*$ is feasible and incentive compatible. This implies that if $V$ ever crosses this threshold, the entrepreneur will be guaranteed the unconstrained efficient level of capital at any future date. In other words, in this region borrowing is not constrained. Lemma 1 in appendix shows that

$$\widetilde{V} = \frac{pR(k^*)}{1 - \delta}.$$

Notice that this value entitlement can be implemented by giving the entrepreneur the efficient capital advancement $k^*$ in every period, with no need of repayment. This also corresponds to a situation where the entrepreneur has a positive balance equal to $k^*/(1 - \delta)$ in the bank at an interest rate $r = (1 - \delta)/\delta$. Such balance is exactly enough to finance the project at full scale in all contingencies.

The results proved in this section are summarized in the following Proposition.[11]

**Proposition 1** The value function $W(V)$ is increasing and concave. There exists values $0 < V_r < \widetilde{V} \equiv pR(k^*)/(1 - \delta)$, such that:

1. The firm is liquidated with probability $(V_r - V)/(V_r)$ when $V \le V_r$;

2. $W(V)$ is linear for $V < V_r$, strictly increasing for $V < \widetilde{V}$, and equal to $\widetilde{W}$ for $V \ge \widetilde{V}$.

## III. Properties of the Optimal Lending Contract

This section characterizes the provisions of the optimal contract when $V \in [V_r, \widetilde{V})$ (region (ii)). We start by showing (see Proposition 2) that in this region the entrepreneur is borrowing-constrained. That is, the optimal contract guarantees him an amount of capital which is always strictly lower than the unconstrained efficient level.

**Proposition 2** i) The optimal capital advancement policy $k(V)$ is single valued and continuous. ii) $V < \widetilde{V}$ implies $k(V) < k^*$; (iii) $V \ge \widetilde{V}$ implies $k(V) = k^*$.

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11. The proofs not included in the body of the paper can be found in Appendix.

Although this result does not depend on a particular repayment policy, its economic intuition is clearest in the case in which $\tau = R(k)$.[12] Lemma 2 in appendix shows that when this is the case, the incentive compatibility constraint (3) binds for all $V < \widetilde{V}$. Therefore, $R(k) = \delta(V^H - V^L)$, implying that the level of capital advancement is tied to the spread in equity values $(V^H - V^L)$. Increasing the level of working capital from an inefficient level is incentive-compatible only if the spread between continuation values also increases. However, due to the concavity of $W(V)$, doing so is costly (unless the value function $W$ is linear in the relevant portion).

This argument can also be appreciated by noticing that when $\tau = R(k)$, given continuation values $V^H$ and $V^L$ imply that the firm's current cash flow is $\Pi(V^H - V^L)$, where:

$$\Pi(V^H - V^L) \equiv \max_k pR(k) - k$$
$$\text{s.t. } R(k) \leq \delta(V^H - V^L).$$

It is straightforward to show that $\Pi$ is strictly concave and strictly increasing for $V^H - V^L < R(k^*)/\delta$, and constant at the value $pR(k^*) - k^*$ thereafter. Then, for $V \geq V_r$, the optimal contract design problem can be conveniently restated as

(P3)
$$W(V) \equiv \max_{V^H, V^L} \Pi(V^H - V^L) + \delta\left[pW(V^H) + (1-p)W(V^L)\right]$$
$$\text{s.t. } V = \delta(pV^H + (1-p)V^L).$$

This reformulation highlights the nature of the tradeoff described above. As long as $V^H - V^L < R(k^*)/\delta$, mean preserving spreads of the continuation values allow for larger working capital and higher current profits. However, because of concavity of $W(V)$, they also decrease the future value of the firm.

## III.A. Optimal Repayment

Given that both agents are risk-neutral and discount future cash-flows at the same rate, allowing the equity value $V$ to reach the threshold $\widetilde{V}$ in the shortest possible time is optimal. This is achieved by setting $\tau = R(k)$, therefore attributing all cash flows to the lender. The reason is as follows. For given level of working capital $k$ and continuation value $V^L$, it is possible to choose among an infinite number of feasible and incentive compatible pairs $(\tau, V^H)$. Now notice that the current cash-flow to the firm, $R(k) - k$, does not depend on this choice, while the continuation value is weakly increasing in $V^H$. Therefore delaying the distribution of dividends is always optimal. Proposition 3 shows that as long as $V^H < \widetilde{V}$, setting $\tau = R(k)$ is actually a necessary condition for efficiency. Instead, in the region where

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12. Proposition 3 below shows that setting $\tau = R(k)$ is indeed optimal.

$V^H \geq \widetilde{V}$, the optimal repayment policy is indeterminate.[13]

**Proposition 3** It is optimal to set $\tau = R(k)$ for all $V < \widetilde{V}$. When $V^H < \widetilde{V}$, $\tau = R(k)$ is a necessary condition for optimality.

### III.B. The Evolution of Equity

Proposition 2 states that if $V$ ever becomes larger than $\widetilde{V}$, then borrowing constraints will cease forever. On the other end, if $V$ becomes smaller than $V_r$, liquidation will occur with positive probability. The question is whether, starting from an arbitrary initial level, $V$ will ever get either so large or so small. Obviously the answer depends on the stochastic process $\{V_t\}$ for the entrepreneur's claim implied by the optimal contract. It is to the characterization of this process that we now turn.

By Proposition 3, optimality implies that as long as $V^H(V) < \widetilde{V}$ a given sequence of shocks implies a unique path for $V_t$. When this condition does not hold, the path followed by $V_t$ depends on the choice of dividend distribution policy. In this section we choose to characterize the evolution of equity in the case in which $\tau = R(k)$ for all $V < \widetilde{V}$.[14] This choice, along with equation (2), implies that $V_t < pV_{t+1}^H + (1-p)V_{t+1}^L$ (i.e. $V_t < E_t(V_{t+1})$), so that $\{V_t\}$ is a submartingale. This process has two absorbing sets, $V_t = 0$ and $V_t \geq \widetilde{V}$. Eventually, either the first one is reached (and the firm is liquidated),[15] or the second one is reached (and borrowing constraints cease forever). We will now characterize the process for $V_{t+1}$ when $V_r \leq V_t < \widetilde{V}$.

Using (2), (3), and $\tau = R(k)$, we obtain that the law of motion for $V$ is characterized by

$$(5) \qquad V^L = \frac{V - pR(k)}{\delta}, \quad V^H = \frac{V + (1-p)R(k)}{\delta}.$$

Figure III displays the qualitative behavior of the functions $V^H(V)$ and $V^L(V)$. Proposition 4 shows that $V^L < V < V^H$: value increases with a good shock and decreases with a bad shock. Furthermore, the process $\{V_t\}$ displays persistence: no matter the revenue realization, a lower value in a given period results in lower values in the future. This is an immediate implication of Proposition 5, which shows that the functions $V^H(V)$ and $V^L(V)$ are nondecreasing for $V \geq V_r$. The figure also illustrates that, starting from an equity value $V_0 \in [V_r, \widetilde{V})$, a finite sequence of good shocks leads to $\widetilde{V}$. From the same value, a sequence of bad shocks leads to

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13. It should be clear that setting $\tau = R(k)$ for all $V > 0$, therefore postponing payments to the entrepreneurs forever, is not a solution. In fact such policy would deliver to the entrepreneur a value which is identically zero.

14. All relevant properties are independent of the particular selection of repayment policy. In particular, this is true for the optimal capital advancement policy $k(V)$, which is uniquely determined.

15. In Section II we have argued that when equity reaches the randomization zone, its continuation value is either $V = 0$, which implies liquidation, or $V = V_r$.

the randomization region. A further result is that in general there is an asymmetry between the change in equity value following good and bad shocks. In particular, if $p$ and $\delta$ are large, $V - V^L > V^H - V$, while the opposite will hold if $p < 1/2$.

[ Figure III here ]

**Proposition 4** Let $V \in \left[ V_r, \widetilde{V} \right)$. If $\tau = R(k)$ the following holds: (i) $V^L < V < V^H$; (ii) $V < pV^H + (1-p)V^L$.

**Proposition 5** The policy functions $V^H(V)$ and $V^L(V)$ are nondecreasing in $V$.

### III.C.   The Optimal Capital Advancement Policy

In subsection II.A we have shown that, under symmetric information, it would always be optimal for the lender to provide the entrepreneur with the unconstrained efficient amount of capital. Proposition 2 establishes that this ceases to be the case when the lender cannot observe the firm's cash flows: borrowing constraints are indeed a feature of the optimal long-term contract under asymmetric information.

An immediate corollary of Proposition 2 is that the allocation of capital increases with $V$ over some range of values. The numerical results discussed below suggest that this range may include most of the domain, yet general properties are hard to derive. Indeed, monotonicity does not hold throughout the whole range of values. Proposition 6 shows that as equity values get close to the randomization region, capital advancements actually increase. This result is due to the possibility of liquidation. For simplicity, consider first the case of deterministic liquidation. In this scenario, the value function $W$ would be flat in an interval to the right of the origin. This implies that for $V$ small enough, there would be no downside to increasing $k$. The trade-off described in Section III would disappear. This is because decreasing $V_L$ in order to insure incentive-compatibility would not lower the lender's value. With stochastic liquidation, the intuition is similar, but less transparent. As long as $V^L$ lies to the right of the randomization region, concavity of the value function implies that a decrease in $V_L$ triggers an accelerating decrease in firm value. To the left of $V_r$, however, the value function is linear. Therefore, when $V^L$ is in the randomization region, the cost of lowering it further increases only at a constant rate. As a consequence, when $V$ approaches $V_r$ from the right, it is optimal to increase the spread $V^H - V^L$. In turn, this implies an increase in capital advancement and a higher probability of liquidation. Since $W(V)$ is flat to the right of $\widetilde{V}$, Proposition 6 uses a

similar argument to show that the capital advancement increases with $V$ in a neighborhood of $\widetilde{V}$.

**Proposition 6** There exist value entitlements $V^*$ and $V^{**}$, with $V_r < V^* \leq V^{**} < \widetilde{V}$, such that:

1. the policy function $k(V)$ is non-increasing for $V \in [V_r, V^*]$;

2. the policy function $k(V)$ is non-decreasing for $V \in \left[V^{**}, \widetilde{V}\right]$.

We now turn to the results of our numerical experiment.[16] Figure IV shows how the advancement of working capital responds to good and bad realizations of the revenue process. The higher curve corresponds to the growth rate of capital, conditional on success, i.e. $k(V^H)/k(V)$. The lower one corresponds to $k(V^L)/k(V)$. Even though $k(V)$ is not monotone, in the case of our experiment success is always followed by an increase in the capital invested, while failure always triggers a decline. Also, notice how the magnitude of the response of investment to cash-flow innovations changes with $V$. Percentage changes in capital tend to be larger (in absolute value) for small values of $V$ and show a tendency to decline as $V$ increases. When $V$ equals the threshold $\widetilde{V}$, the level of capital invested is independent of cash-flows.


[ Figure IV here ]


Economists have long sought ways to measure the extent to which firm dynamics are affected by borrowing constraints. Starting with Fazzari, Hubbard and Petersen [1988], a number of authors have regressed investment on cash flows and on a number of controls, among which average Tobin's $q$, and interpreted the cash-flow coefficients as measures of the tightness of borrowing constraints. According to some, this approach is justified by the work of Hayashi [1982], who showed that under some conditions the neoclassical model of investment yields average $q$ as a sufficient statistic for investment. Fazzari, Hubbard and Petersen [1988] among others have concluded that statistically significant cash flow coefficients, representing deviations from this prediction, constitute signs of the presence of financing constraints. Such conclusion was prompted by the observation that the coefficients are higher for firms that are smaller, younger, or can be classified as financing-constrained a priori, based on criteria such as their dividend distribution policies, for example. The same scholars have also argued that larger coefficients should signal tighter constraints. In recent years, Kaplan and Zingales

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16. Our parametric assumptions are as follows. We set $R(k) = k^{2/5}$, $p = 0.5$, $\delta = 0.99$, and $S = 1.5$.

[1997], Gomes [2001], and Alti [2003] have shown that the existence of financing constraints is neither a necessary nor a sufficient condition for obtaining significant cash-flow coefficients, and that in general there is no reason to expect that the magnitude of the coefficients is actually increasing in the tightness of the constraints. Here we would like to understand whether the cash flow coefficients would be helpful in detecting financing constraints and measuring their tightness, in case our optimal contract was the data generating process.

In the case of our model, the most appropriate measure of investment at time $t$ is $k_t - k_{t-1}$. We could use our policy functions to generate simulated data and then follow Fazzari, Hubbard and Petersen [1988] by regressing such measure of investment on revenues at time $t$ and on $V_{t-1}$ (or, equivalently, on $W(V_{t-1})$). The results discussed above suggest that for a panel of firms obtaining external finance through our contract, (i) the cash-flow coefficient would be positive and significant when and only when the financing constraint binds, (ii) the magnitude of the coefficient would be higher for more constrained firms, and (iii) the investment-cash flow sensitivity would decrease with both size (as measured by the level of working capital invested) and age. The relation between sensitivity and size obtains because, as we have just argued, $k(V)$ is monotone for most values of $V$. The relationship with age stems from the positive correlation between age and size, outlined in the next section. Notice however that the coefficient will be significant only if there is at least a one period lag between investment and the control (this is actually the case in all regression specifications we are aware of). Conversely, since in our model $V_t$ is sufficient for investment at time $t$, the coefficient will be zero if the two variables are contemporaneous.

The statements we have just made about the interpretation of the cash flow coefficient in our setup are not likely to extend to versions of the model with more general stochastic structures. For example, consider the case in which revenues are $\eta R(k)$ with probability $p$ and identically zero with probability $1 - p$, with $\eta$ being a random variable that follows an autoregressive process and whose realization is public information. In such scenario, as long as the support of $\eta$ is bounded, there would still exist a level of equity such that for higher values the entrepreneur receives the unconstrained efficient amount of capital forever. This level however, will change over time to reflect changes in $\eta$. Therefore there will be firms which are not financing constrained, but will display investment-cash flow sensitivity.

## IV.  Firm Growth and Survival

This section considers some of the implications of the model for firm dynamics. Since detailed implications are very difficult to derive, we complement our analysis with some numerical

results, which suggest that our model is consistent with a set of widely established empirical findings. We proceeded to draw a large number of independent sample paths for firm shocks, assuming all firms start at the same initial value $V_0 \in \left[V_r, \widetilde{V}\right)$. Figure V summarizes the results of our simulation exercise.[17]

Throughout this section we identify firm size with its working capital $k$. By Proposition 4, the equity of surviving firms tends to increase over time. Eventually, all firms that do not exit reach the region where $k = k^*$ forever after. Since $k < k^*$ whenever $V < \widetilde{V}$ (see Proposition 2), in average surviving firms grow with age. Therefore, in accordance with the evidence gathered by Dunne, Roberts and Samuelson [1989] and Troske [1996], firm age and size are positively correlated. Both mean and variance of the growth rate decrease systematically with age, and thus with size. These predictions are qualitatively in line with the findings of the empirical literature on firm dynamics. In particular, we refer the reader to work by Evans [1987] and Hall [1987].

[ Figure V here ]

Our model also predicts that the conditional probability of survival increases with the value of the firm's equity $V$.

**Proposition 7** Let $T$ be the stopping time corresponding to firm exit. Then, for every $t$, $\Pr\left(T > t \mid V\right)$ is strictly increasing in $V$ for $V < \widetilde{V}$ and $\Pr\left(T = \infty \mid V\right) = 1$ for $V \geq \widetilde{V}$.

A firm may exit only after having reached the randomization region. This occurs as a consequence of a series of negative shocks. Since the continuation values $V^H$ and $V^L$ are weakly increasing in current value, any history of shocks - including the outcome of randomization - that leads a firm with an initial value $V$ to exit will lead a firm with a lower initial value to exit as well. Selection occurs as a consequence of exit. This implies that in average older firms will have higher values and therefore lower hazard rates. This prediction is qualitatively in line with the evidence presented by Evans [1987], Audretsch [1991], and Baldwin [1994]. Given the positive correlation between age and size, our model also predicts that the failure rate decreases with size. This is in accordance with the findings of Dunne, Roberts and Samuelson [1988] on the relation between survival probability and plant size.[18]

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17. The parameter values are the same as those listed in Section III.C. The discount rate yields an interest rate $\frac{1}{\delta} - 1 = 1$ percent, which we interpret as the quarterly interest rate. The time series obtained have been aggregate to generate yearly data, and then averaged in the cross-section.

18. It should be emphasized that if we were conditioning on size, our model would imply that growth and survival do not depend on age. This occurs because there is only one source of variation across firms, namely the value of equity $V$.

Notice that, given that the initial value $V_0 > V_r$, it may take a few periods for firms to exit. Correspondingly, hazard rates for exit must rise initially. Eventually, this increase must be followed by a decline, given that hazard rates must converge to zero - their value in the unconstrained region. To our knowledge, the study by Bruderl, Preisendorfer and Ziegler [1992] is the only one to report hazard rates by month. They find that hazard rates increase for most of the first year and decline thereafter.

Our model is clearly not the first to generate implications for firm growth and survival that are qualitatively in line with the empirical evidence. In particular, the frameworks introduced by Jovanovic [1982] and Hopenhayn [1992] produce similar results, without relying on moral hazard.

In Jovanovic [1982], firms differ with respect to the distribution of their productivity process. Information is symmetric, but incomplete. Entrepreneurs start with an unbiased prior estimate of their mean productivity, and use revenue realizations to generate a series of posteriors via bayesian learning. The precision of their estimate increases over time. This implies that the growth process of older firms is less volatile. Having to pay a fixed cost every period, entrepreneurs that turn out to have low productivity decide to exit. This selection process implies that among surviving firms, the older ones are larger, grow at a slower pace, and have a higher survival probability. In Hopenhayn [1992], firm productivity is driven by a persistent stochastic process. Younger firms are smaller because their initial productivity is lower than the incumbents' mean. Their growth is faster and more volatile, because of decreasing returns to scale.

According to us, the main advantage of our approach is that it imposes much less structure on the stochastic process that drives firm productivity: the simplest i.i.d process is sufficient to generate sensible implications. Furthermore, differently from the pre-existing literature, our model generates a non-trivial role for capital structure (i.e. the Modigliani-Miller proposition does not hold).

Recent work by Albuquerque and Hopenhayn [2004] has shown that modelling the relation between borrower and lender as one of incomplete enforcement, rather than asymmetric information, generates implications for firm dynamics that are very close to ours. Section VIII is dedicated to a detailed comparison of the two models.

# V. Long Versus Short-Term Contracts, Debt Limit, and Forgiveness

Can a long term lending contract be replicated by a sequence of short term contracts? This section provides a general answer to this question. An optimal lending contract defines a continuous frontier of values $B(V) = W(V) - V$ on the domain of equity values $\mathcal{V}$. The level of debt $B$ is a sufficient statistic if and only if the function $B(V)$ is strictly decreasing on $\mathcal{V}$. In such case, the inverse function $V(B)$ gives the entrepreneur's value as a function of the outstanding debt. The optimal contract can thus be formulated as a debt management problem, as in Gertler [1992]. More precisely, letting $\bar{B} = \sup_{V \in \mathcal{V}} B(V)$, the optimal contract satisfies the following dynamic programming problem:

$$V(B) = \max_{k, \tau, B^H, B^L} p(R(k) - \tau) + \delta \left[ pV(B^H) + (1-p)V(B^L) \right]$$
$$\text{s.t. } B^L, B^H \leq \bar{B},$$

(6) $$B = p\tau - k + \delta \left[ pB^H + (1-p)B^L \right],$$

(7) $$\tau \leq \delta \left( V(B^H) - V(B^L) \right).$$

With $B^H$ and $B^L$ we denote the continuation values for the debt, contingent on high and low realization, respectively. In the region where borrowing constraints are effective, $W'(V) = B'(V) + 1 > 0$. This implies that $B'(V) > -1$ and thus $V'(B) < -1$. As a consequence, $V(B^H) - V(B^L) > B^L - B^H > 0$. Substituting the incentive constraint (7) in (6), we obtain that

$$
\begin{aligned}
B &= p\tau - k + \delta B^L + \delta p \left( B^H - B^L \right) \\
&> p\tau - k + \delta B^L - \delta p \left( V(B^H) - V(B^L) \right) \\
&= -k + \delta B^L,
\end{aligned}
$$

where the last equality holds since the incentive constraint binds. It follows that

$$\delta B^L < B + k.$$

Following a low revenue realization, the discounted value of next period debt is lower than the sum of pre-existing debt and current capital advancement. This implies that the contract provides for some debt forgiveness. This is of course compensated by debt falling by less than the value of repayment in the high state. These remarks apply even when $B(V)$ is not strictly decreasing on all its domain, limitedly to the region where it is.

An immediate consequence of our analysis is that by rolling over the debt, a sequence of one-period contingent contracts is sufficient to implement the solution to the above dynamic

program. There is no need for a long term contract, in the sense that knowing the stock of debt outstanding is sufficient to determine the size of the capital advancement, the repayment contingent on high revenue realization, and the continuation values for the debt.

Notice that $B(V)$ is strictly decreasing on $\mathcal{V}$ if and only if the optimal long-term contract is renegotiation-proof. Hence, renegotiation proofness characterizes financial contracts that can be implemented by one-period contingent debt contracts. This conclusion does not necessarily apply to our model, since for low values of equity it is possible that $B'(V) > 0$. This is the case, for example, when $S = 0$ and $R'(0) = \infty$. However, for a sufficiently high scrap value, $B(V)$ will be strictly decreasing. To see this, first note that by concavity $W'(V)$ is decreasing, and so is $B'(V) = W'(V) - 1$. Hence, $B(V)$ will be strictly decreasing if and only $W'(V) \leq 1$ in the randomization area. This will occur for high levels of $S$.[19] Incidentally, this condition is equivalent to saying that the liquidation decision is renegotiation-proof.[20]

As indicated above, there is a maximum level of sustainable debt $\bar{B}$, which corresponds to the lowest level of equity in the region where $B'(V) < 0$. There are two possible cases: the corresponding value of equity is either zero or positive. In the first case (see the right panel in Figure VI), the contract is renegotiation proof and $\bar{B} = S$. The entrepreneur's debt is fully collateralized. In the second case (see the left panel), $\bar{B} > S$. The debt can exceed the value of collateral.


[ Figure VI here ]


The results of this section are summarized in the following proposition.

**Proposition 8** i) The optimal contract can be implemented by a sequence of one-period contingent debt contracts if and only if it is renegotiation-proof. ii) in any region where there is no incentive for renegotiation the contract involves debt forgiveness in the low state. iii) The maximum sustainable debt $\bar{B} \geq S$, where $\bar{B} = S$ if and only if the contract is renegotiation-proof.

A further interesting question is whether the sequence of short-term contingent contracts that we have characterized can be implemented by means of simpler financial instruments.

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19. It is obviously true for $S = \widetilde{W}$, since in that case $W(V) = \widetilde{W}$ and $W'(V) = 0$. By continuity, it will also hold for lower values of $S$.

20. Quadrini [2003] shows that it is always possible to force a contract to be renegotiation-proof, by imposing a higher lower bound on the entrepreneur's claim $V$. Furthermore (see his Proposition 4), liquidation occurs with strictly positive probability in all contracts resulting from the introduction of this further constraint, for all $S \geq 0$.

DeMarzo and Fishman [2001] have shown that this is the case for a specialized model where scale (i.e. capital) is not a choice variable. In that scenario, the optimal contract is implemented by the combination of long term debt with a constant coupon and a line of credit. The project is liquidated with certainty when the line of credit is exhausted, while the possibility of liquidation disappears when the line's balance goes to zero. As we now show, their result does not extend to our model, where scale is an endogenous variable.

For simplicity of exposition, consider the case where $S = 0$. The limit to the line of credit is $\widetilde{V}$ and the balance at time $t$ is given by $l_t = (\widetilde{V} - V_t)$. Recall that the firm ceases to produce when $V_t$ reaches zero. This is precisely when the line of credit is exhausted. Recall also that while in operation, the cash flow of the project is $R(k_t) - k_t$ if successful and $-k_t$ if not. From condition (5) it follows that:

$$\delta V_{t+1}^h = V_t + R(k_t) - k_t - [pR(k_t) - k_t],$$
$$\delta V_{t+1}^l = V_t - k_t - [pR(k_t) - k_t]$$

Letting $s_t$ denote the payments to long term debt, the evolution of the account balance is given by:

$$\delta l_{t+1}^h = l_t + k_t - R(k_t) + s_t,$$
$$\delta l_{t+1}^l = l_t + k_t + s_t.$$

Finally, using these relations, we obtain:

$$s_t = -(1 - \delta)\widetilde{V} + pR(k_t) - k_t.$$

This term is only constant when $k_t$ is constant, that is when the scale is not variable, as in DeMarzo and Fishman [2001]. With endogenous determination of scale, the payment to long term debt is neither constant nor deterministic, as in the optimal contract $k_t$ depends on all past realizations of the project.[21]

# VI. The Role of Collateral

The scrap value $S$ constitutes the resale value of the project and thus its collateral. In this section we investigate how an increase in the value of $S$ affects the lending contract.

Obviously, a larger $S$ implies a larger surplus $W$ for every $V < \widetilde{V}$. In turn, this implies that the set of pairs $(I_0, M)$ that insure financial feasibility is also larger. Our analysis in Section

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21. In the case of DeMarzo and Fishman [2001], the project requires no investment of working capital ($k_t = 0$ at all $t$) and yields revenues $R^H$ with probability $p$ and $R^L$ with probability $(1 - p)$, $R^H > R^L > 0$. In such scenario, $\delta V_{t+1}^h = V_t + (1 - p)(R^H - R^L)$, $\delta V_{t+1}^l = V_t - p(R^H - R^L)$, and $\widetilde{V} = p(R^H - R^L)/(1 - \delta)$. It follows that $s_t = R^L$ for all $t$.

VII also implies that when the lending market is competitive a larger collateral translates into an higher initial equity value $V_0$.

A more subtle yet interesting implication can be inferred by inspection of Figure VII.[22] As shown in the left panel, an increase in $S$ decreases the degree of concavity of the function $W(V)$. As a consequence, one may expect the capital advancement $k$ to increase for every $V < \widetilde{V}$. There is an intuitive explanation. As explained in Section III.C, higher capital advancements are costly since by lowering $V^L$ they increase the risk of future (inefficient) liquidation. A larger collateral $S$ makes this risk less costly, resulting in a higher capital advancement. The central panel of Figure VII illustrates this effect. In the case of our numerical experiment, the elasticity of capital advancement to changes in the value of collateral can be larger than 1.

[ Figure VII here ]

It is also of interest to study the effect of different collateral values on firm survival. In this respect, the analysis that we have conducted so far is inconclusive. While a larger $S$ makes liquidation more appealing, it also has the effects of increasing the initial equity $V_0$ and the continuation value of the project.[23] The right panel illustrates that from our simulation exercise does not emerge an ordering of the survival functions. However, in the limit survival increases with $S$.

## VII. Financial Feasibility

Our previous analysis did not consider the initial distribution of surplus between lender and borrower. Letting $V_0$ denote the initial value to the entrepreneur, $B_0(V_0) = W(V_0) - V_0$ is the value to the lender. Given the entrepreneur's initial net worth $M < I_0$, the project requires an initial contribution from the lender $I_0 - M$. The lender will be willing to provide financing to the entrepreneur, as long as $V_0$ satisfies

$$[W(V_0) - V_0] - [I_0 - M] \geq 0.$$

The entrepreneur will be willing to invest as long as $V_0 > M$. The project is financially feasible if there exists an initial allocation of value $(B_0, V_0)$ that satisfies both constraints.

If the maximum sustainable debt $\bar{B} < I_0 - M$, the project is not financially feasible. Otherwise, let $\underline{V}$ denote the largest $V_0 \geq 0$ such that $B(V_0) = I_0 - M$. The project is

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22. The right-most panel of the figure is the result of a simulation exercise conducted as described in Section IV. The initial equity values were pinned down by assuming that the lending market is competitive.

23. The analysis carried out in Section VII implies that if the lending market is competitive, the initial equity $V_0$ is larger when the value function $W(V)$ is uniformly less concave.

financially feasible if an only if $\underline{V} \geq M$. The value $\underline{V}$ also defines the equilibrium starting level of equity in a market with competing lenders. This initial equity $\underline{V}$ is strictly increasing in the net worth of the entrepreneur $M$.

It is instructive to examine again the role of collateral. As indicated above, for large collateral, $\bar{B} = B(0)$ and initial lending is constrained by the value of collateral. However, when the collateral is relatively low, initial lending will exceed its value. In the latter case, the contract cannot be implemented by one-period contingent debt. This suggests the importance of long term lending when the value of collateral is small compared to initial setup costs, which is the case for investments that involve large sunk costs.

# VIII. A Comparison to Incomplete Enforcement

Several papers in the literature on financing contracts consider an alternative formulation, where borrowing constraints result from incomplete enforcement. For example, see Eaton and Gersowitz [1981] and Albuquerque and Hopenhayn [2004]. There are no informational asymmetries, but borrowers can default on their obligations and take an outside option. This section provides a sharp distinction between the two alternative approaches.

The incomplete enforcement model that we describe is the natural counterpart to our asymmetric information framework. Take the same revenue function $R(k)$ and process for the shocks, but now assume that the lender observes the outcome of the project. However, the entrepreneur can default by keeping the revenues and not paying back the loan. The project is then liquidated, giving the lender the liquidation value $S$. By defaulting, the entrepreneur gets a value $R(k)$ if the productive state had realized, while he gets nothing otherwise.

Letting $V^H$ denote the continuation value in the productive state, the no-default constraint is given by

$$R(k) \leq \delta V^H.$$

Define now the indirect profit function as

$$\Pi\left(V^H\right) \equiv \max_{k} pR(k) - k$$
$$\text{s.t. } R(k) \leq \delta V^H.$$

The optimal no-default contract solves the following dynamic programming equation:

(P4) $$W(V) = \max_{V^H,V^L} \Pi\left(V^H\right) + \delta\left[pW\left(V^H\right) + (1-p)W\left(V^L\right)\right]$$
$$\text{s.t. } V \geq \delta\left(pV^H + (1-p)V^L\right).$$

This problem is the analogue of (P3). The only difference is that the indirect profit function is an increasing and concave function of $V^H$, rather than of $V^H - V^L$. Proposition 9 shows that

in the case of incomplete enforcement the equity value increases in response to good shocks but never decreases! As a consequence, the firm never shrinks nor exits.

**Proposition 9** In the case of incomplete enforcement, $V^L = V < V^H$.

In the case of asymmetric information, widening the spread $V^H - V^L$ is necessary condition for increasing the capital advancement. Decreasing $V^L$ contributes to accomplishing this task both directly and indirectly (via the promise-keeping constraint). With incomplete enforcement instead, a decrease in $V^L$ has no direct effect on the ability to increase the level of capital. The only impact is the indirect one, via the constraint.

# IX. Final Remarks

There exist a large variety of lending contracts with implicit and unwritten contingencies and clauses. This makes it very difficult to assess directly the empirical implications of borrowing constraints. Existing empirical tests are indirect, focusing mostly on the sensitivity of investment to cash-flows and on other features of firm dynamics. It is our view that theory can contribute to this debate through a better understanding of borrowing constraints and their empirical implications. This requires theories that derive borrowing constraints from first principles and are suitable for the analysis of firm dynamics. Gertler [1992] was clearly a move in this direction. Our paper contributes to this research program.

We have chosen to keep the model as simple as possible, with the minimal elements needed to derive endogenous borrowing constraints in a dynamic setup. In spite of its simplicity, our model matches most of the qualitative properties of firm dynamics that have been recently documented. Moral hazard thus seems to be a good foundation for the analysis of borrowing constraints.

Our model is also useful to better understand the value and importance of long term lending contracts. We have shown that projects with high collateral are renegotiation-proof and are implementable through a sequence of one-period (contingent) loans. Borrowing is limited by the value of the collateral itself. In contrast, when the collateral is low relative to the total value of the project, initial lending can exceed its value. In the latter case, the optimal contract requires commitment of the lender to no-renegotiation and cannot be implemented by a sequence of short-term contracts. Long term lending is thus more valuable for investments that involve high sunk costs.

The restriction to renegotiation-proof contracts can limit lending opportunities. In some cases, it could lead to the extreme of making the project financially infeasible. The ability

of lenders to commit to long term contracts can thus have considerable value. The same can be said of institutions that facilitate reputation building and the establishment of long-term relationships.

It is interesting to compare the implications of our model to theories of incomplete contracts where borrowing constraints arise from incomplete enforcement, as in Albuquerque and Hopenhayn [2004]. We have shown that, differently from our setup, in the incomplete enforcement analogue of our model firm size never decreases. In the environment studied by Albuquerque and Hopenhayn [2004], the stochastic structure is such that firm size does decrease in response to low productivity shocks. Conditional on a given productivity shock, however, size never decreases. While these these predictions are difficult to test, we still conjecture that the two classes of models would differ considerably in their quantitative implications.

We have assumed that both lender and borrower are risk neutral. In spite of this, risk considerations arise in the optimal contract design as a result of the interaction of limited liability and a concave revenue function. The resulting degree of risk aversion is not constant and depends on the entrepreneur's equity. We find that risk taking is encouraged at the extremes, close to the liquidation area and for large values of equity. In our model the project and its risk are taken exogenously. Allowing for risk to be a choice variable is an interesting direction in which our theory could be extended.

There are many other interesting extensions. Multiple shocks can be easily accommodated, as in Quadrini [2003]. A model with partial liquidation (downsizing) has been studied by De-Marzo and Fishman [2001]. The dynamics of capacity utilization may be studied adopting the timing of Gertler [1992]. Finally, we suggest that our model may have interesting applications in macroeconomics. Starting with Bernanke and Gertler [1989], there has been a growing interest in understanding to what extent various forms of frictions in financial markets may generate and/or amplify macroeconomic fluctuations. However, the financial arrangements that are considered in this literature are never intertemporally optimal, meaning that contracts' provisions are not contingent on all public information. The only exception is recent work by Cooley, Marimon and Quadrini [2004], that embed the model by Albuquerque and Hopenhayn [2004] in a general equilibrium framework. It would definitely be of interest to perform a similar exercise with the model developed in this paper, and then contrast the predictions generated by the two approaches.

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# I. Unobservable Investment

This section considers an alternative source of moral hazard. We assume that the lender can monitor the outcome of the project each period, but not the use of funds. The entrepreneur faces an outside opportunity that provides utility $V_a(k)$. We assume that $V_a$ is strictly increasing and strictly concave. The incentive compatibility constraint for this problem is given by

$$pR(k) - \tau + \delta\left(pV^H + (1-p)V^L\right) \geq V_a(k) + \delta V^L.$$

The results described in the main text still apply, once we define

$$\Pi\left(V^H - V^L\right) \equiv \max_k pR(k) - k$$
$$\text{s.t. } V_a(k) \leq \delta p\left(V^H - V^L\right).$$

In fact, our previous analysis uses only two properties of the indirect profit function, namely monotonicity and concavity. The first property is immediate from the above definition. Implicit differentiation of $\Pi$ reveals that $pR''(k)/[pR'(k) - 1] - V_0''(k)/V_0'(k) < 0$ is a sufficient condition for concavity. Notice that in the special case where $V_a(k) = pR(k)$, the two indirect profit functions are identical.

# II. Proofs and Lemmas

**Lemma 1** $\widetilde{V} = pR(k^*)/(1-\delta)$.

<u>Proof of Lemma 1</u> By definition, the value $\widetilde{V}$ solves the following problem:

$$\widetilde{V} \equiv \min_{\tau, V^H, V^L} p\left(R(k^*) - \tau\right) + \delta\left(pV^H + (1-p)V^L\right),$$
$$\tau \leq \delta\left(V^H - V^L\right),$$
$$\tau \leq R(k^*),$$
$$V^L \geq \widetilde{V}, \; V^H \geq \widetilde{V}.$$

Rewriting the objective function as $p(R(k^*) - \tau) + \delta[p(V^H - V^L) + V^L]$, one realizes that $V_L = \widetilde{V}$ and $\tau = \delta(V^H - V^L)$ are necessary conditions for a minimum. By the latter condition, any value $\tau$ such that $\tau \leq R(k^*)$ will be optimal. Choosing $\tau = 0$ yields $\widetilde{V} = pR(k^*)/(1-\delta)$.

<div align="right">QED</div>

<u>Proof of Proposition 1</u> All but one of the statements of the proposition were proven in the text. It is left to show that $W(V)$ is strictly increasing for $V < \widetilde{V}$. By contradiction,

assume that there exists a pair $(V', V)$ such that $V' < V < \widetilde{V}$ and $W(V') = W(V)$. Then, by concavity of the value function it must be that $W(V') = W(V) = \widetilde{W}$, which is a contradiction. QED

Proof of Proposition 2 i) It follows from standard dynamic programming arguments, since (P1) defines a concave programming problem which is strictly concave in $k$. ii) By contradiction, assume that there exists some $V < \widetilde{V}$ such that $k(V) = k^*$. Then, since $\widetilde{W} = [pR(k^*) - k^*]/(1 - \delta)$, $W(V) = (1 - \delta)\widetilde{W} + \delta \left[ pW(V^H) + (1 - p)W(V^L) \right]$. By Proposition 1, it must be the case that $W(V) < \widetilde{W}$, and therefore either $V^H < \widetilde{V}$, or $V^L < \widetilde{V}$, or both. If $W'(V^H) < W'(V^L)$, consider lowering slightly $k$ and $\tau$ in such a way that $R(k) - \tau$ does not change. The marginal effect of this change on the surplus is zero, since $pR'(k^*) = 1$. However, it has the effect of relaxing the incentive compatibility constraint. By Jensen inequality, total surplus can be increased by lowering $V^H$ and raising $V^L$, in such a way that the promise-keeping constraint still hold. If $W'(V^H) = W'(V^L)$ then the value function is linear over the range $(V^L, V^H)$. By Proposition 1, it must be that $V^H < \widetilde{V}$. Therefore, by Proposition 3 it follows that $\tau = R(k^*)$. Thus, $pV^H + (1 - p)V^L = V/\delta$. By linearity, it follows that $pW(V^H) + (1 - p)W(V^L) = W(V/\delta)$. On the other hand, $W(V) < \widetilde{W}$ implies $\left[ pW(V^H) + (1 - p)W(V^L) \right] > W(V)$. Summarizing, we have that $\left[ pW(V^H) + (1 - p)W(V^L) \right] = W(V/\delta) > W(V) > \left[ pW(V^H) + (1 - p)W(V^L) \right]$, which is a contradiction. iii) Again by Proposition 1, $W(V) = [pR(k^*) - k^*]/(1 - \delta)$ for $V \geq \widetilde{V}$. The only way such surplus can be achieved is by setting $k(V) = k^*$.                      QED

**Lemma 2** When $\tau = R(k)$, then the IC constraint (3) binds for every $V < \widetilde{V}$.

Proof of Lemma 2 By Proposition 2, $V < \widetilde{V}$ implies that $k(V) < k^*$. For the sake of contradiction, assume there exists $V$ such that $R(k) < \delta \left( V^H - V^L \right)$. Then by strict monotonicity of the revenue function it is possible to increase the surplus of the match strictly just by raising $k$. But this contradicts optimality.                      QED

Proof of Proposition 3 For part (i), consider the case where $V^H < \widetilde{V}$. If $\tau < R(k)$, then $V^H$ and $\tau$ can be increased in a way that all constraints are still satisfied. This results in a strictly higher value for $W(V^H)$ and, correspondingly, a higher current firm value $W(V)$. Part (ii) follows immediately from the monotonicity of the value function.                      QED

Proof of Proposition 4 Refer to problem (P3). Since $W$ is concave, it is almost everywhere differentiable. Let $\lambda$ denote the Lagrange multiplier for the constraint. Then, the necessary

conditions for optimality are given by

$$\text{(8)} \qquad \Pi'\left(V^H - V^L\right) + \delta p W'\left(V^H\right) - \delta p \lambda = 0$$

$$\text{(9)} \qquad -\Pi'\left(V^H - V^L\right) + \delta\left(1 - p\right) W'\left(V^L\right) - \delta\left(1 - p\right)\lambda = 0$$

and by the envelope condition

$$\text{(10)} \qquad \lambda = W'\left(V\right).$$

Together with Lemma 2, Proposition 2 implies that $\Pi'\left(V^H - V^L\right) > 0$ for $V < \widetilde{V}$. Using (8), (9) and (10) we obtain that

$$W'\left(V^H\right) < W'\left(V\right) < W'\left(V^L\right).$$

By concavity of the value function, it follows that

$$\text{(11)} \qquad V^L < V < V^H.$$

<div align="right">QED</div>

Proof of Proposition 5   We begin by proving that the function $V^H(V)$ is nondecreasing. Take any $V$, $V < \widetilde{V}$. Consider raising $V$. By concavity of the value function, $W'(V)$ decreases weakly. By contradiction, assume that $V^H$ is now strictly lower. It follows that $W'(V^H)$ is weakly higher. For constraint (2) to hold, $V^L$ must increase strictly. Since $\Pi$ is strictly concave in the relevant range, $\Pi'(V^H - V^L)$ must also increase strictly. However, this contradicts condition (8). The proof of monotonicity of $V^L(V)$ is very similar. Take any $V$, $V < \widetilde{V}$. Consider raising $V$. By concavity of $W$, $W'(V)$ decreases weakly. By contradiction, assume that $V^L$ is now strictly lower. It follows that $W'(V^L)$ is weakly higher. For constraint (2) to hold, $V^H$ must increase strictly. Since $\Pi$ is strictly concave in the relevant range, $\Pi'(V^H - V^L)$ increases strictly. However, this contradicts condition (9).                QED

Proof of Proposition 6   Combining conditions (8) and (9), one obtains that a necessary condition for the optimal contract is $p(1 - p)\left(W'(V^L) - W'(V^H)\right) = \Pi'(V^H - V^L) = 1 - 1/[pR'(k)]$.

1) By Proposition 5, $V^L\left(V_r\right) < V_r$. By continuity of the function $V^L\left(\cdot\right)$, there exists $V > V_r$ such that $V^L\left(V\right) < V_r$. Combining this fact with monotonicity of $V^H\left(\cdot\right)$ and concavity of the value function yields the prediction that $\left(W'\left(V^L\right) - W'(V^H)\right)$ is non-decreasing and thus $k\left(\cdot\right)$ non-increasing on $[V_r, V]$.

2) We know that $V^H\left(\widetilde{V}\right) > \widetilde{V}$. By continuity of the function $V^H\left(\cdot\right)$, there exists $V < \widetilde{V}$ such that $V^H\left(V\right) > \widetilde{V}$. Combining this fact with monotonicity of $V^L\left(\cdot\right)$ and concavity of the value function yields the prediction that $\left(W'\left(V^L\right) - W'\left(V^H\right)\right)$ is non-increasing and thus $k\left(\cdot\right)$ non-decreasing on $\left[V, \widetilde{V}\right]$. QED

Proof of Proposition 9 Refer to problem (P4). Since $W$ is concave, it is almost everywhere differentiable. Let $\lambda$ denote the Lagrange multiplier for the constraint. Then, the necessary conditions for optimality are given by

$$\Pi'\left(V^H\right) + \delta p W'\left(V^H\right) - \delta p \lambda = 0$$
$$W'\left(V^L\right) - \lambda = 0.$$

Using the envelope condition $\lambda = W'\left(V\right)$, it follows that

$$W'\left(V^H\right) < W'\left(V\right) = W'\left(V^L\right),$$

which implies that

$$V^L = V < V^H.$$

QED

# References

Albuquerque, Rui and Hugo Hopenhayn, "Optimal Lending Contracts and Firm Dynamics," Review of Economic Studies, LXXI (2004), 285–315.

Alti, Aydogan, "How Sensitive Is Investment To Cash Flow When Financing Is Frictionless?," Journal of Finance, LVIII (2003), 707–722.

Atkeson, Andrew, "International Lending with Moral Hazard and Risk of Repudiation," Econometrica, LIX (1991), 1069–1089.

Atkeson, Andrew and Robert Lucas, "On Efficient Distribution with Private Information," Review of Economic Studies, LIX (1992), 427–453.

Audretsch, David B., "New-Firm Survival and the Technological Regime," Review of Economics and Statistics, LXXIII (1991), 441–450.

Baldwin, John, The Dynamics of Industrial Competition (Cambridge, MA: Cambridge University Press, 1994).

Bernanke, Ben and Mark Gertler, "Agency Costs, Net Worth and Business Fluctuations," American Economic Review, LXXIX (1989), 14–31.

Bolton, Patrick and David S. Scharfstein, "A Theory of Predation Based on Agency Problems in Financial Contracting," American Economic Review, LXXX (1990), 93–105.

Bruderl, Josef, Peter Preisendorfer, and Rolf Ziegler, "Survival Chances of Newly Founded Business Organizations," American Sociological Review, LVII (1992), 227–242.

Caves, Richard E., "Industrial Organization and New Findings on the Turnover and Mobility of Firms," Journal of Economic Literature, XXXVI (1998), 1947–82.

Cooley, Thomas F., Ramon Marimon, and Vincenzo Quadrini, "Aggregate Consequences of Limited Contract Enforceability," Journal of Political Economy, CXII (2004), 817–847.

DeMarzo, Peter and Michael Fishman, Optimal Long-Term Financial Contracting with Privately Observed Cash Flows" (2001)." Graduate School of Business, Stanford University.

Dunne, Timothy, Mark Roberts, and Larry Samuelson, "Patterns of Firm Entry and Exit in US Manufacturing Industries," Rand Journal of Economics, XIX (1988), 495–515.

Dunne, Timothy, Mark Roberts, and Larry Samuelson, "The Growth and Failure of US Manufacturing Plants," Quarterly Journal of Economics, CIV (1989), 671–698.

Eaton and Gersowitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," Review of Economic Studies, XLVIII (1981), 289–309.

Evans, David, "The Relationship between Firm Growth, Size and Age: Estimates for 100 Manufacturing Firms," Journal of Industrial Economics, XXXV (1987), 567–581.

Fazzari, Steven, Glenn Hubbard, and Bruce Petersen, "Financing Constraints and Corporate Investment," Brooking Papers on Economic Activity, I (1988), 141–206.

Gertler, Mark, "Financial Capacity and Output Fluctuations in an Economy with Multi-Period Financial Relationships," Review of Economic Studies, LIX (1992), 455–472.

Gertler, Mark and Simon Gilchrist, "Monetary Policy, Business Cycles, and the Behavior of Small Manufacturing Firms," Quarterly Journal of Economics, CIX (1994), 309–340.

Gilchrist, Simon and Charles Himmelberg, "Evidence on the Role of Cash Flow for Investment," Journal of Monetary Economics, XXXVI (1994), 541–572.

Gomes, Joao, "Financing Investment," American Economic Review, XCI (2001), 1263–1285.

Green, Edward, "Lending and the Smoothing of Uninsurable Income," in "Contractual Arrangements for Intertemporal Trade" (Minneapolis, MN: University of Minnesota Press, 1987).

Hall, Bronwyn, "The Relationship between Firm Size and Firm Growth in the US Manufacturing Sector," Journal of Industrial Economics, XXXV (1987), 583–606.

Hayashi, Fumio, "Tobin's Marginal q and Average q: A Neoclassical Interpretation," Econometrica, L (1982), 213–224.

Hopenhayn, Hugo A., "Entry, Exit and Firm Dynamics in Long Run Equilibrium," Econometrica, LX (1992), 1127–1150.

Hubbard, Glenn, "Capital Market Imperfection and Investment," Journal of Economic Literature, XXXVI (1998), 193–225.

Jovanovic, Boyan, "Selection and The Evolution of Industry," Econometrica, L (1982), 649–670.

Kaplan, Steven and Luigi Zingales, "Do Investment-Cash Flow Sensitivities Provide Useful Measures of Financing Constraints?" Quarterly Journal of Economics, CXII (1997), 169–215.

Marcet, Albert and Ramon Marimon, "Communication, Commitment, and Growth," Journal of Economic Theory, LVIII (1992), 219–249.

Perez-Quiros, Gabriel and Allan Timmermann, "Firm Size and Cyclical Variations in stock Returns," Journal of Finance, LV (2000), 1229–1262.

Quadrini, Vincenzo, "Investment and Liquidation in Renegotiation-proof Contracts with Moral Hazard," Journal of Monetary Economics, LI (2003), 713–751.

Radner, Roy, "Repeated Principal Agent Games with Discounting," Econometrica, LIII (1985), 1173–1198.

Rogerson, William, "Repeated Moral Hazard," Econometrica, LIII (1985), 69–76.

Spear, Steve and Sanjay Srivastava, "On Repeated Moral Hazard with Discounting," Review of Economic Studies, LIV (1987), 599–617.

Stein, Jeremy, "Agency, Information, and Corporate Investment," in "Handbook of Economics of Finance" (Elsevier North-Holland, 2003).

Troske, Kenneth, "The Dynamic Adjustment of Firm Entry and Exit in Manufacturing and Finance, Insurance, and Real Estate," Journal of Law and Economics, XXXIX (1996), 705–735.

Whited, Toni, "Debt, Liquidity Constraints, and Corporate Investment: Evidence from Panel Data," Journal of Finance, XLVII (1992), 1425–1460.
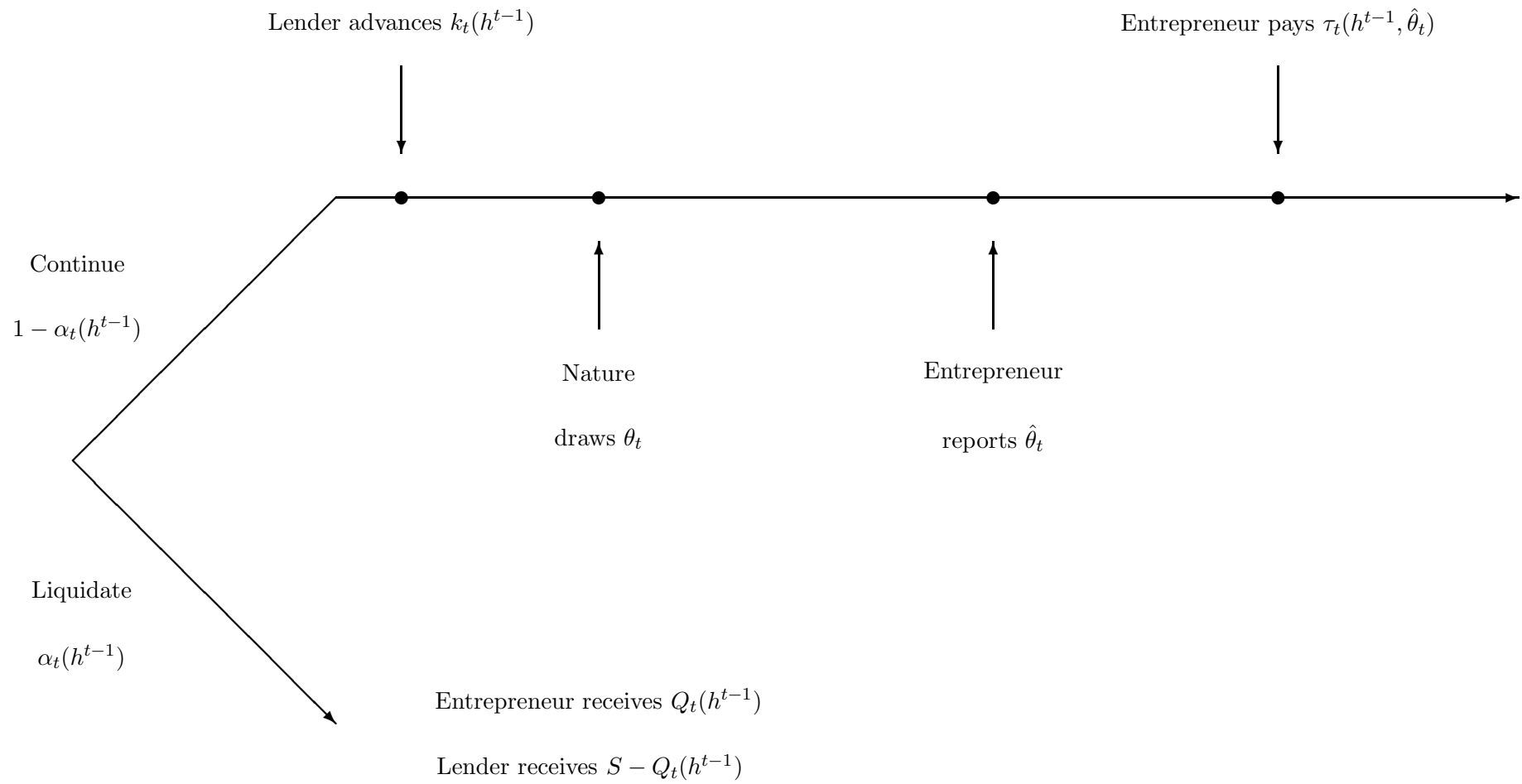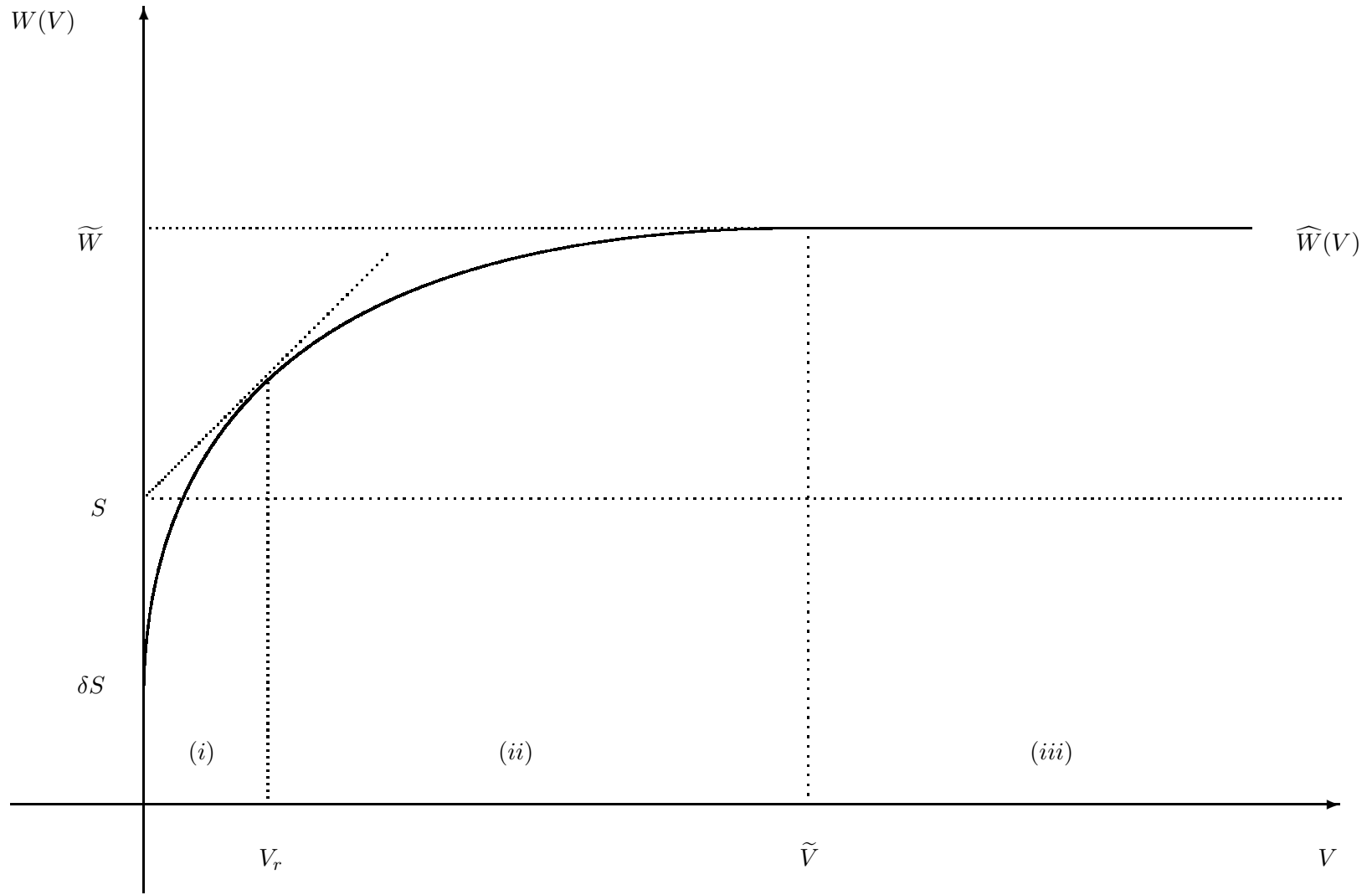
Lender advances $k_t(h^{t-1})$

Entrepreneur pays $\tau_t(h^{t-1}, \hat{\theta}_t)$

Continue

$1 - \alpha_t(h^{t-1})$

Nature

draws $\theta_t$

Entrepreneur

reports $\hat{\theta}_t$

Liquidate

$\alpha_t(h^{t-1})$

Entrepreneur receives $Q_t(h^{t-1})$

Lender receives $S - Q_t(h^{t-1})$

Figure I: The timing.

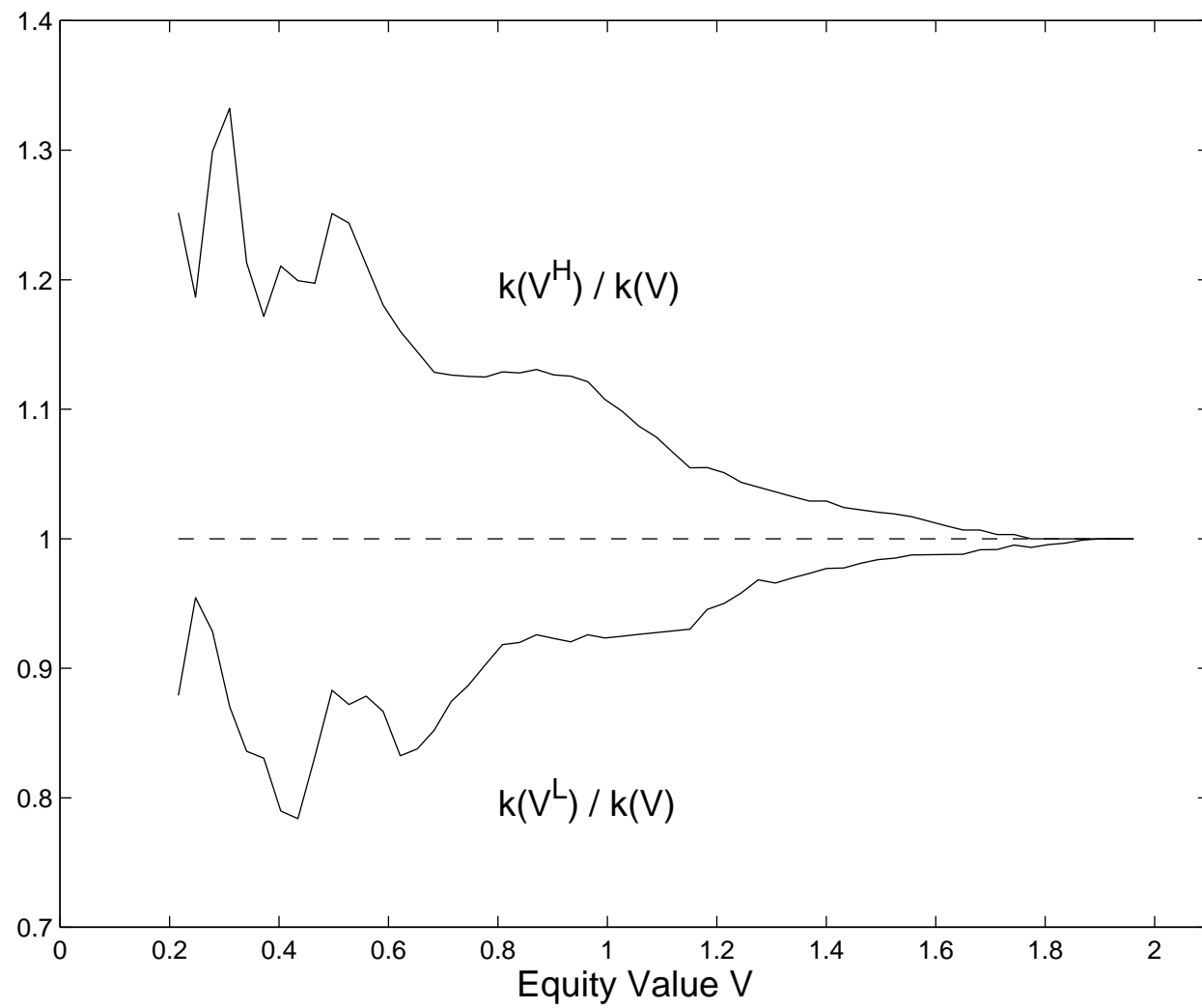Figure II: The value function.
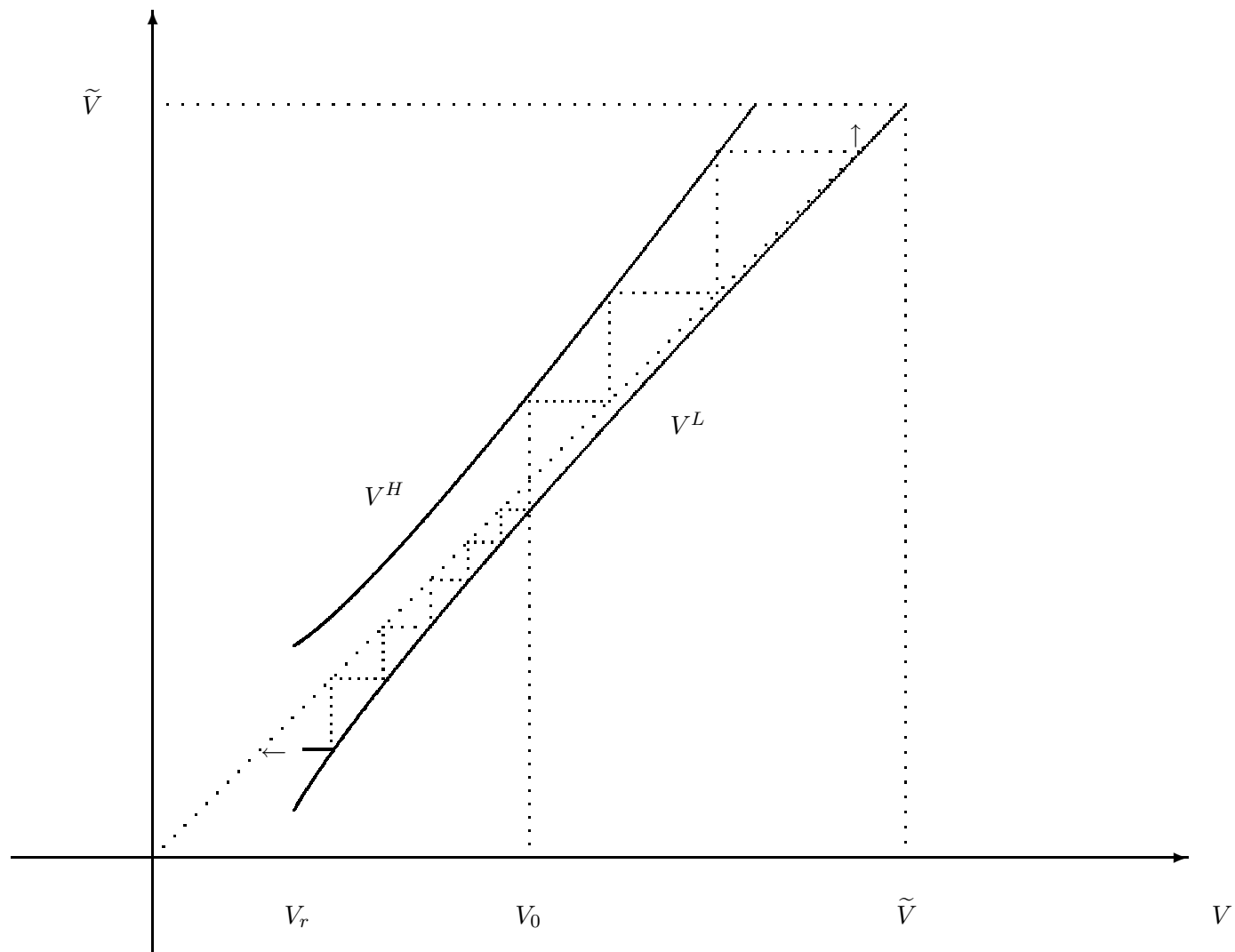
Figure III: The dynamics of equity.

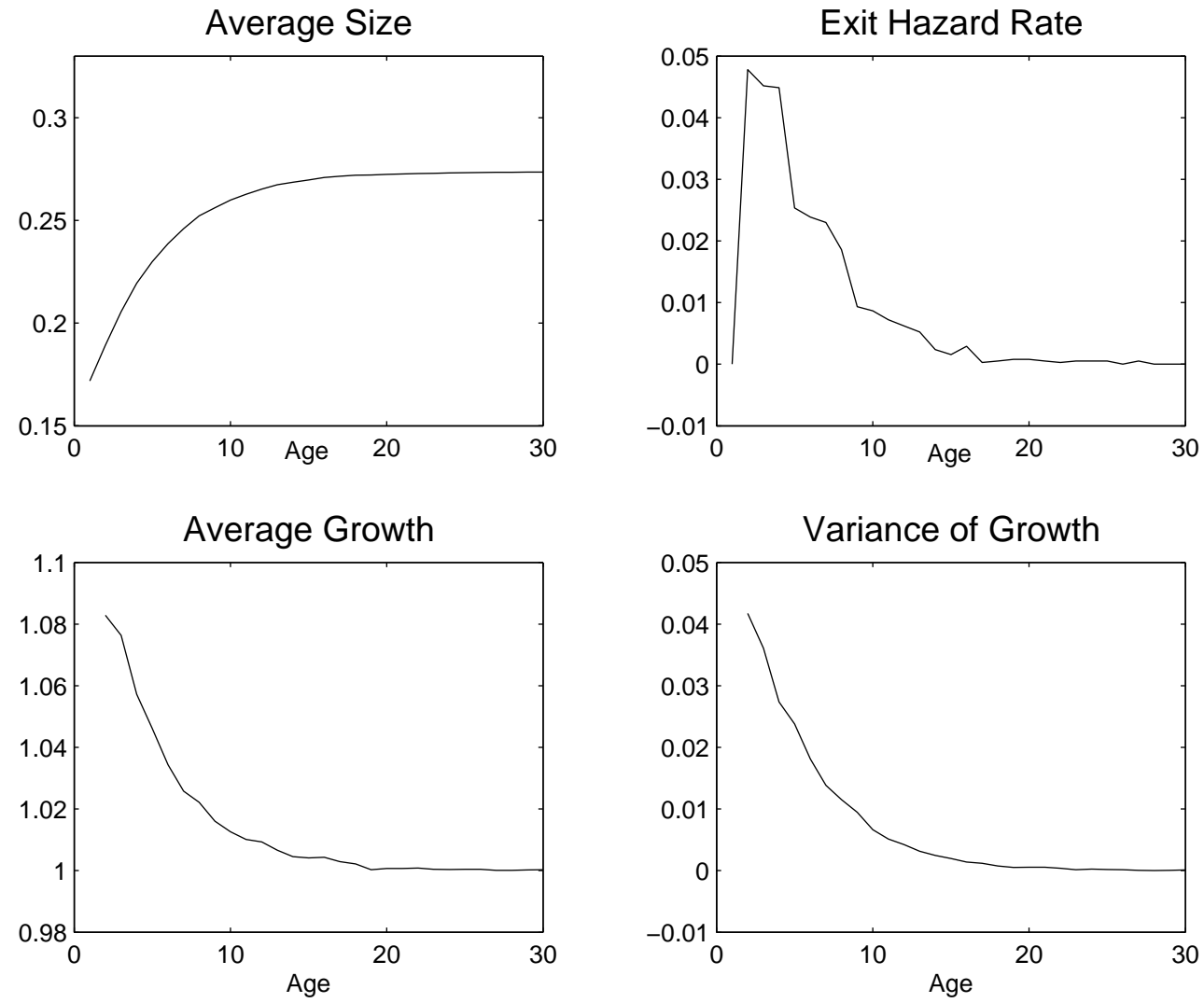Figure IV: Growth rates of capital conditional on revenue realization.

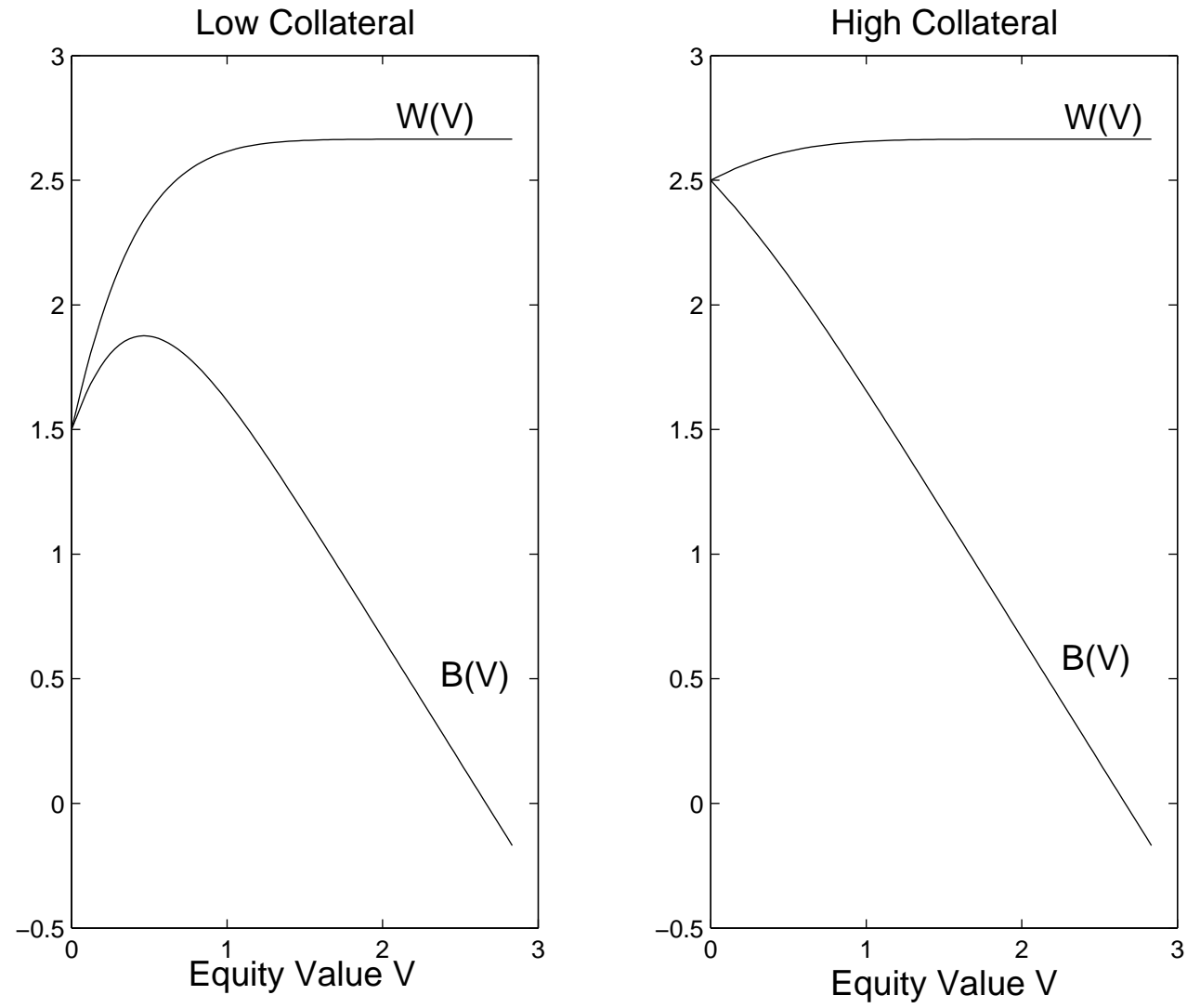Figure V: Dynamics of growth and survival.
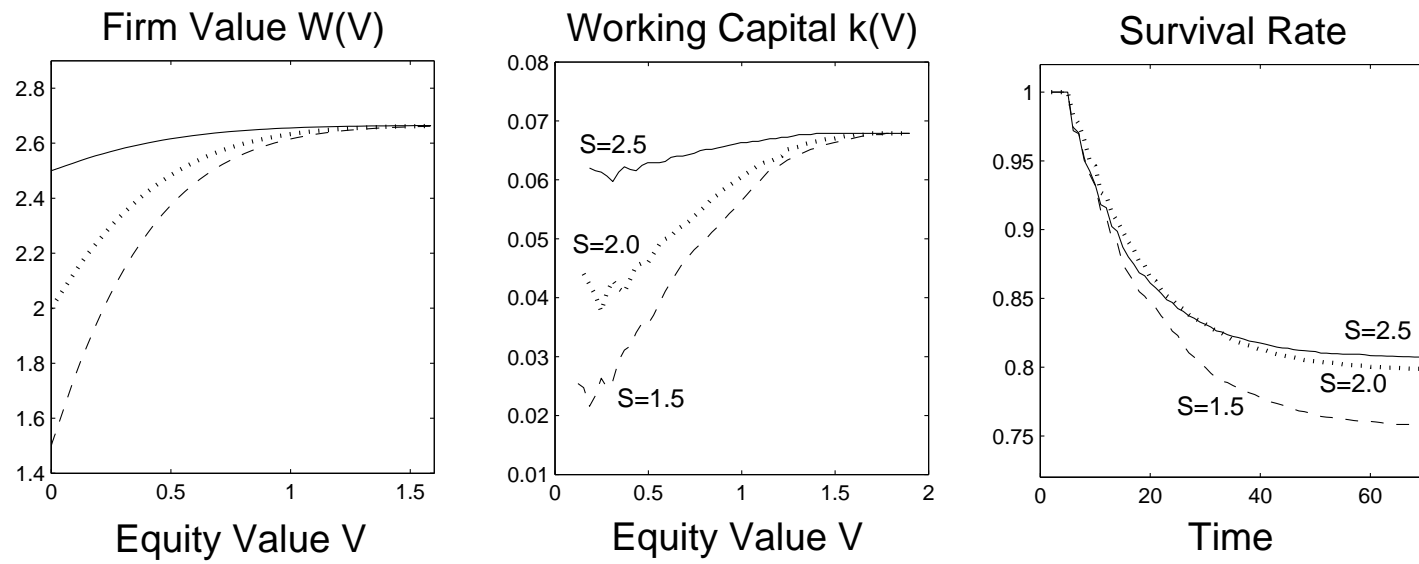
Figure VI: Implementation and Renegotiation-Proofness.

Figure VII: Comparative statics with respect to the value of collateral.