American Options

Detailed outline

1. Payoff process
2. Exercise policy (stopping time)
3. Optimal stopping problem
4. Option value (super-replication, Snell envelope)
5. No-early-exercise condition
6. Markovian setting
   (a) Continuation region, exercise region
   (b) Bellman equation, variational inequality
   (c) Exercise boundary, smooth-pasting condition
7. American puts
8. American calls

Readings

Domenico Cuoco's lecture notes, part V.

Duffie, chapter 8, sections G-H.

Karatzas and Shreve, 1998, chapter 2, sections 5-7.

Merton, R., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141-183.

Karatzas, I., 1988, On the pricing of American options, *Applied Mathematics and Optimization* 17, 37-60.

Kim, I., 1990, The analytic valuation of American options, *Review of Financial Studies* 3, 547-572.

Jacka, S., 1991, Optimal stopping and the American put, *Mathematical Finance* 1, 1-14.

Carr, P., R. Jarrow, and R. Myneni, 1992, Alternative characterizations of American put options, *Mathematical Finance* 2, 87-106.

Problems

1. Prove that if an American option's discounted payoff process βG is a $\mathcal{P}^*$-submartingale, then it is optimal not to exercise early.

2. Consider a complete, continuous-time financial market where B_t^* is a standard 2-dimensional Brownian motion under the martingale measure $\mathcal{P}^*$. The interest rate r_t is a nonnegative one-factor diffusion described by the equation

$$dr_t = \mu(r_t, t)\, dt + \sigma_r(r_t, t)\, dB_{1,t}^* \ , \tag{1}$$

where μ and σ_r are continuous and satisfy Lipschitz and linear growth conditions. The price of a stock, S_t, satisfies

$$\frac{dS_t}{S_t} = (r_t - \delta)\, dt + \sigma\, dB_{2,t}^* \ , \tag{2}$$

where $\delta > 0$ and $\sigma \neq 0$ are constants. Let $c(S_t, r_t, t)$ be the time t price of an American call on the stock with strike price k and expiration date T.

(a) Prove that for each $(r, t) \in \mathcal{R}^+ \times [0, T)$ and for all $s_2 > s_1 > 0$,

$$0 \leq \frac{c(s_2, r, t) - c(s_1, r, t)}{s_2 - s_1} \leq 1 \ . \tag{3}$$

(b) Prove that for each $(r, t) \in \mathcal{R}^+ \times [0, T)$, if it is optimal to continue at (s_2, r, t), then it is optimal to continue at (s_1, r, t) for all $s_1 < s_2$.