Implications of Heterogeneity in Preferences, Beliefs and Asset Trading Technologies for the Macroeconomy

Extremely Preliminary and Incomplete

August 1, 2013

Yili Chien, Harold Cole and Hanno Lustig

August 1, 2013

1 Introduction

This paper extends the methodology developed in Chien, Cole and Lustig (2011, 2012) (hereafter CCL2011 and CCL2012) to analyze and compute the equilibria of economies with large numbers of agents who have different asset trading technologies which are design to replicate the portfolio behavior we see in the data. The different asset trading technologies fall into two classes. *Active traders* manage the composition of their portfolios among a given set of assets along with choosing how much to save. *Passive traders* take their portfolio composition as given and choose only how much to save. Within each class there can be a wide variety of different cases. For active traders, the trading technology varies depending upon the set of assets that they can use. While for passive traders it varies with the specific portfolio composition rule. In CCL2011 and CCL2012 all of our agents had to have the same CRRA flow utility functions, discount rate and beliefs. In this extension, we relax this restriction, greatly extending the set of economies to which our method applies. This richer degree of heterogeneity, especially with respect to beliefs, allows the model to match up with a number of key features of the data.

To compute and characterize equilibria we utilize a recursive multiplier method and analytic aggregation results with respect to consumption shares and the stochastic discount factor that rely on a single cross-sectional moment of the multiplier distribution. As a result, we can compute equilibria via an iterative method in which we guess and recover a transition rule for the updating of this moment. In these iterations we do not need to compute "market clearing prices" since we know them analytically as a function of the updating rule. Instead, we simply solve the analog of a dynamic programming problem for each type of agent with respect to their individual multipliers to determine their individual multiplier updating rule. The only input into this problem is the stochastic discount rate implied by the conjectured updating rule for the key cross-sectional multiplier moment. We then pass their individual multiplier updating rules through a stochastic panel of aggregate and individual shocks to determine the implied updates of the key cross-sectional multiplier moment. If we recover the same updating rule we conjectured, then we have solved for an equilibrium of our economy. Otherwise, we use the implied updating rule to modify our conjectured rule and continue to iterate on the aggregate multiplier transition rule.

In CCL2011 and CCL2012 we relied on the homogeneity of the inverse of the marginal utility of consumption when all households have common CRRA preferences, discount rates and beliefs. In our extension, we create a parallel economy of reference traders who have common CRRA preferences, discount rates and beliefs, and a mapping rule which maps our standard household's multiplier into a multiplier for the reference traders such that their consumptions are equal history-state-by-history-state if state prices are the same. All of our aggregation results hold in the parallel economy. Our extended methodology takes advantage of this by using a guess and recover method for the transition rule for the key cross-sectional moment of the reference economy multipliers. We use the regular economy to determine updating rule for their individual multipliers, given the state prices implied by the transition rule for the key moment of the reference trader's mulitplier distribution. We then use these inidividual updating rules to determine the realized individual multipliers in a stochastic panel. We map these multipliers into the multipliers for our reference traders, and then determine the cross-sectional moment and updated transition rule in the parallel economy. The new procedure essentially adds one small step to the original algorithm. However, the set of economies that can be handled includes any for which we can construct a multiplier mapping rule for the reference traders. This turns out to be a very broad set.

Our methodology is well suited to exploring the implications of a rapidly growing literature on household finance which studies the portfolio decisions of households for the macroeconomy. This literature finds that many households do not use asset markets as our standard theory would predict: both the extent of the assets they use and how they utilize the ones they do use differs in important ways (see Guiso and Sodini 2012 for a survey of this literature). First, many households do not utilize all of the available assets. Second, even households which hold equities make very few adjustments in their financial positions. Third, many households who do adjust their portfolios seem to do so in a backward looking manner that leads them to systematically mistime the market. These empirical finds suggest that many households are either completely unresponsive to variations in the pricing of risk or respond in the wrong direction. This pattern of asset usage by households is potentially important since it creates a form of market segmentation which can have wide ranging implications for many aspects of our models' predictions such as household consumption behavior, the distribution of wealth, and, most directly, asset prices.

CCL2011 and CCL2012 used this methodolgy to impose portfolio behavior on passive households in the model to evaluate its impact. We found that having a large number of investors who invest only in low risk / low return portfolios means that other investors have to take on more risk, and in particular aggregate risk. We also found that having a large number of investors who invest in equities very passively, allowing the equity share in their portfolios to rise and fall with excess returns on equity means that the equity investments of other traders who do adjust their equity positions must move in a counter-cyclical manner, rising when the price of risk is high and falling when it is low. We found that the concentration of aggregate risk through equities on a small number of active traders has the potential to generate both a high and a highly volatile market price of risk, which has been a challenge for our asset pricing models (see Lettau and Ludvigson 2010). Also, having a large number of households who do not use assets well can explain their failure to smooth their consumption to the degree that the richness of actual financial markets would allow. Finally, the fact that households are realizing very different returns on their investments can explain the distribution of wealth: both the fact that it is highly skewed relative to income and that equity investment is highly correlated with wealth.

We apply this new methodology to several quantitative experiments. In the first experiment we examine the impact of belief heterogeneity on asset prices and trading behavior. In looking at data on mutual equity mutual fund investments, we find that many investors seem to systematically rely on recent returns to determine their current investments. In our sample this leads them to mistime the market. We conjecture that this investment behavior is motivated by changes in their beliefs about future outcomes due to their heavy reliance on recent events in formulating their forecasts. To examine this conjecture, we include volatilte-belief households (VB) who have a different assessment of the likelihood of a good or bad growth rate realization than the other actively trading households (ST). Our VB households believe that the data generating mechanims may have recently changed and put additional weight on more recent observations. The ST households assume that the world is stationary and hence weight all of the historically observed transition frequencies equally. Disagreements of this sort seem natural in a world in which the stationary of the mechanism generating our data is not obvious. Because the VB households become more (less) optimistic about the likelihood of a good growth shock precisely when the price of risk is rising (falling), their investment behavior is less responsive to the price of risk, forcing the ST households to have large counter-cyclical aggregate risk exposure. This leads the VB traders to miss time the market and substantially increases the counter-cyclical volatility of the market price of risk. This mechanism is complementary to the mechanisms we explored in CCL2011 and CCL2012 because it involves a different set of traders and a similar cyclical concentration of risk for the ST households.

In the second experiment we examine the role of difference in discount rates on the investment behavior of our active traders. We the behavior of the less patient and more patient traders to be surprisingly similar. The less patient active traders ends up essentially mirroring the more patient agents except that they act as if they had a lower wealth target. As a result, the asset pricing implication of our model are not substantially changed by their inclusion, though the predictions for individuals portfolio and savings behavior are.

In the third experiment we show how our methodolgy can be extended to include traders with recursive preferences. This allows us to distinguish between risk aversion (RA) and the intertemporal rate of substitution (IES). We then compare the implications of changing these two preference factors for the asset pricing, portfolio and wealth implications of our model.

One striking finding of our experiments is that households with incorrect beliefs or low discount factors do not become economically negligible. In fact they remain large and important through out. The reason for this is two-fold. First, the presence of idiosyncratic risk and net wealth bounds implies that these households have large precautionary savings motivations at low wealth levels. Second, the low risk-free rate implies that exerts a downward force on household wealth. The combination of these factors leads to an ergodically stable wealth distribution for our economy. Moreover, if the standard active traders even became economically completely dominate, then asset prices would have to reflect this, which would lead to a low price of risk and a high risk-free rate. This in turn would make the other trading technologies more effective and the consequence of incorrect beliefs less substantial. Hence, prices end up adjusting so that these ST households never become dominate.

2 Literature Review-under construction

3 Model

This an infinite horizon endowment economy with a large number of households and a single nonstorable consumption good in each period. Time is discrete, infinite, and indexed by $t = 0, 1, 2, ...$ The first period, $t = 0$, is a planning period in which financial contracting takes place. There are a unit measure of households who come in a finite set of household types indexed by $i \in \{1, ..., I\}$. These types differ in terms of their preferences, beliefs and asset trading technologies. All households have additively separable utility functions over consumption with constant discount rates. The aggregate endowment of this good is stochastic, and households are subject to both aggregate and idiosyncratic income shocks as well as idiosyncratic taste shocks. Asset markets are complete, and both a stock and a risk-free bond are traded. A household's portfolio selection is limited by its trading technology.

3.1 Endowments and Shocks

Our economy is subject to two kinds of shocks. The first is an aggregate growth rate shock $z_t \in Z$. The second is an idiosyncratic household-specific income shock $\eta_t \in N$ in period t, and $\theta_t \in \Theta$ to denote the idiosyncratic taste shock. We will assume that Z and Θ are finite. z^t denotes the history of aggregate shocks and η^t denotes the history of idiosyncratic income. The history state for a household is (z^t, η^t)

The aggregate per capita supply of the endowment good is given by $Y_t(z^t)$, and it evolves according to

$$Y_t(z^t) = \exp\{z_t\} Y_{t-1}(z^{t-1}), \tag{1}$$

with $Y_0(z^0) = 1$. Households' endowment of this good comes in the form of two Lucas fruit trees. The first tree yields *diversifiable income* $(1-\gamma)Y_t(z^t)$. The household is free to trade claims directly on this income and hence it forms the basis for positive net supply financial wealth. The second tree yields non-diversifiable income $\gamma Y_t(z^t)\eta_t$ and it is subject to the household's idiosyncratic income shock. We assume that the household cannot directly trade away its claim to non-diversifiable income and hence its ability to do so will depend whether or not its asset trading technology allows it to trade claims which are conditional on η_t. We will assume that the expected value of η_t is 1 and that the law of large numbers applies. So, these idiosyncratic income shocks will average out in the population and that per capita supply of non-diversifiable income is $\gamma Y_t(z^t)$, and hence γ is the overall share of income that is non-diversifiable. We will discuss taste shocks when we layout the household's preferences.

The idiosyncratic income shocks are i.i.d. across households, and have mean one. For simplicity we will assume that all of the households face the same stochastic process for idiosyncratic shock. We use $\pi(z^t, \eta^t)$ to denote the unconditional probability of state (z^t, η^t) being realized. In a slight abuse of notation we also use $\pi(\eta^t)$ to denote the unconditional probability of personal history η^t. Since we can appeal to a law of large number, $\pi(\eta^t)$ also denotes the fraction of agents in state z^t that have drawn an idiosyncratic history η^t.

The events are first-order Markov, and, continuing our pattern of notation abuse, we assume that the conditional probability of (z^{t+1}, η^{t+1}) is given by

$$\pi(z^{t+1}, \eta^{t+1}|z^t, \eta^t) = \pi(z_{t+1}|z_t)\pi(\eta_{t+1}|\eta_t, z_{t+1}). \tag{2}$$

Thus we allow the distribution of idiosyncratic shocks to depend upon the aggregate shock, but not the reverse.

We introduce some additional notation: $z^{t+1} \succ z^t$ or $\eta^{t+1} \succ \eta^t$ means that the left hand side node is a successor node to the right hand side node. We denote by $\{z^\tau \succ z^t\}$ the set of successor aggregate histories for z^t including those many periods in the future; ditto for $\{\eta^\tau \succ \eta^t\}$. When we use $\succeq$, we include the current nodes z^t or η^t in the set.

3.2 Preferences

A household of type i has a flow utility function $u^i(c)$ where u^i is strictly concave and C^2. Households discount future utility at a constant rate β_i. The measure of type i households is given by μ_i. The households are expected utility maximizers, however, we allow their beliefs to differ from the true probabilities. A household of type i has preferences over stochastic consumption sequences given by

$$\sum_{t\geq 1,(z^t,\eta^t)}^{\infty} (\beta_i)^t\, u^i(c_t)\tilde{\pi}^i(z^t,\eta^t),$$

where $\tilde{\pi}^i(z^t,\eta^t)$ denotes the probabilities that agent i assigns to state (z^t,η^t). We will assume that $\tilde{\pi}^i(z^t,\eta^t)$ is separable between z^t and η^t, as in (2), and that all agents have common beliefs about the agent's personal outcomes η^t given by the actual probabilities $\pi(\eta_{t+1}|\eta_t,z_{t+1})$. However, agents may differ with respect to the aggregate probabilities, with $\tilde{\pi}^i(z_{t+1}|z_t)$ denoting agent i's beliefs.[1]

3.3 Asset Markets

Households trade assets in securities markets that re-open every period. There can be a wide range of assets traded, including one-period Arrow securities, a risk-free bond and a levered equity claim.

We denote the price of a unit claim to the final good in aggregate state z^{t+1} acquired in aggregate state z^t by $Q_t(z_{t+1},z^t)$. We will take the price of a unit claim in state (z^{t+1},η^{t+1}) to be

$$Q\left((z_{t+1},\eta_{t+1}),(z^t,\eta^t)\right) = Q_t(z_{t+1},z^t)\pi(\eta_{t+1}|\eta_t,z_{t+1}). \tag{3}$$

If a trader can hedge his idiosyncratic risk, then this price is implied by arbitrage, while if not it is a innocuous assumption since he will only care about the aggregate state price $Q(z_{t+1},z^t)$.

Starting from aggregate state-contingent Arrow bond prices, we can back out the aggregate present-value prices recursively as follows:

$$\widetilde{P}(z^t) = Q(z_t,z^{t-1})Q(z_{t-1},z^{t-2})\cdots Q(z_1,z^0)Q(z_0).$$

From the present-value price $\widetilde{P}(z^t)$, we can construct the state prices as

$$P(z^t) = \frac{\widetilde{P}(z^t)}{\pi(z^t)}.$$

Consistent with (3), we let $\widetilde{P}(z^t,\eta^t) = \widetilde{P}(z^t)\pi(\eta^t|z^t)$.[2] Finally, we let $m(z^{t+1}|z^t) = P(z^{t+1})/P(z^t)$ denote the stochastic discount factor that prices any random payoffs. We will assume that there is a positive measure of traders who can freely trade aggregate the aggregate state-contingent security to guarantee the uniqueness of the stochastic discount factor.

In addition to state-contingent bonds, we split the aggregate diversifiable income stream, $(1-\gamma)Y_t(z^t)$, into payments on a risk-free debt and an equity claim. For simplicity, the bonds are taken to be one-period risk-free

[1] Without this assumption of common beliefs about idiosyncratic risk, we cannot price this risk in the simple manner assumed in equation (3)

[2] The state price is independent of the realization of the idiosyncratic history.

bonds. Since we assume a constant leverage ratio ψ, the supply of one-period non-contingent bonds $B_t^s(z^t)$ in each period needs to adjust such that:

$$B_t^s = \psi\left[(1-\gamma)V_t[\{Y\}] - B_t^s\right],$$

where $V_t[\{Y\}](z^t)$ denotes the value of a claim to aggregate income in node z^t. The payout to bond holders is given by $R_t^f(z^{t-1})B_{t-1}^s(z^{t-1}) - B_t^s(z^t)$, where $R_t^f(z^{t-1})$ is the risk-free rate between $t-1$ and t in node z^{t-1}. The payments to shareholders, $D_t(z^t)$, are then determined residually as:

$$D_t = (1-\gamma)Y_t - R_t^f(z^{t-1})B_{t-1}^s + B_t^s.$$

In our model, the supply of shares is constant and all equity payouts come exclusively in the form of dividends. We denote the value of the equity claim as $V_t[\{D\}](z^t)$. $R_{t,t-1}[\{D\}](z^t)$ denotes the gross return on the dividend claim between $t-1$ and t. A trader who invests a fraction $\psi/(1+\psi)$ in bonds and the rest in debt is holding the market portfolio. Note that both equities and the risk-free bond are in positive net supply because they add up to a claim to diversifiable wealth.

The equity payout/output ratio is given by the following expression:

$$\frac{D_t}{Y_t} = (1-\gamma)\left(1 + \frac{\psi}{1+\psi}\left[\frac{V_t[\{Y\}]}{Y_t} - (1+R_{t,t-1}[1])\frac{V_{t-1}[\{Y\}]}{Y_{t-1}}\exp\{-z_t\}\right]\right). \quad (4)$$

As can easily be verified, the payout/output ratio is pro-cyclical provided that the price-dividend ratio of a claim to aggregate (or diversifiable) output is. Since our calibrated benchmark model produces procyclical price/dividend ratios, the equity payout/output ratio inherits this property, as in the data.

3.4 Asset Trading Technologies

Our households face different asset trading technologies which take the form of portfolio restrictions and net wealth constraints. Portfolio restrictions fix the relative amounts that the household can invest in certain groups of assets. As a result, a household can only choose how much to invest in such an assets group. Since the relative shares within a group will determine the return on the total amount invested in the group, given the asset returns within the group, each group becomes in effect a single asset from the perspective of a restricted household. The total number of groups available to a household will then determine the extent to which it can allocate financial wealth tomorrow across states of the world.

To ensure that the stochastic discount factor is uniquely determined, at least with respect to the aggregate shock, we will assume that a positive measure of our agents can freely trade an aggregate state-contingent bond. This will imply that the return on any traded can asset can be expressed n terms of the return on an Arrow security due to arbitrage. Hence, we can simply think of all of our households as trading the state-contingent Arrow securities subject to a linear portfolio restriction.

We denote by $b^i\left[(z^t,\eta^t),(z_{t+1},\eta_{t+1})\right]$ the number of state contingent bonds purchased in state (z^t,η^t) and paying off one unit in state $([z^t,z_{t+1}],[\eta^t,\eta_{t+1}])$ by a household of type i. We denote the *vector* of bond positions acquired in state (z^t,η^t) by $b^i_{z_{t+1},\eta_{t+1}}\left[(z^t,\eta^t)\right]$.(We will use this notation whenever we are converting a function into a conditional vector.) A household's trading technology implies a savings choice $\sigma(z^t,\eta^t)$ which has the same dimension as the number of assets groups the trader can choose among. Their savings technology implies a matrix $\mathcal{A}^i(z^t)$ of asset group returns between z^t and $z^{t+1}\succ z^t$, which is of dimension $\#\left[Z\times\Theta\right]\times\#\sigma(z^t,\eta^t)$ (i.e. the dimension of shocks $\times$ the dimension of the number of asset choices). The portfolio restrictions of these restricted households can be expressed as the requirement that

$$b^i_{z_{t+1},\eta_{t+1}}\left[(z^t,\eta^t)\right]=\mathcal{A}^i(z^t)\times\sigma(z^t,\eta^t).^3$$

This linear restriction can encompass a wide range of standard cases.

Active traders have multiple groups and hence have genuine portfolio allocation decisions. Two examples of this class of traders are complete and aggregate-complete traders. *Complete traders* can freely trade Arrow securities, the dimension of $\sigma(z^t,\eta^t)$ is $\#\left[Z\times\Theta\right]$ and $\mathcal{A}^i(z^t)=I$. For *aggregate-complete traders* who can only trade in aggregate state contingent bonds, $\sigma(z^t,\eta^t)$ has dimension $\#Z$ and $\mathcal{A}^i(z^t)$ equals one for those row elements that correspond a particular realization of the aggregate shock z_{t+1}. This leads to a collection of linear restrictions of dimension $\#Z$ for each (z^t,η^t) which have the form

$$b^i\left[(z^t,\eta^t),(z_{t+1},\eta_{t+1})\right]-\sigma(z^{t+1},\eta^t)=0.$$

Passive Traders have only one group so there is no allocation decision, only an overall savings decision, and we will refer to these investors as *passive* since they do not actively manage their portfolios. Two examples of this class of traders are nonparticipants and fixed-portfolio traders. For *nonparticipant traders* who can only trade a risk-free bond, $\sigma(z^t,\eta^t)$ has dimension 1 and $\mathcal{A}^i(z^t)$ is a vector whose elements equal the risk-free rate $(1+r^f(z^t))$. This leads to a series of linear restrictions of dimension 1 for each (z^t,η^t) with the form

$$b^i\left[(z^t,\eta^t),(z_{t+1},\eta_{t+1})\right]-(1+r^f(z^t))\sigma(z^t,\eta^t)=0.$$

For *fixed-portfolio* traders whose restrictions imply that they hold a portfolio with fixed values shares of stocks and bonds, then $\sigma(z^t,\eta^t)$ has dimension 1 and $\mathcal{A}^i(z^t)$ is a vector whose elements equal the state-contingent returns on that portfolio $1+r^p(z^{t+1})$. In this case, the restrictions implied by $\mathcal{A}^i(z^t)$ take the form

$$b^i\left[(z^t,\eta^t),(z_{t+1},\eta_{t+1})\right]-(1+r^p(z^{t+1}))\sigma(z^t,\eta^t)=0.$$

For traders who can choose between a risk free bond and stock portfolio, then $\sigma(z^t,\eta^t)$ has dimension 2, with the first element being, say, investment in stocks and the second purchases of bonds, and $\mathcal{A}^i(z^t)$ is a matrix with two columns where the first column is the stock return $1+r^e(z^{t+1})$ for the corresponding aggregate state and the second column is the risk-free rate $1+r^f(z^t)$.

Our trader's will also face a net-wealth constraint which puts a lower bound on the value of their financial position at (z^t,η^t) of $\underline{D}^i_t(z^t)$. This net-wealth constraint constraint plays an important role in the quantitative analysis since

[3] To be clear, we can also write this expression as

$$b^i\left[(z_{t+1},h_{t+1})(z^t,h^t)\right]=\mathcal{A}^i_{(z_{t+1},h_{t+1})}(z^t)\times\sigma(z^t,h^t),$$

which is the vector of realized returns for each asset in state $\left(z^{t+1},h^{t+1}\right)$ times the investment in each the asset, summed up.

it prevents any type from trading away all of their wealth. Note that since this constraint is on net-wealth it does not directly restricted how negative a trader's bond position can be.

3.5 The Savings Function

We will find it convenient to pose the traders problems in terms of feasible consumption sequences $\left\{c^i(z^t,\eta^t)\right\}$. For that reason, we will define the continuation net savings function. This will allow us to expression our portfolio restrictions and net-wealth restrictions in terms of the continuation consumption allocation. We next define his net savings position and relate that to his feasible net asset values.

Denote the *continuation net savings* out of nontradeable income in state (z^t,η^t) for a trader of type i by $\mathcal{S}^i(z^t,\eta^t)$, where

$$\mathcal{S}^i(z^t,\eta^t)=\sum_{\tau\geq t}\sum_{(z^\tau,\eta^\tau)\succeq(z^t,\eta^t)}\widetilde{P}(z^\tau,\eta^\tau)\left[\gamma Y(z^\tau)\eta_\tau-c^i(z^\tau,\eta^\tau)\right],$$

Since the present-value budget constraint from state (z^t,η^t) onwards is given by

$$\mathcal{S}^i(z^t,\eta^t)+b^i(z^t,\eta^t)\widetilde{P}(z^t,\eta^t)=0.$$

Let $\mathcal{S}^i_{z_{t+1},\eta_{t+1}}(z^t,\eta^t)$ denotes the vector of continuation values conditional on the vector or history states $(z^{t+1},\eta^{t+1})\succ(z^t,\eta^t)$. Then our linear restriction on asset positions can be rewritten in terms of net savings. Or

$$-\mathcal{S}^i_{z_{t+1},\eta_{t+1}}\left[(z^t,\eta^t)\right]=\left[\mathcal{A}^i(z^t)\times\sigma(z^t,\eta^t)\right]\circ\left[\widetilde{P}_{z_{t+1},\eta_{t+1}}(z^t,\eta^t)\right].\text{[4]}\tag{5}$$

This will allow us to reformulate our trader's problems solely in terms of consumption sequences.

3.6 Trader i's Problem

We will exploit the equivalence between the sequential choice problem and the present-value problem to formulate our trader's problem as a time-zero choice problem. Let χ denote the multiplier on the present-value budget constraint. Let $\nu(z^t,\eta^t)$ denote the multiplier on the trader's portfolio restrictions, and let $\nu_{z_{t+1},\eta_{t+1}}(z^t,\eta^t)$ denote the vector on continuation values conditional on the history states $(z^{t+1},\eta^{t+1})\succ(z^t,\eta^t)$. Let $\varphi(z^t,\eta^t)$ denote the multiplier on the debt constraint. With this notation, the saddle point problem of trade i can be stated as:

$$\begin{aligned}
L \;=\; & \max_{\{c^i,\sigma^i\}}\min_{\{\chi,\nu,\varphi\}}\sum_{t=1}^{\infty}(\beta_i)^t\sum_{(z^t,\eta^t)}u^i(c^i(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t)\\
&+\chi^i\left\{\sum_{t\geq1}\sum_{(z^t,\eta^t)}\widetilde{P}(z^t,\eta^t)\left[\gamma Y(z^t)\eta_t-c^i(z^t,\eta^t)\right]+\varpi(z^0)\right\}\\
&+\sum_{t\geq1}\sum_{(z^t,\eta^t)}\nu^i_{z_t,\eta_t}(z^{t-1},\eta^{t-1})\cdot\left\{\begin{array}{c}\mathcal{S}^i_{z_t,\eta_t}(z^{t-1},\eta^{t-1})\\ -\left[\mathcal{A}^i(z^{t-1})\times\sigma^i(z^{t-1},\eta^{t-1})\right]\circ\left[\widetilde{P}_{z_t,\eta_t}(z^{t-1},\eta^{t-1})\right]\end{array}\right\}\\
&+\sum_{t\geq1}\sum_{(z^t,\eta^t)}\varphi^i(z^t,\eta^t)\left\{\underline{D}^i_t(z^t)\widetilde{P}(z^t,\eta^t)-\mathcal{S}^i(z^t,\eta^t)\right\}.
\end{aligned}\tag{6}$$

[4] The symbol "$\circ$" denotes element-by-element multiplication, which will result in each asset position for state (z^{t+1},h^{t+1}) being multiplied by the present value price $\tilde{P}(z^{t+1},h^{t+1})$.

The first constraint is the standard present-value budget constraint. We start each household out symmetrically by endowing them with an claim to diversifiable wealth by adding the value of this claim, $\varpi(z^0)$, to their net savings from non-diversifiable income. There is no consumption in period 0, however households use this period to transform their claims into those consistent with their trading technology as dictated by their measurability and borrowing constraints. The second constraint is the household's measurability constraint and it dictates that each period the households continuation net savings, and hence it's beginning of period net wealth, be consistent with the realized returns in z^t as given by $\mathcal{A}^i(z^{t-1})$, and asset investment choices in the prior period $\sigma^i(z^{t-1}, \eta^{t-1})$. This asset investment choice is chosen optimally, but its only role is to dictate the how net savings, $\mathcal{S}^i_{z_t,\eta_t}(z^{t-1}, \eta^{t-1})$, can vary across realized states. This is because the present value budget constraint already requires that overall consumption spending be consistent with the household's overall wealth. The final constraint is the borrowing limit which caps the extent to which wealth can be transferred across states by forcing the present value of net savings in that state not to be too high, and hence the net asset position too low.

This is a standard convex programming problem –the constraint set is still convex, even with the measurability conditions and the solvency constraints. The first order conditions are necessary and sufficient.

The first-order condition for consumption is given by

$$\beta_i^t u^{i\prime}(c^i(z^t, \eta^t))\tilde{\pi}^i(z^t, \eta^t) - \left\{ \chi^i + \sum_{(z^\tau, \eta^\tau) \preceq (z^t, \eta^t)} \left[\nu^i(z^\tau, \eta^\tau) - \varphi^i(z^\tau, \eta^\tau) \right] \right\} \widetilde{P}(z^t, \eta^t) = 0.$$

This condition is common to all of our traders irrespective of their trading technology because differences in their trading technology does not effect the way in which $c(z^t, \eta^t)$ enters the objective function or the constraint. However, it does vary depending on their utility function and discount rate.

To economize on notation, define the recursive multiplier

$$\zeta^i(z^t, \eta^t) = \chi^i + \sum_{(z^\tau, \eta^\tau) \preceq (z^t, \eta^t)} \left[\nu^i(z^\tau, \eta^\tau) - \varphi^i(z^\tau, \eta^\tau) \right].$$

Note that this implies that $\zeta_0 = \chi$ and that ζ evolves over time as follows for all $t \geq 1$:

$$\zeta^i(z^t, \eta^t) = \zeta^i(z^{t-1}, \eta^{t-1}) + \nu^i\left(z^t, \eta^t\right) - \varphi^i(z^t, \eta^t). \tag{7}$$

Given this notation, we can rewrite this first-order condition for consumption in terms of the state price $P(z^t)$ as

$$\beta_i^t \theta_t u'(c(z^t, \eta^t))\tilde{\pi}^i(z^t, \eta^t) = \zeta^i(z^t, \eta^t) P(z^t) \pi(z^t, \eta^t). \tag{8}$$

The first order condition with respect to $\sigma(z^t, \eta^t)$ is given by:

$$\sum_{(z^{t+1}, \eta^{t+1}) \succ (z^t, \eta^t)} \nu\left(z^{t+1}, \eta^{t+1}\right) R(z^{t+1}) \pi(z^{t+1}, \eta^{t+1}) P(z^{t+1}) = 0, \tag{9}$$

where $R(z^{t+1})$ is the state-contingent return on the asset associated with the particular element of $\sigma(z^t, \eta^t)$. This condition is specific to the trading technology.

The recursive rule (7), the two first order conditions, (8) and (9), the measurability condition w.r.t. net savings, (5), and the borrowing constraint written in terms of net savings, determine the updating rule our recursive multiplier.

3.7 The Reference Consumer, Consumption Shares and Asset Prices

We will construct a reference consumer for each household and a mapping from the household's recursive multiplier to a Negishi-type weight for the reference consumer in a static allocation problem that will lead him to consume the same level of consumption as the household given the state prices. These reference consumers will have identical CRRA utility functions, discount rates and beliefs, and will not be subject to preference shocks (i.e. $\theta_t = 1$). As a result, we will be able to exploit an aggregation result along the lines of CCL (*). To do so, we first express the reference consumer's consumption share in terms of their Negishi weight and a single moment of the overall weight distribution in that aggregate state z^t. Given this, we will be able to derive an expression for the stochastic discount factor that depends only on the aggregate growth rate of output and the updating of this single moment of the weight distribution.

We will now construct a series of state allocation problems where the state contingent price of output is $P(z^t)$ and each agent has CRRA flow utility $\bar{u}(c)$, a discount rate β, common beliefs π, and a social planning weight $1/\bar{\zeta}^i(z^t,\eta^t)$. (In an abuse of language, we will refer to $\bar{\zeta}^i(z^t,\eta^t)$ as the social planning weight even through its really it's inverse.) The static allocation problem is given by

$$\sum_i \left\{ \beta^t \sum_{(z^t,\eta^t)} \frac{1}{\bar{\zeta}^i(z^t,\eta^t)} \bar{u}(\bar{c}(z^t,\eta^t))\pi(z^t,\eta^t) - P(z^t)\bar{c}(z^t,\eta^t) \right\} \mu_i.$$

The first-order condition for consumption is then given by

$$\frac{\beta^t(\bar{c}(z^t,\eta^t))^{-\bar{\alpha}}}{P(z^t)} = \bar{\zeta}(z^t,\eta^t),$$

where $\bar{\alpha}$ is the CRRA factor for the reference trader. Then, note that if we set

$$\bar{\zeta}^i(z^t,\eta^t) : \left(\frac{\bar{\zeta}^i(z^t,\eta^t)P(z^t)}{\beta^t}\right)^{-1/\bar{\alpha}} = u'^{-1}\left(\frac{\zeta^i(z^t,\eta^t)\pi(z^t,\eta^t)P(z^t)}{\beta_i^t \tilde{\pi}^i(z^t,\eta^t)}\right), \tag{10}$$

then both the type i trader and the reference consumer with recursive multiplier $\bar{\zeta}^i(z^t,\eta^t)$ would make the same consumption choice.

Remark 1 *This mapping takes a very simple form in case where $u^i(c^i(z^t,\eta^t))$ is CRRA and $\pi(z^t,\eta^t) = \tilde{\pi}^i(z^t,\eta^t)$. In this case, our mapping rule becomes*

$$\left(\frac{\bar{\zeta}^i(z^t,\eta^t)P(z^t)}{\beta^t}\right)^{-1/\bar{\alpha}} = \left(\frac{\zeta^i(z^t,\eta^t)P(z^t)}{\beta_i^t}\right)^{-1/\alpha_i},$$

where α_i is the CRRA coefficient for type i.

Assume that we constructed the social planning weight for each type of trader as dictated by (10). Then, note that since aggregate consumption is the sum of individual consumptions, it is given by

$$C(z^t) = \sum_i \left\{ \sum_{z^t,\eta^t} \left(\frac{\bar{\zeta}^i(z^t,\eta^t)P(z^t)}{\beta^t}\right)^{-1/\bar{\alpha}} \pi(z^t,\eta^t) \right\} \mu_i.$$

Note next that the consumption share of the type i reference consumer is given by

$$\begin{aligned}
\frac{c^i(z^t,\eta^t)}{C(z^t)} &= \frac{\left(\frac{\bar{\zeta}^i(z^t,\eta^t)P(z^t)}{\beta^t}\right)^{-1/\bar{\alpha}}}{\sum_i \left\{\sum_{z^t,\eta^t}\left(\frac{\bar{\zeta}^i(z^t,\eta^t)P(z^t)}{\beta^t}\right)^{-1/\bar{\alpha}}\pi(z^t,\eta^t)\right\}\mu_i} \\
&= \frac{\bar{\zeta}^i(z^t,\eta^t)^{-1/\bar{\alpha}}}{\sum_i \left\{\sum_{z^t,\eta^t}\bar{\zeta}^i(z^t,\eta^t)^{-1/\bar{\alpha}}\pi(z^t,\eta^t)\right\}\mu_i}.
\end{aligned}$$

Hence, the consumption share of the consumer is determined solely by the ratio of his social planning weight raised to the $-1/\bar{\alpha}$ relative to the average of this ratio in the population. Define this average as

$$\xi(z^t) = \sum_i \left\{\sum_{z^t,\eta^t}\bar{\zeta}^i(z^t,\eta^t)^{-1/\bar{\alpha}}\pi(z^t,\eta^t)\right\}\mu_i. \tag{11}$$

Assume that we had constructed the reference multipliers at all dates and states, and note that for each reference type

$$\frac{\beta\frac{(c^i(z^{t+1},\eta^{t+1}))^{-\bar{\alpha}}}{P(z^{t+1})}}{\frac{(c^i(z^t,\eta^t))^{-\bar{\alpha}}}{P(z^t)}} = \frac{\bar{\zeta}^i(z^{t+1},\eta^{t+1})}{\bar{\zeta}^i(z^t,\eta^t)}.$$

If we substitute in for consumption using the prior result on consumption shares, we get that

$$\frac{\beta\frac{\left(\frac{\bar{\zeta}^i(z^{t+1},\eta^{t+1})^{-1/\alpha}}{\xi(z^{t+1})}C(z^{t+1})\right)^{-\bar{\alpha}}}{P(z^t)}}{\frac{\left(\frac{\bar{\zeta}^i(z^t,\eta^t)^{-1/\alpha}}{\xi(z^t)}C(z^t)\right)^{-\bar{\alpha}}}{P(z^t)}} = \frac{\bar{\zeta}^i(z^{t+1},\eta^{t+1})}{\bar{\zeta}^i(z^t,\eta^t)},$$

which simplifies to

$$\frac{P(z^{t+1})}{P(z^t)} = \beta\left(\frac{\xi(z^{t+1})}{\xi(z^t)}\right)^{\bar{\alpha}}\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\bar{\alpha}}. \tag{12}$$

This leads to the following proposition.

Proposition 2 *Relative state prices in this economy are determined by the growth rate of aggregate consumption and the $-1/\bar{\alpha}$ population average of the reference social planning weight.*

This result implies that we can think of the equilibrium of our economy in terms of a simple fixed point. If $\Xi(z^{t+1}) = \xi(z^{t+1})/\xi(z^{t+1})$ is the updating rule for the $-1/\bar{\alpha}$ population average of the reference recursive multiplier, then prices are given by (12) and, when we construct the updating rule for $\zeta^i(z^t,\eta^t)$ and thereby for $\bar{\zeta}^i(z^t,\eta^t)$, the resulting values of

$$\sum_i \left\{\sum_{z^t,\eta^t}\bar{\zeta}^i(z^t,\eta^t)^{-1/\alpha}\pi(z^t,\eta^t)\right\}\mu_i$$

must satisfy (11)

3.8 Computational Algorithm

Here we discuss how to construct a computational algorithm to compute an equilibrium of our model. To keep things simple, we will assume preferences are CRRA (through they may different in terms of their degree of risk aversion). Note that agent's beliefs only differ with respect to the aggregate state's evolution.

The algorithm has two major steps. First, a low of motion for the updating of the multiplier moment, $\Xi(z^{t+1}) = \xi(z^{t+1})/\xi(z^{t+1})$, is posited. Note that this updating rule is w.r.t. the cross-sectional moment of the reference traders. Given this updating rule, we can determine relative state prices $Q(z^{t+1}) = P(z^{t+1})/P(z^t)$ using (12). Second, we solve for the household's savings function and multiplier updating rule for the standard traders. Third, we map the sequences of household multipliers into the implied sequences of reference trader's multipliers and compute the implied updating rule for their multiplier moment. Fourth we compare our original guess of $\Xi(z^{t+1})$ to its realized value: if they are the same, we have an equilibrium and if not we need to iterate on $\Xi(z^{t+1})$ in some fashion.

To describe step 2 in more detail note that we can recursively represent the savings functions as

$$\mathcal{S}^i(\zeta, \eta_t, z^t) = \left[\gamma\eta_t Y(z^t) - c^i(\zeta, z^t)\right] + \beta \sum_{\substack{z^{t+1}|z^t \\ \eta_{t+1}}} \mathcal{S}^i(\zeta_{t+1}\left(z_{t+1}, \eta_{t+1}\right), \eta_{t+1}, z^{t+1}) Q(z^{t+1}) \pi\left(z^{t+1}|z^t\right),$$

where

$$\beta_i^t u'(c^i(\zeta, z^t)) \frac{\tilde{\pi}^i(z^t, \eta^t)}{\pi(z^t, \eta^t)} = \zeta P(z^0) \left[\prod_{z^\tau \preceq z^t} Q(z^\tau)\right].$$

To close the system we need to specify how the household's recursive multiplier ζ is updated. In addition, there is an initial degree of freedom in the choice of $P(z^0)$ which is chosen to clear the first period consumption market.

In addition, updating of the individual's multiplier $\zeta_{t+1}^i\left(z_{t+1}, \eta_{t+1}\right)$,

$$\zeta_{t+1}^i(z_{t+1}, \eta_{t+1}) = \zeta_t^i + \nu^i\left(z_{t+1}, \eta_{t+1}\right) - \varphi^i(z_{t+1}, \eta_{t+1})$$

must be consistent with our (i) saving's measurability condition (5), (ii) debt bound, and (iii) optimality condition (9). To satisfy (i) and (ii), there must exist $\sigma^i(z^t, \eta^t)$

$$-\mathcal{S}^i_{z_{t+1},\eta_{t+1}}(\zeta_{t+1}\left(z_{t+1}, \eta_{t+1}\right), z^t) = \left[\mathcal{A}^i(z^t) \times \sigma^i(z^t, \eta^t)\right] \circ \left[\widetilde{P}_{z_{t+1},\eta_{t+1}}(z^t, \eta^t)\right],$$

(where $\mathcal{S}^i_{z_{t+1},\eta_{t+1}}(\zeta^i_{t+1}\left(z_{t+1}, \eta_{t+1}\right), z^t)$ is a conditional vector and vector and $\zeta^i_{t+1}\left(z_{t+1}, \eta_{t+1}\right)$ is understood to vary with this conditioning) and

$$\underline{D}^i_{t+1}(z^{t+1})\widetilde{P}(z^{t+1}, \eta^{t+1}) - \mathcal{S}^i(\zeta^i_{t+1}\left(z_{t+1}, \eta_{t+1}\right), \eta_{t+1}, z^{t+1}) \geq 0.$$

To satisfy (iii) it must be the case that

$$\sum_{(z^{t+1},\eta^{t+1}) \succ (z^t,\eta^t)} \left[\zeta^i_{t+1}\left(z_{t+1}, \eta_{t+1}\right) - \zeta^i_t\right] R(z^{t+1}) \pi(z^{t+1}, \eta^{t+1}) P(z^{t+1}) = 0$$

if the debt bound does not bind and

$$\zeta^i_{t+1}\left(z_{t+1}, \eta_{t+1}\right) = \zeta^i_t + \nu^i\left(z^{t+1}, \eta^{t+1}\right).$$

Whereas, if the debt bound binds, then $\zeta^i_{t+1}(z_{t+1}, \eta_{t+1})$ must be such as to satisfy the bound and the debt bound multiplier $\varphi^i(z^{t+1}, \eta^{t+1})$ and $\nu^i\left(z^{t+1}, \eta^{t+1}\right)$ are chosen so that the updating rule and (9) are jointly satisfied. The outcome of step 2 is an operator $T^i(\zeta_t, \eta_t, z^t) = \zeta_{t+1}(z_{t+1}, \eta_{t+1}, \zeta)$ which given the law of motion for the recursive multiplier ζ for a household of type i.

In step 3 we compute the implied sequence of multipliers $\left\{\zeta^i_t(z^t, \eta^t)\right\}_{t=0}^{\infty}$ for each of our types. We then use our mapping rule (10) to map these sequences into sequences of multipliers for our reference traders $\left\{\bar{\zeta}^i_t(z^t, \eta^t)\right\}_{t=0}^{\infty}$. Then we can compute the implied values of the key moment of the distribution in each state (z^t, η^t) using (11), and thereby complete step 3 by computing the implied value of $\Xi(z^{t+1}) = \xi(z^{t+1})/\xi(z^{t+1})$.

This computation is made much easier by first stationarizing the economy, and second making a key assumption. The law of motion $\Xi(z^{t+1})$ depends in principal upon the either history z^{t+1}. However, we have found that a sufficiently long but finite history allows one to compute the equilibrium with a very high degree of accuracy. Let j denote the length of the utilized history and let $\mathbf{z} \in Z^j$ denote an element of these j-period histories. Then the moment updating and relative state prices - $\Xi(\mathbf{z}, \mathbf{z}')$ and $Q(\mathbf{z}, \mathbf{z}')$ - are simply functions of the history transition. In addition, the saving function $\mathcal{S}^i(\zeta, \eta, \mathbf{z})$ and multiplier updating rule $T^i(\zeta, \eta, \mathbf{z})$ are functions of the aggregate history state, the current idiosyncratic shock and the recursive multiplier ζ. Hence, computing them amounts to a simple fixed point operation.

Given all this, our algorithm works by first positing $\Xi^0(\mathbf{z}, \mathbf{z}')$ for all $\mathbf{z}, \mathbf{z}' \in Z^j$ and $\Pr(\mathbf{z}'|\mathbf{z}) > 0$ (i.e. positive probability history transitions). Then it works as follows:

1. Draw a long sample of aggregate shocks $\{z_t\}$ and a corresponding large panel of idiosyncratic shocks $\left\{\eta^{ij}_t\right\}$ where i corresponds to the trader's type and j distinguishes among traders of a given type. The proportions of trader types is determined by μ_i. Construct the implied history state sequence from $\{\mathbf{z}_t\}$ from $\{z_t\}$.

2. Construct the sequence of relative prices $Q^0(\mathbf{z}, \mathbf{z}')$ for each positive probability finite history transition using $\Xi^0(\mathbf{z}, \mathbf{z}')$ and the pricing rule (12).

3. For each type i, use the stationary first-order conditions and savings function $\mathcal{S}^i(\zeta, \mathbf{z}, \eta_t, \theta_t)$, along with the budget constraint and measurability conditions to construct the transition rule for their recursive multiplier $T : \zeta' = T^i(\mathbf{z}, \mathbf{z}', \eta, \zeta)$.

   - With recursive preferences both the recursive savings function and the appropriate ratio of future utility to expect future utility must be constructed. $M(\mathbf{z}, \mathbf{z}', \eta', \zeta')$.

4. Use $T^i(\mathbf{z}, \mathbf{z}', \eta, \zeta)$ and (10) to construct the implies sequence of reference recursive multiplier $\left\{\bar{\zeta}^{ij}_t\right\}$ for panel. Note that at point we normalize these reference multipliers so that the consumption shares average one, and the ratio of the actual consumption share average and one is a natural measure of the allocation error (and hence of the accuracy of our approximation).

5. Construct the panel of implied reference recursive multiplier and use them to estimate $\Xi^{n+1}(\mathbf{z}, \mathbf{z}')$. If $\mathbf{z}$ takes on a small set of values this can be done using simple conditional means.

6. If the accuracy is high and $\left\|\Xi^{n+1}(\mathbf{z},\mathbf{z}') - \Xi^{n}(\mathbf{z},\mathbf{z}')\right\|$ is small stop. Otherwise return to step 2 with a new guess for Ξ.

3.9 Calibration of Economy

The baseline model is calibrated to annual data. We set the value of the coefficient of relative risk aversion α to five and the time discount factor β to .95. These preference parameters allow us to match the collateralizable wealth to income ratio in the data when the tradeable or collateralizable income share $1-\gamma$ is 10%, as discussed below. The average ratio of household wealth to aggregate income in the US is 4.30 between 1950 and 2005. The wealth measure is total net wealth of households and non-profit organizations (Flow of Funds Tables). With a 10% collateralizable income share, the implied ratio of wealth to consumption is 5.28 in the model's benchmark calibration.[5]

Our benchmark model is calibrated to match the aggregate consumption growth moments from AlvarezJ01. The average consumption growth rate is 1.8% and the standard deviation is 3.16%. These are the moments of U.S. per capita aggregate consumption between 1889-1978 used in MP85's seminal paper. Recessions are less frequent than expansions: 27% of realizations are low aggregate consumption growth states. The first-order autocorrelation coefficient of aggregate consumption growth (ρ_z) is set to zero so all propagation is internally generated. Consequently, aggregate consumption growth is identically and independently distributed over time: $\pi(z'|z) = \pi(z')$. The elements of the discretized process for z are $\{0.9602, 1.0402\}$.

We calibrate the labor income process as in STY06, except that we eliminate the counter-cyclical variation (CCV) of labor income risk. Hence, the variance of labor income risk is constant in our model. The Markov process for the log of the labor income share $\log\eta$ has a standard deviation of 0.71, and its autocorrelation is 0.89. We use a 4-state discretization for both aggregate and idiosyncratic risk. The elements of the discretized process for η are $\{0.3894, 1.6106\}$.

Equity in our model is simply a leveraged claim to diversifiable income. In the Flow of Funds, the ratio of corporate debt-to-net worth is around 0.65, suggesting a leverage parameter ψ of 2. However, lam90 report that the standard deviation of the growth rate of dividends is at least 3.6 times that of aggregate consumption, suggesting that the appropriate leverage level is over 3. Following abel:99 and bansal-yaron:04, we choose to set the leverage parameter ψ to 3. We set the debt bound at zero meaning that households cannot borrow against their idiosyncratic income.

We consider three types of trading technologies that can be easily ranked in terms of their sophistication. The first and most sophisticated are *aggregate-complete* active traders who can trade Arrow bonds conditional on aggregate growth shocks as well as equity and risk-free debt claims. The second and least sophisticated are *nonparticipants* - i.e. passive traders who hold only risk-free bonds. The third technology is that of the *disversified* traders who hold debt and equity claims in the same fixed proportion as the exist in the market. These traders are following

[5] As is standard in this literature, we compare the ratio of total outside wealth to aggregate non-durable consumption in our endowment economy to the ratio of total tradeable wealth to aggregate income in the data. Aggregate income exceeds aggregate non-durable consumption because of durable consumption and investment.

the standard advice of holding the market and will earn a higher return than the nonparticipants to the extent that the economy exhibits and equity premium, which it will. That is why we rank the sophistication of their trading technology in the middle between the other two.

In the most recent Survey of Consumer Finances, 51.1% of households reported owning stocks directly or indirectly. Therefore, the fraction of passive traders with zero equity holding (non-participants) is calibrated to 50%. In order to deliver a large equity premium, a small fraction of active traders need to bear the residual aggregate risk created by non-participant. We set the share of active traders equal to 10%. We set the share of passive traders who hold both equities and debt to 40%, and had them hold these claims in the same proportion as the market. Note here that these shares represent fractions of total human wealth (i.e. nondiversifiable wealth), not financial wealth, owned by each trader type.[6]

3.10 Baseline Results

We begin by computing the equilibrium outcomes for our baseline economy. In this benchmark all of our traders have common preferences CRRA preferences and beliefs, and households differ only with respect to their asset trading technologies. The average value of the market price of risk (MPR) of 0.40 and the risk-free rate of 1.9% are very close to what financial economists estimate as being in the data. The low volatility of the risk-free also accords well with the data. Figure 3, shows a plot of the Sharpe ratio for our equity asset for a fix interval from the simulation panel, indicates that the Sharpe ratio (and the MPR) are counter-cyclical, as they are in the data. However, the volatility of the MPR is much lower than the is commonly estimated.

As one would expect, the active traders have the most sophisticated trading technology and earn the highest return on their investments and have an excess return of 4.8%. The diversified traders have an excess return of 1.5%, while the nonparticipants, with the least sophisticated trading technology by construction have an excess return of 0. The active traders have a much more volatile excess return - the standard deviation of their excess return is 11.8 while that of the diversified traders is 3.8.

[6]Because of the homogeneity of the investor's optimization problem, we can reallocate human wealth within each trader pool without affecting equilibrium asset prices.

Table X: Baseline Results

$\frac{\sigma(m)}{E(m)}$	$Std(\frac{\sigma_t(m)}{E_t(m)})$	$E(R_f)$	$\sigma(R_f)$
0.40	2.77	1.93	0.06
$E(R_z^W - R_f)$	$E(R_{div}^W - R_f)$	$\frac{E(R_z^W - R_f)}{\sigma(R_z^W - R_f)}$	$\frac{E(R_z^W - R_f)}{\sigma(R_z^W - R_f)}$
4.8	1.5	0.41	0.41
$E\left(W_z/W\right)$	$\sigma\left(W_z/W\right)$	$E\left(\omega_z\right)$	$\sigma\left(\omega_z\right)$
2.15	0.57	0.80	0.11
$corr\left(\omega_z, SR\right)$	$\frac{\sigma(\triangle\log(c_z))}{\sigma(\triangle\log(C))}$	$\sigma\left(\triangle\log\left(C_z\right)\right)$	%error
0.93	2.99	6.99	0.38
$E(W_{div}/W)$	$\sigma(W_{div}/W)$	$E(W_{np}/W)$	$\sigma(W_{np}/W)$
1.03	0.03	0.84	0.10
$\frac{\sigma(\triangle\log(c_{div}))}{\sigma(\triangle\log(C))}$	$\sigma\left(\triangle\log\left(C_{div}\right)\right)$	$\frac{\sigma(\triangle\log(c_{np}))}{\sigma(\triangle\log(C))}$	$\sigma\left(\triangle\log\left(C_{np}\right)\right)$
3.38	3.63	3.59	2.55

The active traders achieve their high and volatile returns by heavily investing in risk assets, with equities making up 80% of their portfolios on average. Moreover, their equity exposure is very tightly connected to cyclical fluctuations in the pricing of risk, and as a result the correlation of their equity share and the Sharpe ratio on equities is roughly 1. The diversified traders hold the market and hence earn the risk-return trade-off offered by the market. Both the active and diversified traders have an average Sharpe ratio on their investments of 0.41. For the diversified traders they hold the portfolio offered by the market and simply earn the implied return. In contrast, the active traders are loading up on aggregate risk when expected excess returns are high, but since this is also when the variance of excess returns are high, these two factors offset and they end up achieving the same risk-return trade-off as the diversified traders.

The differences in investment returns impacts on the wealth distribution. The active traders have more than twice as much wealth per capita as the average individual. The diversified traders average wealth level is equal to average per capita wealth, while the nonparticipants on average have only 84% of per capita wealth. We report the standard deviation of the group wealth ratio because this averages out idiosyncratic risk. It is highest for the active traders as a group indicating that they have the most exposure to aggregate risk. Both the diversified and the nonparticipants have much lower wealth standard deviation; 0.03 and 0.10 respectively.

In terms of consumption, the agents with the most sophisticated trading technology have the lowest standard deviation of the consumption growth , while those with the least sophisticated technology have the highest. The ratio of the standard deviation of individual consumption growth to aggregate consumption growth is 3.0 for the active traders, 3.4 for the diversified traders and 3.6 for the nonparticipants respectively. However, when look at the volatility of average consumption for these groups, the results are reversed. This is because the averaging removes the impact idiosyncratic income shocks and thereby reveals that the agents with the more sophisticated trading technology are using this technology to take on more aggregate risk.

Figure 4 shows several plots of the dynamics of our economy for a fixed 100 period simulation interval. Low aggregate growth rate realizations are in the grey shaded panes. The first pane illustrates the cyclical volatility of the Sharpe ratio on equities and the high degree to which it is countercyclical, as it is in the data. The second and third panels in figure 4 show plots of the average equity share and the average wealth-to-per-capita wealth for the active traders for our fixed simulation interval. It indicates that both the average equity exposure is counter-cyclical while the wealth ratio is procyclical.

These plots indicate the fundamental dynamics that are driving asset pricing in our baseline economy. The active traders absorb the aggregate risk created by having investors who hold low risk assets. They do this by holding large amounts of the equity claims and low amounts of risk-free debt. At certain points their average holdings are greater than 100%, indicating that they are shorting the risk-free security in order to meet its demand. At the same time, high realizations of the aggregate growth rate shock lead them to become relative rich, while low realizations lead to the reverse. When they are relatively poor, the active traders are induced to hold large amounts of equity by a high Sharpe ratio. These fluctuations in the relative wealth of the active traders and the resulting price they demand in order to bear the amount of aggregate risk being pushed at them by the market leads to the aggregate dynamics in the model. If all of our traders were either active traders or passive traders who held the market, then their would be no dynamics in the startionarized economy. As a result, each group would simply "hold the market" in their portfolios and have a constant and identical consumption share.

These outcomes we report are very similar to results reported in our earlier work; in particular CCL (2011), where we discuss them in greater detail.

4 Volatile Aggregate Belief Traders

4.1 Definition

A natural issue with respect to forming beliefs about the aggregates shock is how much to rely on the full time series history. This is because it is unclear how strong a stationarity assumption to impose and statistically it can be hard to determine whether and when a structural break in a time series has occurred. In forecasting models it is common practice to allow for the possibility of structural breaks by (i) allowing for a moving window in the estimation, (ii) allowing explicitly for parameter drift, or (iii) down weighting older observations (See Clements and Hendrey "Forecasting with Breaks", Handbook of Forecasting 2006). Procedures such as these will lead to parameter estimates that are more responsive to recent data than to older data, and suggest that this greater responsiveness can have a rational basis. We therefore allow for some traders who put more weight recent data, through the following beliefs.

Definition 3 *A trader with volatile beliefs believes with probability κ the current aggregate transition $z_t \to z_{t+1}$ is governed by $\pi\left(z_{t+1}|z^t\right)$ and with probability $1-\kappa$ it is governed by a new Markov transition matrix about which he has a uninformative prior. Conditional on a shift in the transition matrix occurring, his best estimate of the new* *transition matrix would be the empirically observed transition frequencies since the assumed potential shift date $t-j$. Let $\Pi\left(\cdot|\cdot; z_{t-j}^{t}\right)$ denote these conditional transition frequencies for the history z_{t-j}^{t}. Then, the trader's conditional probability distribution over z_{t+1} would be given by*

$$\tilde{\pi}\left(z_{t+1}|z^{t}\right) = \kappa\pi\left(z_{t+1}|z^{t}\right) + (1-\kappa)\Pi\left(z_{t+1}|z_{t}; z_{t-j}^{t}\right).$$ [7]

The volatility of beliefs is governed by the parameter κ. With $\kappa = 1$, these traders rely solely on the ergodic transition probabilities and have stable beliefs which do not vary over time. With $\kappa = 0$, they are using a rolling window which completely ignores older data. While with $\kappa \in (0,1)$, traders with volatile beliefs down weight older data through the use of two tiers - older than j years and j years or less.

4.2 Volatile Belief Experiment

The first experiment involves giving half of our active traders slightly volatile beliefs. We set the belief parameter to $\kappa = 0.75$, which implies that 75% of the probability mass is placed upon the ergodic transition matrix, while the remaining mass is placed upon the observed transition frequencies during the past 5 years. The results are reported in the second column of table 1 and in figure 5.

Table XX: Volatile Belief Results $\kappa = 0.75$

(z^* denotes nonstandard active traders)

$\frac{\sigma(m)}{E(m)}$	$Std(\frac{\sigma_t(m)}{E_t(m)})$	$E(R_f)$	$\sigma(R_f)$
0.41	8.56	2.03	0.04
$E(R_z^W - R_f)$	$E(R_{z^*}^W - R_f)$	$\frac{E(R_z^W - R_f)}{\sigma(R_z^W - R_f)}$	$\frac{E(R_{z^*}^W - R_f)}{\sigma(R_{z^*}^W - R_f)}$
5.1	3.3	0.39	0.35
$E\left(W_z/W\right)$	$\sigma\left(W_z/W\right)$	$E\left(\omega_z\right)$	$\sigma\left(\omega_z\right)$
2.40	0.54	0.90	0.26
$corr\left(\omega_z, SR\right)$	$\frac{\sigma(\triangle \log(c_z))}{\sigma(\triangle \log(C))}$	$\sigma\left(\triangle \log\left(C_z\right)\right)$	%error
0.98	2.90	7.15	
$E\left(W_{z^*}/W\right)$	$\sigma\left(W_{z^*}/W\right)$	$E\left(\omega_{z^*}\right)$	$\sigma\left(\omega_{z^*}\right)$
1.88	0.48	0.66	0.14
$corr\left(\omega_{z^*}, SR\right)$	$\frac{\sigma(\triangle \log(c_{z^*}))}{\sigma(\triangle \log(C))}$	$\sigma\left(\triangle \log\left(C_{z^*}\right)\right)$	
-0.97	3.07	6.95	

From table 1, one can see that the average MPR and the average risk-free rate are virtually unchanged. However, the volatility of the MPR has increased by factor of almost 3 at the cost of a fairly modest increase in the risk-free rate. The second panel in figure 5 displays the Sharpe ratio and one can see by inspection that its fluctuations are negatively related to the aggregate growth rate shocks. Hence, the presence of traders with volatile beliefs can lead to more of the kind of cyclical volatility in asset prices that we see in the data.

The increased volatility in asset prices lowers the excess returns of our traders. The average excess return of the standard active traders falls to 5.1%, while that of the VB traders is 3.3% and the diversified traders is 1.3%. The lower excess return for the VB traders is coming in part from the fact that their average equity share is much lower (0.66 vs. 0.90) and more stable (a standard deviation of 0.14 vs. 0.26), and the correlation of their equity share with the Sharpe ratio is the mirror image of the normal active traders: -1 vs. +1. Thus the VB traders are systematically miss timing the market and this also depresses the excess return on their portfolio. At the same time, the standard deviation of their excess return is also substantially lower than that of the standard active traders: 9.3% vs. 13.1%. However, the fact that their equity share is more stable means that the extent to which they move their equity share the wrong is not that large. This is born out by the Sharpe ratio on their portfolio return. It is 0.35, which is only slightly lower than the 0.39 and 0.38 values for the standard active and diversified traders respectively.

The return differences lead to differences in wealth accumulation and consumption for out two types of active traders. The average ratio of the per capita wealth of our standard active traders to overall per capita wealth is 2.4 while that of the VB traders is 1.9, while the standard deviation of this ratio is 0.45 and 0.48 respectively. Interestingly, the standard deviations of consumption growth are both very similar which indicates similar degree of overall consumption smoothing. In addition, the volatility of the ratio of group average consumption to per capita consumption are also very similar, which indicates that there exposure to aggregate risk is also essentially the same.

To better understand the dynamics in this first experiment, we display some key series from the same fixed interval in the simulation panel in figure 5. The first panel shows the probability that the two types of active traders assign to the event that the next period's aggregate growth rate shock is high. Because the we assumed that the growth rate shocks are i.i.d., this probability is constant for our normal active traders, but procyclical for our VB active traders. However, the fluctuations are fairly modest, staying between 0.80 and 0.60 and often near the 0.72 ergodic probability. The second panel shows the Sharpe ratio, and one can see from inspection that the VB trader's probability and the Sharpe ratio are strongly negatively correlated. This negative correlation means that the pricing of risk and the likelihood of a good outcome are moving in opposite directions for these traders which simultaneously explains the low and stable equity shares we see for them relative to the normal active traders in the third panel. It is interesting to note that the VB traders appear to exhibit momentum trading: their equity share rises after a series of good shocks/returns, and falls when the reverse is true. The final panel shows the how the per capita wealth levels for these two types of traders (relative to the average per capita wealth) evolves. It is easy to see that the VB traders have lower and smoother wealth levels from this plot.

In the appendix we report on an additional experiment in which we set the belief parameter to $\kappa = 0.50$, which makes the beliefs of the VB traders even more volatile. In this case, the volatility of the MPR goes up even further to 12.9, while the other asset pricing facts remain largely unchanged. The portfolio behavior of our active traders becomes more polar, with the average equity share of the ST traders climbing to 0.95 and that of the VB traders falling to 0.35. In addition, belief swings of the VB traders becomes more severe, as does the "momentum trading" aspect of their equity investments. As a result, the excess return on their portfolio falls to 1.25, as opposed to 5.57 for the ST traders and 1.27 for the diversified traders. The Sharpe ratio on their portfolio returns falls to 0.15, as opposed to 0.39 for the ST traders. Overall, this indicate that their investment behavior is now leading to substantial return loses.

4.2.1 Data Discussion

The aggregate data coming from mutual funds suggest that not only do many investors not increase their exposure to equities when the price of risk rises, but rather that they tend to go the other direction, over responding to the recent history of past returns.

During the stock market rally from 1990 to 1998, the share of U.S. equity mutual funds in total assets of the mutual fund industry increased from 23% to 62%. Between 1998 and 2002, after the end of this rally, the equity share dropped to 43%, only to recover and reach 60% in 2006. Between 2007 and 2008, the share dropped again to 40%. (Source: http://www.ici.org/pdf/2011$_factbook.pdf ICIFactbook, Table4, year-endtotalnetassetsbyinvestmentclassification).Broadlys$

We next want to examine what longer term data show about the risk exposure of mutual fund traders and how it responds to recent events. To do so, we use the aggregate mutual fund data from ICI and, following Dichev and Yu (2012), the definition of asset weighted return, r_{dw}, given a sample period T as

$$\frac{A_T}{(1+r_{dw})^T} = A_0 + \sum_{t=1}^{T} \frac{CF_t}{(1+r_{dw})^t} \tag{13}$$

where A_t denotes the period t asset holding and CF_t for the flow of asset derived form the following equation:

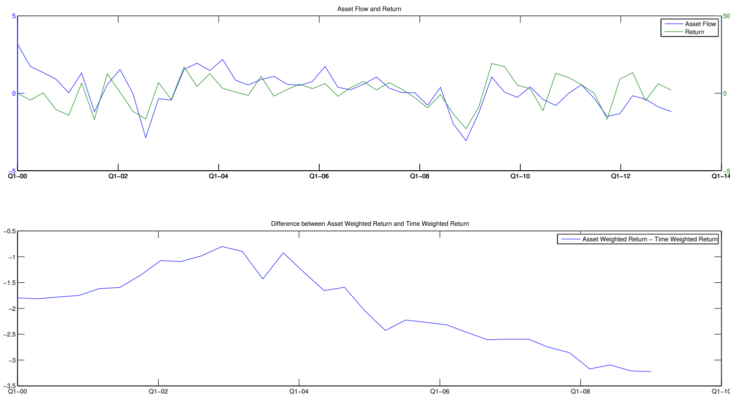
$$CF_t = A_t - A_{t-1}(1+r_t). \tag{14}$$

The r_t is the return of asset in period t. The time average return is the average of r_t given the same sample period T. If an investor do not change her asset position during the sample period T, meaning the flow is zero for all periods. Then asset weighted return is the same as time weighted return. Therefore, the asset weight return defined in equation (13) considers the impact to returns from the time-varying asset positions. If investors time the market correctly, their asset weighted return should outperform the time weighted return, otherwise their weighted return should be lower than those of buy and hold investors.

We present our results in Figure 1. The first panel shows the mutual fund returns we inferred from looking at aggregate mutual fund balances net of inflows between Q1 2000 to Q4 2012, along with the net inflows. The inflows are generally positively correlated with the most recent realized return with a positive correlation 0.49. Moreover, the size of the variation in flows also moves positively w.r.t. to the magnitude of the changes in returns. The outflow of equity holding is especially large whenever a very poor return occurs. Clearly, these mutual fund investors miss the market timing: tend to sell low and buy high. How costly to these investor by the miss timing behavior? In order to measure the cost, we compare the asset weighted return to time weighted return in our sample period. The second panel of Figure 1 plots the annual return difference between time weighted return and asset weighted return over a five year rolling windows period over our entire sample. Namely, each point of the figure represents the return difference between time weighted and asset weighted returns in a five year period. Clearly, the asset weighted return is dominated by time weight return over our entire sample by 1% to 2.5% and with sample average 1.98%. This

Figure 1: Comparing Asset and Time Weighted Mutual Fund Returns

Asset and Time Weighted Returns from ICI Mutual Fund data Q1 2000 to Q4 2012, using a 5 year rolling window.



number represent the average cost to mutual fund investors who adjust their asset position in the wrong direction compared to the buy and hold investors. Our findings are consist with several reports from the Morningstar, a major mutual fund and investment finance company: Morningstar estimated that investors lost 1.46% of their returns by examining the gap between asset weighted (which they called dollar-weighted) and time weighted returns.

5 Reduced Patience Traders

This quantitative experiment involves lowering the discount rate for half of our active traders from 0.95 to 0.925, thereby making them less patient (LP) than the normal active traders. What is surprising about this experiment is how little impact this change had on the outcomes either at the aggregate or the individual level.

At the aggregate level asset prices were virtually unchanged relative to the baseline model. At the individual level the two types of active traders had very similar average equity shares, similar volatility in their equity shares and a similar correlation of their equity share and the Sharpe ratio. The LP traders had a lower and less volatile per capita wealth, and their consumption growth rate was also more volatile. However, the volatility of the average growth rate of consumption for the two groups was very similar, reflecting the fact that they had similar exposure to aggregate risk. The similarity in the outcomes of the two types of traders and the level differences in wealth can be seen clearly in the second and third panels of figure 6.

To understand what is going on here, note the following. The fact that both trader's have similar exposure to idiosyncratic risk and the same borrowing constraint leads to similar demands for precautionary saving. However differences in patience imply differences in the extent to which the lower average MPR leads to a downward drift in their per capita wealth. As a result, the point at which the two forces offset is lower for the LP traders and hence their average wealth level is also lower.

Table XXX: Reduced Patience Results $\beta = 0.925$

(z^* denotes nonstandard active traders)

$\frac{\sigma(m)}{E(m)}$	$Std(\frac{\sigma_t(m)}{E_t(m)})$	$E(R_f)$	$\sigma(R_f)$
0.42	2.79	1.98	0.08
$E(R_z^W - R_f)$	$E(R_{z^*}^W - R_f)$	$\frac{E(R_z^W - R_f)}{\sigma(R_z^W - R_f)}$	$\frac{E(R_{z^*}^W - R_f)}{\sigma(R_{z^*}^W - R_f)}$
4.85	5.14	0.39	0.39
$E\left(W_z/W\right)$	$\sigma\left(W_z/W\right)$	$E\left(\omega_z\right)$	$\sigma\left(\omega_z\right)$
2.38	0.67	0.82	0.11
$corr\left(\omega_z, SR\right)$	$\frac{\sigma(\triangle \log(c_z))}{\sigma(\triangle \log(C))}$	$\sigma\left(\triangle \log\left(C_z\right)\right)$	%error
0.92	3.18	5.14	
$E\left(W_{z^*}/W\right)$	$\sigma\left(W_{z^*}/W\right)$	$E\left(\omega_{z^*}\right)$	$\sigma\left(\omega_{z^*}\right)$
1.66	0.40	0.87	0.12
$corr\left(\omega_{z^*}, SR\right)$	$\frac{\sigma(\triangle \log(c_{z^*}))}{\sigma(\triangle \log(C))}$	$\sigma\left(\triangle \log\left(C_{z^*}\right)\right)$	
0.94	6.83	11.71	

6 Lowering Risk Aversion v.s. IES

In this section, we indicate how our methodology can be extended to allow for recursive preferences which distinguish between the degree of risk aversion (RA) and the intertemporal elasticity of substitution (IES). We then use this extenstion to examine the implications of these variations for our economy.

6.1 Recursive Preferences

The methodology that we have developed can be extended to allow for households with different recursive preferences. To demonstrate this, assume that a household of type i has recursive preferences along the lines of Epstein-Zin or Kreps-Porteous, and these preferences are given by

$$V_t = \left[(1-\beta)c_t^{1-\rho} + \beta\left(\mathcal{R}_t V_{t+1}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where $\mathcal{R}_t$ denotes the operator

$$\mathcal{R}_t V_{t+1} = \left\{ \sum_{(z^t,\eta^t)\preceq(z^{t+1},\eta^{t+1})} V_{t+1}(z^{t+1},\eta^{t+1})^{1-\alpha}\tilde{\pi}\left(z^{t+1},\eta^{t+1}|z^t,\eta^t\right)\right\}^{\frac{1}{1-\alpha}}.$$

With these preferences ρ controls the substitutability between consumption today and future utility as measured by V_{t+1}. And, α controls the willingness to have future utility vary across states of the world tomorrow; i.e. risk. To economize on notation, we will not index ρ, α, β by i, however it should be understood that they can differ with the trader's type. Also, since we will allow for different beliefs, we will distinguish between the trader's beliefs and the reference trader's by putting a "˜" over the trader's probabilities, just as before.

We can replace the objective in Trader i's problem (6) with this objective. We can then determine the first-order condition by recursive constructing the derivative of V_t with respect to $c(z^t, \eta^t)$ as

$$\begin{aligned}\frac{\partial V_t}{\partial c_{t+1}(z^{t+1}, \eta^{t+1})} &= V_t^{\rho} \beta \left(\mathcal{R}_t V_{t+1}\right)^{\alpha-\rho} V_{t+1}^{\rho-\alpha}(1-\beta) c_{t+1}^{-\rho} \pi(z^{t+1}, \eta^{t+1} | z^t, \eta^t) \\ &= V_t^{\rho}(1-\beta) M_{t+1} c_{t+1}^{-\rho} \pi(z^{t+1}, \eta^{t+1} | z^t, \eta^t),\end{aligned}$$

where

$$M_{t+1}(z^{t+1}, \eta^{t+1}) = \beta \left(\frac{V_{t+1}(z^{t+1}, \eta^{t+1})}{\mathcal{R}_t V_{t+1}} \right)^{\rho-\alpha}.$$

By backward induction, we get

$$\frac{\partial V_0}{\partial c_{t+1}(z^{t+1}, \eta^{t+1})} = V_0^{\rho} \mathcal{M}_{t+1}(z^{t+1}, \eta^{t+1})(1-\beta) c_{t+1}(z^{t+1}, \eta^{t+1})^{-\rho} \pi(z^{t+1}, \eta^{t+1}),$$

where

$$\mathcal{M}_{t+1}(z^{t+1}, \eta^{t+1}) = \prod_{(z^{\tau}, \eta^{\tau}) \preceq (z^{t+1}, \eta^{t+1})} M_{\tau}(z^{\tau}, \eta^{\tau}).$$

With these results, we can write the first-order condition for our trader's consumption $c_t(z^t, \eta^t)$ as

$$\begin{aligned} & V_0^{\rho} \mathcal{M}_t(z^t, \eta^t)(1-\beta) c_t(z^t, \eta^t)^{-\rho} \tilde{\pi}(z^t, \eta^t) \\ = \quad & \zeta(z^t, \eta^t) P(z^t) \pi(z^t, \eta^t). \end{aligned} \tag{15}$$

The recursive rule for the multiplier ζ is still 7), and the first order condition with respect to $\sigma(z^t, \eta^t)$ is still given by (9).

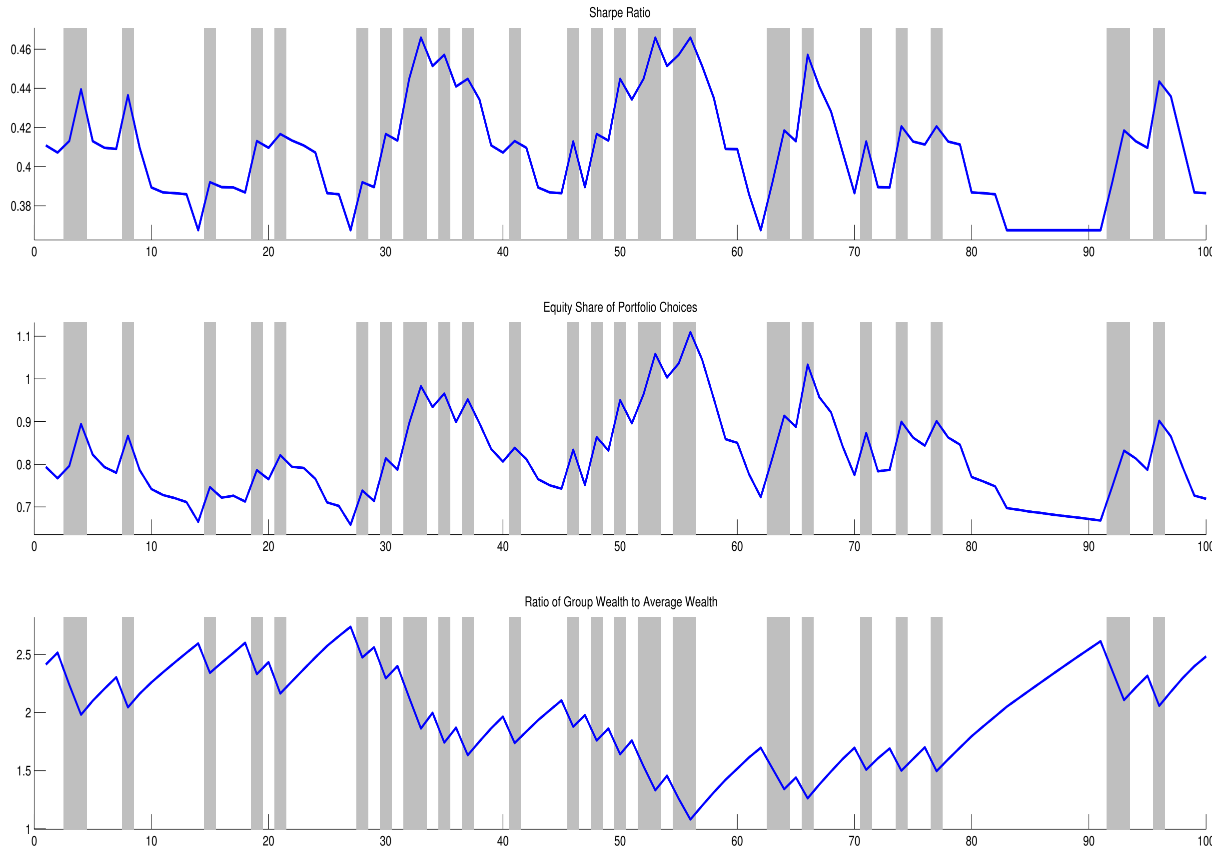
The mapping between the type i trader and his reference trader is easy to compute here, and is given by solving the following equation for $\bar{\zeta}(z^t, \eta^t)$:

$$c_t(z^t, \eta^t) = \left[\frac{\zeta(z^t, \eta^t) P(z^t) \pi(z^t, \eta^t)}{V_0^{\rho} \mathcal{M}_t(z^t, \eta^t)(1-\beta) \tilde{\pi}(z^t, \eta^t)} \right]^{-1/\rho} = \left[\frac{\bar{\zeta}(z^t, \eta^t) P(z^t)}{\beta^t} \right]^{-1/\bar{\alpha}}. \tag{16}$$

6.2 Quantitative Experiments

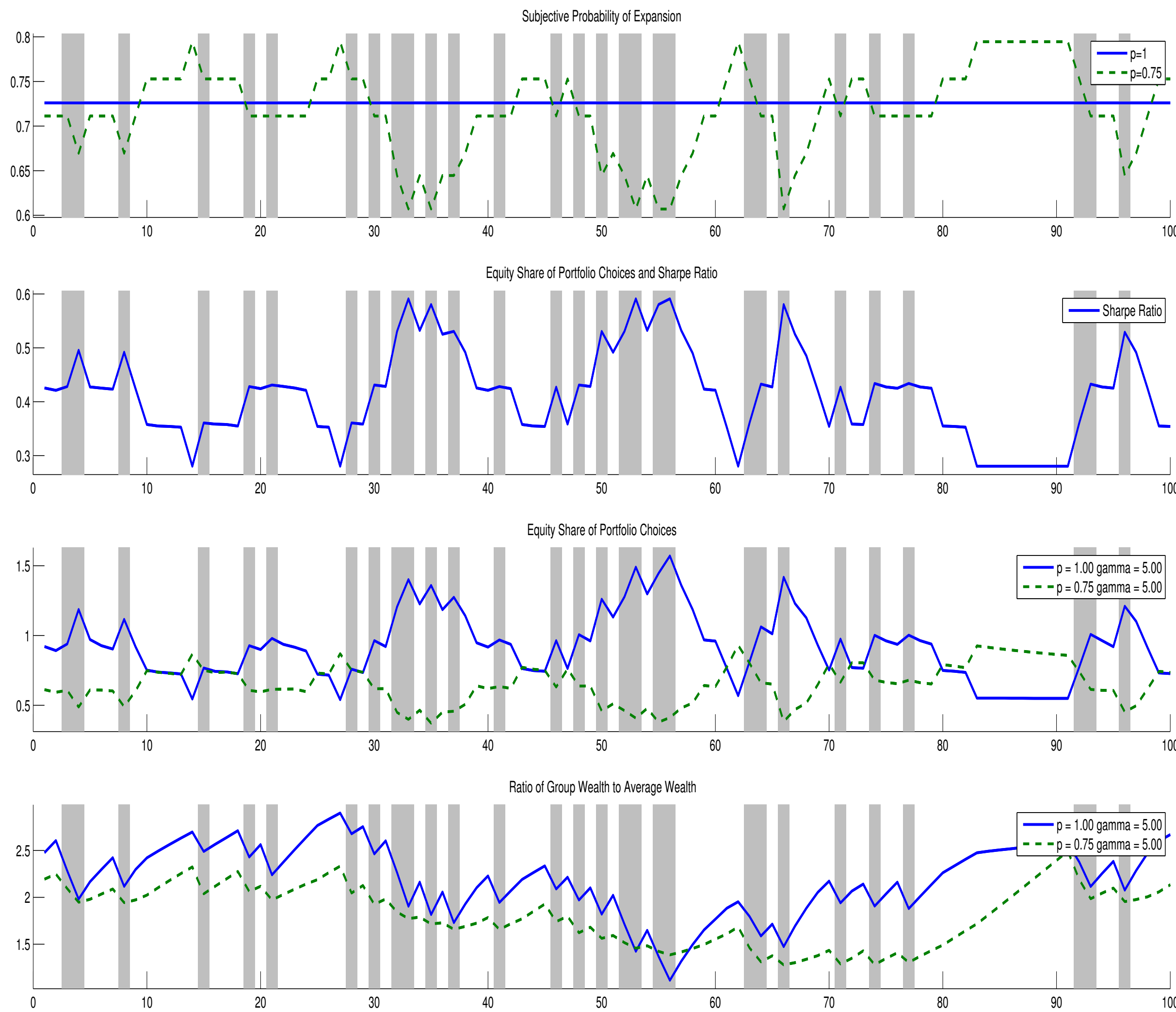
Cases	Lower RA	Higher IES
Z complete: $\gamma = 5$	5%	5%
Z^* *complete:* $RA = 2.5, IES = 1/5$	5%	0%
Z^* *complete:* $RA = 5, IES = 1/2.5$	0%	5%
$\frac{\sigma(m)}{E(m)}$		
$Std(\frac{\sigma_t(m)}{E_t(m)})$		
$E(R_f)$		
$\sigma(R_f)$		
$E\left(W_z/W\right)$		
$E\left(W_{z^*}/W\right)$		
$\sigma\left(W_z/W\right)$		
$\sigma\left(W_{z^*}/W\right)$		
$E\left(\omega_z\right)$		
$E\left(\omega_{z^*}\right)$		
$\sigma\left(\omega_z\right)$		
$\sigma\left(\omega_{z^*}\right)$		
$corr\left(\omega_z, SR\right)$		
$corr\left(\omega_{z^*}, SR\right)$		
$\sigma\left(\triangle \log\left(c_z\right)\right)/\sigma\left(\triangle \log\left(C\right)\right)$		
$\sigma\left(\triangle \log\left(c_{z^*}\right)\right)/\sigma\left(\triangle \log\left(C\right)\right)$		
$\sigma\left(\triangle \log\left(C_z\right)\right)$		
$\sigma\left(\triangle \log\left(C_{z^*}\right)\right)$		
accuracy (% allocation error)		

Figure 2: Baseline Case



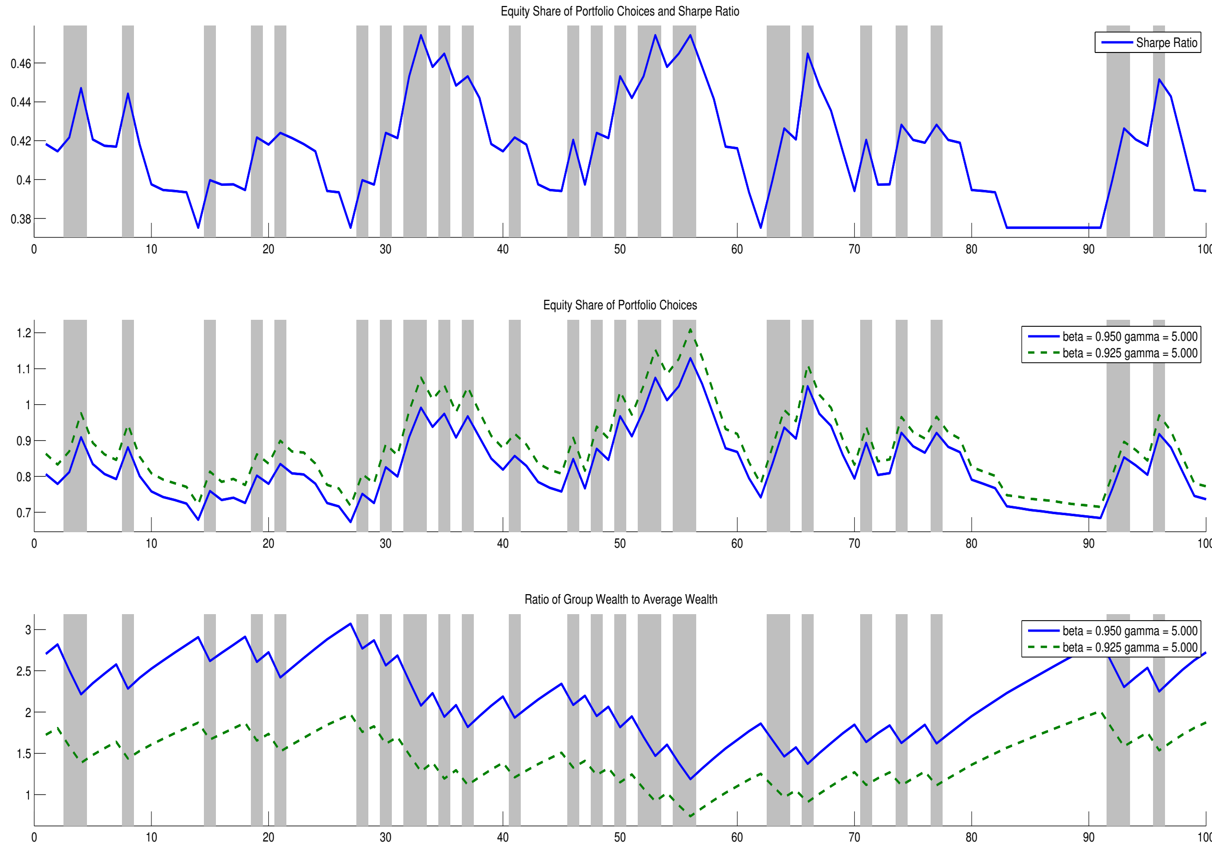
Notes: The shaded areas indicate recessions.

Figure 3: Variation in Beliefs



Notes: The shaded areas indicate recessions.

Figure 4: Variation in Discount Rates



Notes: The shaded areas indicate recessions.

7 Appendix

7.1 Stationarizing the Economy

For any agent with CRRA preferences, we can readily express his payoff in terms of his consumption share. This allows us to construct a stationary analog to our economy. Assume that

$$u^i(c^i(z^t,\eta^t)) = \frac{c^i(z^t,\eta^t)^{1-\alpha_i}}{1-\alpha_i},$$

and let $\hat{c}^i(z^t,\eta^t) = c^i(z^t,\eta^t)/C(z^t)$ denote the individual's consumption share. Then

$$\begin{aligned}
\sum_{t=1}^{\infty}(\beta_i)^t \sum_{(z^t,\eta^t)} u^i(c^i(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t) &= \beta_i \sum_{(z^1,\eta^1)} u^i(z_1\hat{c}^i(z^1,\eta^1)\tilde{\pi}^i(z^1,\eta^1) \\
&\quad + \sum_{t=2}^{\infty}(\beta_i)^t \sum_{(z^t,\eta^t)} u^i(c^i(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t), \\
&= \beta_i \sum_{(z^1,\eta^1)} u^i(z_1\hat{c}^i(z^1,\eta^1)\tilde{\pi}^i(z^1,\eta^1) \\
&\quad + \sum_{(z^2,\eta^2)} (\beta_i)^2 C(z^1)^{1-\alpha_i} u^i(z_2\hat{c}^i(z^2,\eta^2)\tilde{\pi}^i(z^2,\eta^2) \\
&\quad + \sum_{t=3}^{\infty}(\beta_i)^t \sum_{(z^t,\eta^t)} u^i(c^i(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t).
\end{aligned}$$

Continuing in this fashion, we get that

$$\sum_{t=1}^{\infty}(\beta_i)^t \sum_{(z^t,\eta^t)} u^i(c^i(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t) = \sum_{t=1}^{\infty}\sum_{(z^t,\eta^t)} \beta_i\left(z^{t-1}\right) u^i(z_t\hat{c}^i(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t),$$

where

$$\beta_i\left(z^{t-1}\right) = (\beta_i)^t C(z^{t-1})^{1-\alpha_i},$$

and take $C(z^0) = 1$. Note that this normalization does depend upon the CRRA coefficient of the individual's preferences, but is independent of his probabilities or his preference shock. We can exploit this change in variables to express the individual's f.o.c. (8) as

$$(\beta_i)^t u'(z_t\hat{c}(z^t,\eta^t))\tilde{\pi}^i(z^t,\eta^t) = \hat{\zeta}^i(z^t,\eta^t)P(z^t)\pi(z^t,\eta^t),$$

where $\hat{\zeta}^i(z^t,\eta^t) = \zeta^i(z^t,\eta^t/\left[C(z^{t-1})^{1-\alpha_i}\right]$. The transformed version of our law of motion for the recursive multiplier, (7), is

$$\hat{\zeta}^i(z^t,\eta^t) = \hat{\zeta}^i(z^{t-1},\eta^{t-1})/\left(z_{t-1}\right)^{1-\alpha_i} + \hat{\nu}^i\left(z^t,\eta^t\right) - \hat{\varphi}^i(z^t,\eta^t),$$

where $\hat{\nu}^i\left(z^t,\eta^t\right) = \nu^i\left(z^t,\eta^t\right)/\left[C(z^{t-1})^{1-\alpha_i}\right]$ and $\hat{\varphi}^i(z^t,\eta^t) = \varphi^i(z^t,\eta^t)/\left[C(z^{t-1})^{1-\alpha_i}\right]$. The transformed version of (9) is simply

$$\sum_{\eta^{t+1}\succ\eta^t} \hat{\nu}\left(z^{t+1},\eta^{t+1}\right) R(z^{t+1})\pi(z^{t+1},\eta^{t+1})P(z^{t+1}) = 0.$$

The individual's net saving function can also be made stationary and expressed in terms of the consumption share:

$$\widehat{\mathcal{S}}^i(z^t,\eta^t) = \frac{\mathcal{S}^i(z^t,\eta^t)}{Y(z^t)} = \sum_{\tau\geq t}\sum_{(z^\tau,\eta^\tau)\succeq(z^t,\eta^t)} \widetilde{P}(z^\tau,\eta^\tau)\left[\gamma\eta_\tau - \hat{c}^i(z^\tau,\eta^\tau)\right]\left[\prod_{j=t+1}^{\tau} z_j\right].$$

And, finally, the the analog of the mapping rule (10) is given by

$$\bar{\zeta}^i(z^t,\eta^t) : \left(\frac{\bar{\zeta}^i(z^t,\eta^t)P(z^t)}{\beta^t C(z^{t-1})^{1-\alpha}}\right)^{-1/\alpha} = u'^{-1}\left(\frac{\zeta^i(z^t,\eta^t)\pi(z^t,\eta^t)P(z^t)}{(\beta_i)^t\, C(z^{t-1})^{1-\alpha_i}\tilde{\pi}^i(z^t,\eta^t)}\right). \tag{17}$$