Internet Appendix for
"Risk Choice Under High-Water Marks"

This Internet Appendix serves as a companion to the paper "Risk Choice Under High-Water Marks". It reports results that were not included in the main text in order to conserve space.

IA.1 Additional Results on Risk Limits and Walk-away

If walk-away is not optimal (corresponding to $C_w^* < C$ in the proof of Proposition 7) then the manager may still be fully constrained. Alternatively he may be partially constrained or unconstrained. If he is fully constrained then the solution takes the same form as in equation (A.8.1), but now the boundary condition becomes $G(C) = g$ and (A.8.2) no longer holds. To verify if this is indeed the solution, we must check then check whether, consistent with our assumption, the risk limit binds everywhere for this candidate G solution. If this is the case, then the resulting G function does indeed give the value function. If the risk limit does not bind everywhere, then the either the constraint only partially binds, or the manager is unconstrained.

Before moving on to solve the partially constrained case, consider the case where the unconstrained manager has $D_0 < 0$ (to this point we have thought about a managers with an unconstrained $D_0 > 0$). Since in this case the unconstrained $\pi^*(X_t)$ is decreasing in X_t, in the presence of the constraint the solution to the problem will have $\pi^*(X_t)$ *constrained* on a region $[C, \mathbb{X}]$ and unconstrained on $(\mathbb{X}, 1]$. If $\mathbb{X} = 1$, then the manager is constrained on the whole interval $[C, 1]$ ($[C^*, 1]$ if there is optimal walk-away) and the solution proceeds as discussed above. On the other hand, if $\mathbb{X} < 1$, then this is the case of a partially binding constraint, which is discussed next.

The partially binding case can be separated into two sub-cases, depending on whether the unconstrained manager would have $D_0 \geq 0$ or $D_0 \leq 0$. If $D_0 \geq 0$, then either the constraint binds on $(\mathbb{X}, 1]$ where $C < \mathbb{X} < 1$, or the manager is unconstrained. Suppose that the manager is sometimes constrained, so that $C < \mathbb{X} < 1$. Since the manager is unconstrained on the lower region, walk-away is *not* optimal, as shown above. The form of V remains the same as in equation (9), but now the solution for G splits into two regions. Let $\underline{G}(X_t)$ be the solution on the region $[C, \mathbb{X}]$ and $\overline{G}(X_t)$ be the solution on $(\mathbb{X}, 1]$. Then $\underline{G}(X_t)$ has the form given in equation (10) since the HJB equation on this region corresponds to an unconstrained manager. Correspondingly, $\overline{G}(X_t)$ takes the form (A.8.1) as the HJB equation on this region corresponds to a constrained manager. In addition to $\underline{G}(C) = g$ and $\overline{G}(1) = 1$, there are now also three other boundary conditions,

1. $\underline{G}(\mathbb{X}) = \overline{G}(\mathbb{X})$
2. $\underline{G}_X(\mathbb{X}) = \overline{G}_X(\mathbb{X})$
3. $0 = V_X \mathbb{X}(\mu - r) + \overline{\pi}\mathbb{X}^2\sigma^2 V_{XX}$

The first two conditions match the value and first derivative of the two parts of G across the change of regions. The third condition says that at the point $\mathbb{X}$, where the constraint begins to bind, $\pi_t^* = \overline{\pi}$. This means the risk-choice of the manager is continuous. The five boundary conditions jointly pin down the values of the five constants, D_0, D_1, A_1, A_2, and $\mathbb{X}$. It then remains to check the assumptions that $C < \mathbb{X} < 1$ and that the position limit (19) indeed binds on $(\mathbb{X}, 1]$.

The top panel of Figure IA.1 plots $V(X_t)$ for examples of managers who are unconstrained (circles), partially constrained (dash-dot line), fully constrained without walk-away (dashed line), and fully constrained with walk-away (solid line). The bottom panel plots the corresponding $G(X_t)$ function. Note that the fully constrained managers' value functions need *not* be concave, in contrast to that of the unconstrained manager. Moreover, when there is optimal walk-away (i.e., $C^* > C$), then the manager's value function *must* be convex at the walk-away point. To see this, note that the Lagrange multiplier ψ on the position limit (19) must be positive. Since ψ is given by $V_X X(\mu - r) + \overline{\pi} X^2 \sigma^2 V_{XX}$, and since $V_X(C^*) = 0$, we must have $V_{XX}(C^*) > 0$.

The only difference between the managers in the figure is the value of the position limit $\overline{\pi}$. The lowest value of $\overline{\pi}$ corresponds to the manager who is fully constrained and walks away. As $\overline{\pi}$ is increased, it is no longer optimal to walk-away. As $\overline{\pi}$ increases further, the limit starts to bind only partially, and $\mathbb{X}$ increases from a low value. When $\overline{\pi}$ has increased sufficiently that $\mathbb{X}$ increases to 1, the manager becomes unconstrained.

The top plot shows how $V(X_t)$ increases as the risk-limit $\overline{\pi}$ is raised, while the bottom plot shows how $G(X_t)$ becomes increasingly concave. Note that the slope of G is zero at the walk-away point in the constrained case with walk-away (solid line), while it is positive in the other cases. Figure IA.2 plots the corresponding optimal risk choice policies $\pi^*(X_t)$. Note that for low X_t, the risk-choice of the globally constrained manager (dashed line) is actually *bigger* than for the unconstrained and partially-constrained manager.

It remains to take care of the case corresponding to an unconstrained $D_0 < 0$. In this case a partially-binding constraint binds on $[C, \mathbb{X}]$, with $C < \mathbb{X} < 1$. Again, the form of $V(X_t)$ remains equation (9) and the solution for G is split into two regions. Let $\underline{G}(X_t)$ be the solution on the region $[C, \mathbb{X}]$ and $\overline{G}(X_t)$ be the solution on $(\mathbb{X}, 1]$. Now, it is $\underline{G}(X_t)$ that has the form (A.8.1), since the manager is constrained on the lower region, while $\overline{G}(X_t)$ takes the form (10) since the manager is unconstrained on the upper region. The boundary conditions include the same five as above: $\underline{G}(C) = g$, $\overline{G}(1) = 1$, $\underline{G}(\mathbb{X}) = \overline{G}(\mathbb{X})$, $\underline{G}_X(\mathbb{X}) = \overline{G}_X(\mathbb{X})$, and $0 = V_X \mathbb{X}(\mu - r) + \overline{\pi}\mathbb{X}^2\sigma^2 V_{XX}$. However, because the constraint binds on the lower region, it *is* now possible that there will be optimal walk-away, $C^* > C$. If this is the case, C is replaced with C^* above, and C^* satisfies the smooth-pasting condition, $\underline{G}_X(C^*) = 0$.

IA.2 Calculation of Figure 3

Let

$$P(X_t) = E[1_{\tau_H < \infty} | X = X_t]$$

be the probability of reaching the current HWM prior to termination under the risk-choice policy π_t. Then $P(X)$ satisfies the following ordinary differential equation

$$P_X X \pi_t(\mu - r) + \frac{1}{2} P_{XX} X^2 \pi_t^2 \sigma^2 = 0$$

with boundary conditions

$$P(C) = 0$$
$$P(1) = 1$$

Substitution of the optimal risk-choice policy gives

$$P_X X \frac{(\mu - r)^2}{\sigma^2} \frac{1}{1-\eta} \frac{X_t - D_0}{X_t} + \frac{1}{2} P_{XX} X^2 \frac{(\mu - r)^2}{\sigma^2} \frac{1}{(1-\eta)^2} \frac{(X_t - D_0)^2}{X_t^2} = 0$$

The general solution to this is

$$P(X) = P_1 (X_t - D_0)^{\gamma_1} + P_2 (X_t - D_0)^{\gamma_2}$$

The constants γ_1 and γ_2 are found by substituting the solution into the ODE and solving the resulting quadratic equation. Imposing the two boundary conditions then provides the solution to the constants P_1 and P_2.

Next we calculate the certainty-equivalent expected waiting times. To calculate $E_t[e^{-\rho(\tau_H - t)} | \tau_H < \infty]$ we calculate $E_t[e^{-\rho(\tau_H - t)}]$ and then divide by $P(X_t)$. Let $F(X_t) = E_t[e^{-\rho(\tau_H - t)}]$. Then $F(X_t)$ satisfies the following ordinary differential equation

$$-\rho F + F_X X \pi_t (\mu - r) + \frac{1}{2} F_{XX} X^2 \pi_t^2 \sigma^2 = 0$$

with boundary conditions

$$F(C) = 0$$
$$F(1) = 1$$

Substitution of the optimal risk-choice policy gives

$$-\rho F + F_X X \frac{(\mu - r)^2}{\sigma^2} \frac{1}{1-\eta} \frac{X_t - D_0}{X_t} + \frac{1}{2} F_{XX} X^2 \frac{(\mu - r)^2}{\sigma^2} \frac{1}{(1-\eta)^2} \frac{(X_t - D_0)^2}{X_t^2} = 0$$

The general solution to this is

$$F(X) = F_1 (X_t - D_0)^{\gamma_1} + F_2 (X_t - D_0)^{\gamma_2}$$

The constants γ_1 and γ_2 are found by substituting the solution into the ODE and solving the resulting quadratic equation. Imposing the two boundary conditions then provides the solution to the constants F_1 and F_2.

To find $E_t[e^{-\rho(\tau_C - t)}]$, repeat the same calculations but now with the boundary conditions given by $F(C) = 1$ and $F(1) = 0$.

Figure IA.1: Value Function with Risk Limits

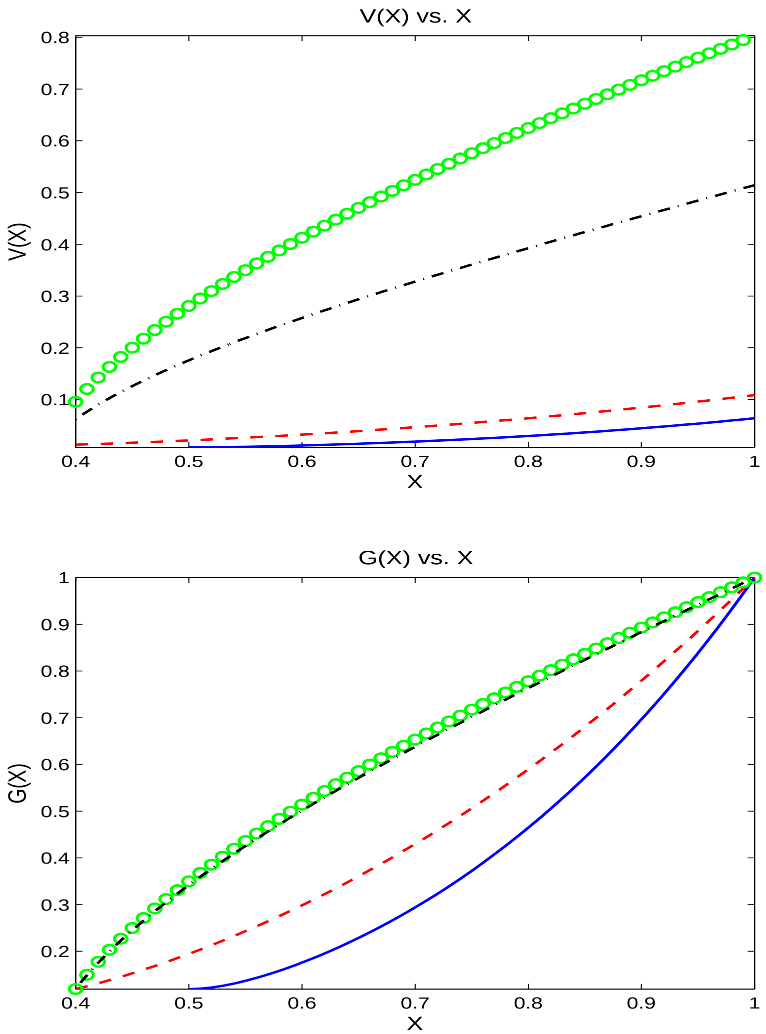

Figure IA.1 plots $V(X_t)$ (top panel) and $G(X_t)$ (bottom panel) for a manager who is completely unconstrained (circles), partially constrained (dash-dot line, $\overline{\pi} = 2.0$), completely constrained but does not walk away (dashed line, $\overline{\pi} = 0.75$), and completely constrained with optimal walk-away (solid line, $\overline{\pi} = 0.50$). The optimal walk-away point in the last case is $C_w = 0.5$. The remaining parameters are $C = 0.40$, $\phi = 0.11$, $\rho = 0.03$, $\lambda = 0$, $g = 0.35$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$

Figure IA.2: Optimal Risk Choice with Risk Limits

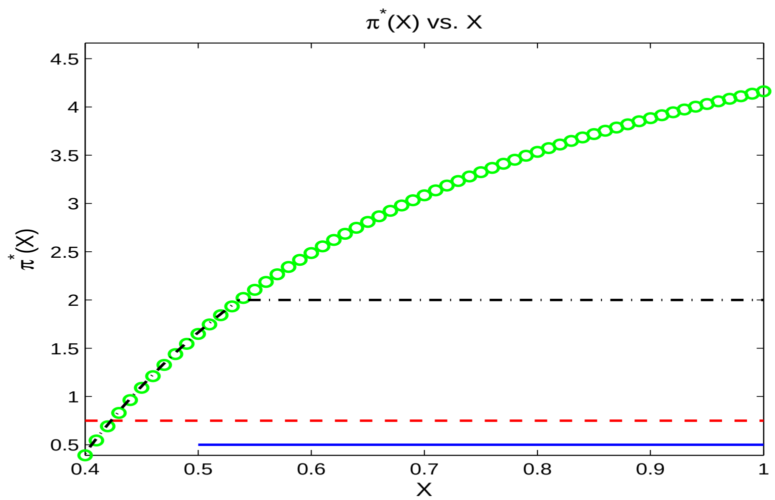

Figure IA.2 plots $\pi^*(X_t)$ for the four cases in Figure IA.1.