Equilibrium in a Pure Exchange Economy

Detailed outline

1. Primitives–agents, endowments, preferences
2. Arrow-Debreu equilibrium
   (a) Characterization of state-price density process
   (b) Existence and uniqueness of equilibrium
   (c) Representative agent
3. Security market equilibrium
4. Consumption CAPM

Readings

Duffie, chapter 10.

Karatzas and Shreve, 1998, chapter 4.

Merton, R., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867-888.

Breeden, D., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *Journal of Financial Economics* 7, 265-296.

Karatzas, I., J.P. Lehoczky, and S. Shreve, 1990, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model, *Mathematics of Operations Research* 15, 80-128.

Problems

1. Define the representative agent utility function u by

$$u(c,t) \equiv \max_{(c_1,\ldots,c_m)} \sum_{j=1}^{m} \lambda_j u_j(c_j,t) \;\; \text{s.t.} \; \sum_{j=1}^{m} c_j \le c \; , \tag{1}$$

where for all $j = 1,\ldots,m$, $u_j(\cdot,t)$ is C^3, $u_j'(c,t) > 0$, $u_j'(c,t) \to \infty$ as $c \to 0$, $u_j'(c,t) \to 0$ as $c \to \infty$, and $u_j''(c,t) < 0$. Let $(c_1^*(c),\ldots,c_m^*(c))$ be the optimal solution. Prove that $u(\cdot,t) : (0,\infty) \to \mathcal{R}$ has the following properties:

   (a) $u(\cdot,t)$ is C^2,

   (b) $u'(c,t) = \lambda_j u_j'(c_j^*(c),t) \;\; \forall \; j = 1,\ldots,m$,

   (c) $u'(c,t) \to \infty$ as $c \to 0$,

   (d) $u'(c,t) \to 0$ as $c \to \infty$,

   (e) $u''(c,t) < 0$.

2. Consider an economy with m agents with preferences as described above and endowment processes $e_t^j, j = 1,\ldots,m$ on $[0,T]$. Suppose the state-price density process H_t and consumption plans $c_t^{j*}, j = 1,\ldots,m$, on $[0,T]$ form an Arrow-Debreu equilibrium. Let $S_0, S_1, ..., S_d$ be the prices of a riskless asset and d risky assets with nonsingular $(d \times d)$-dimensional volatility matrix process σ_t. Suppose these assets are in zero net supply. Suppose that instead of trading Arrow-Debreu consumption plans explicitly, agents continuously trade the $d+1$ assets to implement their optimal consumption. Let the m $(d+1)$-dimensional processes π_t^j be the trading strategies that finance the agents' optimal consumption plans. Prove that the $d+1$ security markets clear, that is, prove

$$\sum_{j=1}^{m} \pi_t^j = 0 \in \mathcal{R}^{d+1}, \;\; \forall \; t \in [0,T] \; .$$

3. Consider an economy with identical agents each deriving expected utility from consumption plan $\{c_t\}$ equal to

$$\mathrm{E} \int_0^T e^{-\rho t} \frac{c_t^\gamma}{\gamma} \, dt \ , \ \gamma \in \ (0, 1) \ . \tag{2}$$

The aggregate endowment process, e_t, satisfies

$$\frac{de_t}{e_t} = \mu_e(t) \, dt + \sigma_e(t) \, dB_t \tag{3}$$

where B_t is d-dimensional Brownian motion and μ_e and σ_e are bounded, progressively measurable processes taking values in $\mathcal{R}$ and $\mathcal{R}^d$, respectively.

(a) Determine the interest rate process r_t in the Arrow-Debreu equilibrium for this economy.

(b) Suppose $\hat{r}_t$ is an arbitrary bounded, nonnegative, progressively measurable process. Construct an equilibrium in which the interest rate is $\hat{r}_t$.

(c) Suppose $\hat{r}_t$ above is an Itô process with $d\hat{r}_t = \mu_r(t) \, dt + \sigma_r(t) \, dB_t$. What is the drift of $\hat{r}_t$ under the martingale measure $\mathcal{P}^*$ in the equilibrium you constructed?