STAT-UB.0103 Notes on elasticity.

Here is a quick take on the elasticity concept. This is usually the topic for an economics course, so we'll just mention it briefly. The material after the *** line was not done in class.

Let's think of role of b_1 in fitted model $\hat{Y} = b_0 + b_1 x$. This says that as x goes up by one unit, Y (tends to) go up by b_1. In fact, it's precisely what we call $\frac{dY}{dx}$ in calculus. This measures the sensitivity of Y to x. Suppose that x is the price at which something is sold and Y is the quantity that clears the market. It helps to rewrite this as $Q = b_0 + b_1 P$. We'd expect b_1 to be negative, since the quantities consumed will usually decrease as the price rises. If currently the price is P_0 and currently the quantity is Q_0, then $Q_0 = b_0 + b_1 P_0$.

A change of price from P_0 to $P_0 + \theta$ leads to a change in quantity from Q_0 to $b_0 + b_1 (P_0 + \theta) = b_0 + b_1 P_0 + b_1\theta = Q_0 + b_1 \theta$. Thus, the change in quantity is $b_1 \theta$.

The proportional change ratio - $\dfrac{\frac{\text{change in } Q}{Q_0}}{\frac{\text{change in } P}{P_0}}$ is called an elasticity. (We use minus sign since they tend to move in opposite direction, and we like positive elasticities.) Of course, we can simplify this a bit, writing it finally as

$$elasticity = - b_1 \frac{P_0}{Q_0}$$

This worked for the straight-line relationship between Q and P. If it's not a straight line, replace b_1 by $\left.\left(\frac{dQ}{dP}\right)\right|_{(P,Q)=(P_0,Q_0)}$ and then use $elasticity = -\left.\left(\frac{dQ}{dP}\right)\right|_{(P,Q)=(P_0,Q_0)} \times \frac{P_0}{Q_0}$. The symbol $\left.\left(\frac{dQ}{dP}\right)\right|_{(P,Q)=(P_0,Q_0)}$ asks for the derivative to be evaluated at the point (P_0, Q_0).

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The P, Q relationship is not usually a straight line. In such a story, we'd use *elasticity* $= -\left.\left(\frac{dQ}{dP}\right)\right|_{(P,Q)=(P_0,Q_0)} \times \frac{P_0}{Q_0}$. The symbol $\left.\left(\frac{dQ}{dP}\right)\right|_{(P,Q)=(P_0,Q_0)}$ asks for the derivative to be evaluated at the point (P_0, Q_0).

Example: Suppose that a demand "curve" has equation $Q = 400{,}000 - 2{,}000\,P$. (This "curve" is actually a straight line.) Illustrations in most economics texts use straight lines. Suppose that the current price is $P_0 = \$60$. The quantity corresponding to this is $Q_0 = 400{,}000 - 2{,}000 \times 60 = 280{,}000$. Now suppose that the price rises by 1%, to \$60.60. The quantity is now $400{,}000 - 2{,}000 \times 60.60 = 278{,}800$. This is a decrease of 1,200 in quantity consumed, which is a decrease of $\frac{1{,}200}{280{,}000} \approx 0.0043 = 0.43\%$. Thus a 1% increase in price led to a decrease in quantity of 0.43%, so we would give the elasticity as 0.43. This is less than 1, so that the demand is *inelastic*.

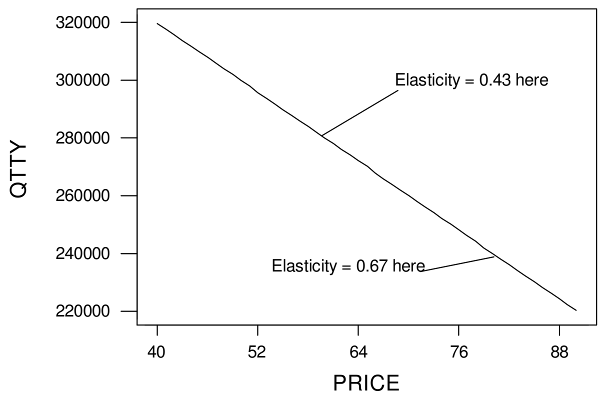
We could of course obtain this as $-\, b_1 \frac{P_0}{Q_0} = 2{,}000 \times \frac{60}{280{,}000} \approx 0.43$.

One approach follows through on the "1% change in price" idea. The other approach simply uses $-\, b_1 \frac{P_0}{Q_0}$. These will not give exactly the same numerical result, but the values will be very close.

Observe that the elasticity calculation depends here on where we started. Suppose that we had started at $P_0 = \$80$. The quantity consumed at this price of \$80 is $Q_0 = 400{,}000 - 2{,}000 \times 80 = 240{,}000$. An increase in price by 1%, meaning to \$80.80, leads to a new quantity of $400{,}000 - 2{,}000 \times 80.80 = 238{,}400$. This is a decrease of 1,600 in quantity consumed. In percentage terms, this decrease is $\frac{1{,}600}{240{,}000} \approx 0.0067 = 0.67\%$. We would give the elasticity as 0.67.

This is also found as $-\, b_1 \frac{P_0}{Q_0} = 2{,}000 \times \frac{80}{240{,}000} \approx 0.67$.

We can see that the elasticity for this demand curve $Q = 400{,}000 - 2{,}000\ P$ depends on where we start.



You might check that at starting price P_0 = \$120, the elasticity would be computed as 1.50, and we would now claim that the demand is (highly) *elastic*.

> Details: Find $Q_0 = 400{,}000 - 2{,}000 \times 120 = 160{,}000$. The 1% increase in price would lead to a consumption decrease of $1.2 \times 2000 = 2{,}400$. This is a percentage decrease of $\dfrac{2{,}400}{160{,}000} = 0.015 = 1.5\%$.

What is the shape of a demand curve on which the elasticity is the same at all prices? This is fun little calculus exercise that gets into simple differential equations. The problem is equivalent to asking about the curve for which $\dfrac{\frac{dQ}{Q}}{\frac{dP}{P}} = -c$, where c is constant. (The minus sign is used simply as a convenience; of course c will be positive.) This condition can be expressed as

$$\frac{dQ}{Q} = -c\,\frac{dP}{P}$$

for which the solution is $\log Q = -c \log P + d$. This uses the calculus fact that $\dfrac{d}{dt}\log f(t) = \dfrac{f'(t)}{f(t)}$.

The result can be re-expressed as

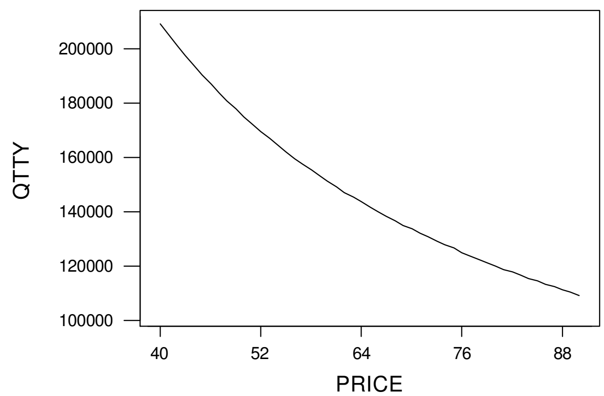
$$e^{\log Q} = e^{-c \log P + d}$$

or

$$Q = mP^{-c}$$

where $m = e^d$.

The picture below shows the graph of $Q = 4{,}000{,}000\ P^{-0.8}$. This curve has elasticity 0.80 at every price.



The equation $\log Q = -c \log P + d$ is a simple linear relationship between $\log Q$ and $\log P$. Thus, if we base our work on a regression of log(quantity) on log(price), then we can claim that the resulting elasticity is the same at every price.

If you began by taking logs of BOTH variables and fitted the regression (log-on-log), you'd get the elasticity directly from the slope (with no need to worry about P_0 or Q_0). That is, in a log-on-log regression, the elasticity is exactly $-b_1$.

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