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Financial Theory IV*

1 Arbitrage and Martingales

These notes are taken primarily from Section III of Domenico Cuoco's lecture notes and Harrison and Kreps (1979). Consider the following economy:

1.1 Probabilistic Setting

There is a finite time horizon $[0, T]$, a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t=0}^T$ satisfying the usual conditions, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Each $\omega \in \Omega$ represents a complete description of what happens from time 0 to T. Each $A \in \mathcal{F}_t$ represents an event distinguishable at time t, i.e., you know whether or not $\omega \in A$ at time t. The measure $\mathcal{P}$ is the subjective probability measure believed by agents in the economy.

1.2 Consumption

There is a single consumption good consumed only at time T. A *consumption plan* $x(\omega)$ is an $\mathcal{F}$-measurable random variable. The *consumption space* $\mathcal{C}$ is $L^p(\mathcal{P})$ for some $p \in [1, \infty)$.

1.3 Preferences

Agents are represented by their preferences $\succeq$ on $\mathcal{C}$ (a preference relation $\succeq$ is a complete, transitive binary relation). W.L.O.G. $\succeq$ represents preferences for time T net trades. (If

*Many thanks to my students for valuable comments and corrections.

original preference $\succeq'$ is defined over total consumption when endowment is $\bar{x}$, then derive $\succeq$ on net trades from $(\succeq', \bar{x})$ by $x \succeq y \Longleftrightarrow x + \bar{x} \succeq' y + \bar{x}$.) Note that $\mathcal{C} = L^p(\mathcal{P})$ implies negative consumption must have meaning. Preferences satisfy

1. *Lower semi-continuity*: the sets $\{x \in \mathcal{C} : \hat{x} \succeq x\}$ are norm-closed $\forall \hat{x} \in \mathcal{C}$.

2. *Convexity*: the sets $\{x \in \mathcal{C} : x \succeq \hat{x}\}$ are convex $\forall \hat{x} \in \mathcal{C}$.

3. *Strict monotonicity*: $x + y \succ x$ for every $x \in \mathcal{C}$ and $y \geq 0$ with $\mathcal{P}\{y > 0\} > 0$, where $\succ$ denotes strict preference.

Remark A preference relation defined by $x \succeq y \Longleftrightarrow \mathrm{E}U(x) \geq \mathrm{E}U(y)$ where U is concave, strictly increasing, and grows at no more than quadratic rate satisfies these conditions.

1.4 The One-Period Market

Let $\mathcal{M} \subset \mathcal{C}$ be the subspace of marketed consumption plans, i.e., consumption plans that agents can buy or sell, and let p_m be the price functional on $\mathcal{M}$. Assume $\mathcal{M}$ is a linear subspace and p_m is a linear functional. I.e., there are no transaction costs, short sale constraints, etc. Suppose there exists $\hat{x} > 0$ a.s. in $\mathcal{M}$.

Definition 1.1 The market is *complete* if $\mathcal{M} = \mathcal{C}$.

Definition 1.2 The price system $(\mathcal{M}, p_m)$ is *viable* if there exist preferences $\succeq$ and net trade $x^* \in \mathcal{M}$ such that $p_m(x^*) \leq 0$ and $x^* \succeq x$ for every $x \in \mathcal{M}$ such that $p_m(x) \leq 0$.

The viability condition (VC) above means that there exists an agent who is able to find an optimal net trade subject to his budget constraint. This is clearly a necessary condition for an equilibrium. It is also sufficient in that one can construct an equilibrium from this as follows. Define $\succeq'$ by $x \succeq' y \Longleftrightarrow x + x^* \succeq y + x^*$. Then 0 is an optimal net trade for an agent with preferences $\succeq'$ so $(\mathcal{M}, p_m)$ is an equilibrium price system for an economy populated by such agents.

Definition 1.3 A *free lunch* or *arbitrage opportunity* is an $x \in \mathcal{M}$ such that $x \geq 0, \mathcal{P}\{x > 0\} > 0$, and $p_m(x) \leq 0$.

Remark There is no arbitrage if and only if p_m is a strictly positive linear functional.

Proposition 1.1 *The price system $(\mathcal{M}, p_m)$ is viable $\Longrightarrow$ there is no arbitrage.*

Proof Homework

If the market is complete, i.e., if $\mathcal{M} = \mathcal{C} = L^p(\mathcal{P})$, and there is no arbitrage, then p_m is a strictly positive *continuous* linear functional p on $L^p(\mathcal{P})$ (see Proposition 7 on page 14 of section I of Domenico Cuoco's lecture notes). Then, by the Riesz Representation Theorem, there exists a unique random variable $\zeta > 0$ a.s. such that

$$p(x) = \int \zeta(\omega)x(\omega)\, d\mathcal{P}(\omega) = \mathrm{E}\{\zeta x\} \;\forall\, x \in \mathcal{M}\ . \tag{1.1}$$

Definition 1.4 A random variable $\zeta > 0$ satisfying equation (1.1) is called a *state-price density* or *stochastic discount factor (sdf).*

If Ω is finite, $\zeta(\omega)$ can be interpreted as the Arrow-Debreu price for state ω per unit of probability, i.e., $p(1_\omega) = \zeta(\omega)\mathcal{P}(\omega)$ so $\zeta(\omega) = p(1_\omega)/\mathcal{P}(\omega)$.

We are interested in knowing when we can represent prices with an sdf. The issue is, if $\mathcal{M}$ is incomplete, does p_m have a continuous extension to $L^p(\mathcal{P})$? That's what viability gives. We use the Separating Hyperplane Theorem.

Theorem 1.1 (Harrison and Kreps) *The price system $(\mathcal{M}, p_m)$ is viable $\Longleftrightarrow$ there exists a strictly positive continuous linear extension p of p_m to all of $\mathcal{C}$.*

Proof $\Longleftarrow$: Suppose such a p exists. Define $\succeq$ on $\mathcal{C}$ by $x \succeq y$ iff $p(x) \geq p(y)$. Then $\succeq$ satisfies preference properties 1-3 and together with $x^* = 0$ satisfies the VC so $(\mathcal{M}, p_m)$ is viable.

$\Longrightarrow$: Suppose $(\mathcal{M}, p_m)$ is viable. Let $\succeq$ and x^* satisfy the VC and W.L.O.G. set $x^* = 0$. Define $A = \{x \in \mathcal{C} : x \succeq 0\}$ and $B = \{y \in \mathcal{M} : p_m(y) \leq 0\}$. The sets A and B are convex by the convexity of preferences and the linearity of p_m. It can be shown that the interior of A is the set $\{x \in \mathcal{C} : x \succ 0\}$, which is nonempty, since it contains $\hat{x}$, for example, and disjoint from B by the VC. Therefore, by the Separating Hyperplane Theorem, $\exists$ a nontrivial continuous linear functional ϕ on $\mathcal{C}$ such that $\phi(x) \geq 0$ on A and $\phi(x) \leq 0$ on B.

Now ϕ is strictly positive: First, recall $\exists\, \hat{x} > 0$ a.s. in $\mathcal{M}$. Also, $\exists\, y \in \mathcal{C}$ such that $\phi(y) > 0$ since ϕ is nontrivial. By continuity of preferences, $\exists$ scalar $\lambda > 0$ such that $\hat{x} - \lambda y \succ 0$. Thus $\phi(\hat{x} - \lambda y) \geq 0 \Longrightarrow \phi(\hat{x}) \geq \lambda \phi(y) > 0$. Second, $\forall\ x \geq 0$ such that $\mathcal{P}\{x > 0\} > 0$, $x \succ 0$ by strict monotonicity of preferences, so $\exists\ \lambda > 0$ such that $x - \lambda \hat{x} \succ 0$. Thus, $\phi(x - \lambda \hat{x}) \geq 0 \Longrightarrow \phi(x) \geq \lambda \phi(\hat{x}) > 0$.

Finally, $\phi(x) = p_m(x)$ on $\mathcal{M}$ up to scaling: Let $x \in \mathcal{M}$ and let $b = -\hat{x} p_m(x)/p_m(\hat{x}) + x$. Both b and $-b$ are in B since $p_m(b) = p_m(-b) = 0$. So $\phi(b) \leq 0$ and $\phi(-b) \leq 0 \Longrightarrow \phi(b) = 0 \Longrightarrow \phi(x) = \phi(\hat{x}) p_m(x)/p_m(\hat{x})$. So let $p \equiv [p_m(\hat{x})/\phi(\hat{x})]\phi$. Then $p|\mathcal{M} = p_m$ and p is a strictly positive continuous linear functional on $\mathcal{C}$. $\square$

It then follows from the Riesz Representation Theorem that

Corollary 1.1 *The price system $(\mathcal{M}, p_m)$ is viable $\Longleftrightarrow$ there exists a sdf.*

1.5 Securities Markets with Multiple Trading Dates

Now suppose there are $n+1$ long-lived securities traded with right-continuous, adapted price processes $S = (S_0, S_1, \ldots, S_n)$ where S_0 is strictly positive and $S_k(t) \in L^p(\mathcal{P})$ for

$k = 1, \ldots, n$ and $t \in [0, T]$. Assume there are no dividends.

Definition 1.5 A *trading strategy* $N = (N_0, N_1, \ldots, N_n)$ is an $n+1$-dimensional row-vector-valued predictable process where $N_k(t)$ denotes the number of shares of security k held at time t.

Definition 1.6 A trading strategy is *simple* if there exists a finite partition $0 = t_0 < t_1 < \cdots < t_J = T$ of $[0, T]$ and random variables $N_{kj} \in \mathcal{F}_{t_j}$ such that

$$N_k(t) = \begin{cases} N_{k0} & \text{if} \quad t \in [t_0, t_1] \\ N_{kj} & \text{if} \quad t \in (t_j, t_{j+1}] \end{cases} \tag{1.2}$$

$\forall\ k = 0, 1, \ldots, n$ and $j = 0, \ldots, J-1$.

We will restrict attention to simple trading strategies for now.

Definition 1.7 A simple trading strategy is *self-financing* if

$$\underbrace{N(t_j)S(t_j)}_{\text{cost of new portfolio at } t_j} \leq \underbrace{N(t_{j-1})S(t_j)}_{\text{proceeds from sale of old portfolio at } t_j} \quad \text{a.s.} \tag{1.3}$$

for each $j = 1, \ldots, J-1$. The trading strategy is *tight* or *tightly self-financing* if

$$N(t_j)S(t_j) = N(t_{j-1})S(t_j) \text{ a.s.} \tag{1.4}$$

Proposition 1.2 *A simple trading strategy is tight* $\Longleftrightarrow$

$$\underbrace{N(t)S(t)}_{\text{ending portfolio value}} = \underbrace{N(0)S(0)}_{\text{beginning value}} + \underbrace{\sum_{i=0}^{j} N(t_i)[S(t_{i+1} \wedge t) - S(t_i)]}_{\text{trading gains}} \text{ a.s.} \tag{1.5}$$

for every $t \in (t_j, t_{j+1}]$ and $j = 0, \ldots, J-1$.

Proof Homework

Remark If S is a continuous process then equation (1.5) can be written as

$$N(t)S(t) = N(0)S(0) + \int_0^t N(u)\, dS(u) \text{ a.s.} \tag{1.6}$$

where the integral is Riemann-Stieltjes path by path.

Definition 1.8 A trading strategy N is *admissible* if it is simple, tight, and its final payoff $N(T)S(T) \in \mathcal{C}$.

Definition 1.9 A consumption plan $x \in \mathcal{C}$ is *marketed* if there exists an admissible trading strategy N such that $N(T)S(T) = x$, in which case N *generates* or *finances* x.

Let $\mathcal{M}$ denote the set of marketed consumption plans. Note that $\mathcal{M}$ is a linear subspace of $\mathcal{C}$.

Definition 1.10 The securities market is *dynamically complete* if $\mathcal{M} = \mathcal{C}$.

Remark With only simple trading strategies, $\mathcal{M}$ will generally be dynamically incomplete unless $\mathcal{C}$ is finite-dimensional.

Definition 1.11 A *free lunch* or *arbitrage opportunity* is an admissible trading strategy N such that $N(0)S(0) \leq 0$, $N(T)S(T) \geq 0$ a.s., and $\mathcal{P}\{N(T)S(T) > 0\} > 0$.

Assume there are no arbitrage opportunities. We want to define the price of $x \in \mathcal{M}$ as the cost $N(0)S(0)$ of the trading strategy that generates x. But what if the trading strategy is not unique?

Proposition 1.3 *Let $x \in \mathcal{M}$ and N_1 and N_2 be admissible trading strategies that finance x. Then the portfolio value processes N_1S and N_2S are indistinguishable.*

Proof Homework

Therefore, the implicit price process S^x of any marketed claim x is uniquely defined by $S^x = NS$ where N is any admissible trading strategy that generates x. Thus we can define the pricing functional p_m on $\mathcal{M}$ by $p_m(x) \equiv S^x(0) = N(0)S(0)$ where N generates x.

Proposition 1.4 *The pricing functional p_m is linear and strictly positive.*

Proof Homework

Proposition 1.5 *Suppose the market is dynamically complete. Then p_m is a continuous linear functional on $L^p(\mathcal{P})$ and hence there exists a unique sdf.*

Definition 1.12 The price system S is *viable* if there exists an admissible trading strategy N^* such that $N^*(0)S(0) \le 0$ and $N^*(T)S(T) \succeq N(T)S(T)$ for every admissible trading strategy N with $N(0)S(0) \le 0$.

Theorem 1.1 implies the price system S is viable if and only if p_m has a strictly positive continuous linear extension p to all of $\mathcal{C}$. Corollary 1.1 implies S is viable if and only if there exists a sdf. Unless the market is dynamically complete, such a sdf is not necessarily unique.

1.6 The Martingale Property

Take security 0 as numeraire and let

$$S^* = \frac{S}{S_0} = (1, \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0}) \ . \tag{1.7}$$

Proposition 1.6 *A simple trading strategy N is tightly self-financing $\Longleftrightarrow$*

$$N(t)S^*(t) = N(0)S^*(0) + \sum_{i=0}^{j} N(t_i)[S^*(t_{i+1} \wedge t) - S^*(t_i)] \ \textit{a.s.} \tag{1.8}$$

for every $t \in (t_j, t_{j+1}]$ and $j = 0, \ldots, J-1$.

For the proof, it suffices to verify that N is tight if and only if $N(t_{j+1})S^*(t_j) = N(t_j)S^*(t_j)$ for every $j = 0, 1, \ldots, J-1$.

Definition 1.13 A probability measure $\mathcal{P}^*$ on $(\Omega, \mathcal{F})$ is an *equivalent martingale measure (emm)* if $\mathcal{P}^* \sim \mathcal{P}$ and S^* is a vector martingale under $\mathcal{P}^*$.

Theorem 1.2 (Harrison and Kreps) *There exists a one-to-one correspondence between sdf's ζ and emm's $\mathcal{P}^*$. The correspondence is given by*

$$\zeta = \frac{S_0(0)}{S_0(T)} \frac{d\mathcal{P}^*}{d\mathcal{P}} \ . \tag{1.9}$$

Proof $\Longleftarrow$: Let $\mathcal{P}^*$ be an emm. Let $\zeta = \frac{S_0(0)}{S_0(T)} \frac{d\mathcal{P}^*}{d\mathcal{P}}$. Take $x \in \mathcal{M}$. Let N be an admissible trading strategy that generates x and let $0 = t_0 < t_1 < \cdots < t_n = T$ be its trading dates. Then

$$\begin{aligned} \mathrm{E}^*\{N(t_{j+1})S^*(t_{j+1})|\mathcal{F}_{t_j}\} &= \mathrm{E}^*\{N(t_j)S^*(t_{j+1})|\mathcal{F}_{t_j}\} && (1.10) \\ &= N(t_j)\mathrm{E}^*\{S^*(t_{j+1})|\mathcal{F}_{t_j}\} && (1.11) \\ &= N(t_j)S^*(t_j) && (1.12) \end{aligned}$$

because N is tight, $N(t_{j+1}) \in \mathcal{F}_{t_j}$, and S^* is a $\mathcal{P}^*$-martingale. Iterating yields

$$\begin{aligned} & \mathrm{E}^*\{N(T)S^*(T)\} = N(0)S^*(0) && (1.13) \\ \Longrightarrow \quad & \mathrm{E}\{N(T)S(T)\tfrac{S_0(0)}{S_0(T)}\tfrac{d\mathcal{P}^*}{d\mathcal{P}}\} = S^x(0) && (1.14) \\ \Longrightarrow \quad & \mathrm{E}\{\zeta x\} = p_m(x) && (1.15) \end{aligned}$$

Thus, ζ is a sdf.

$\Longrightarrow$: Let ζ be a sdf and let $\frac{d\mathcal{P}^*}{d\mathcal{P}} = \frac{S_0(T)}{S_0(0)}\zeta$. Then $\frac{d\mathcal{P}^*}{d\mathcal{P}} > 0$ and $\mathrm{E}\{\frac{d\mathcal{P}^*}{d\mathcal{P}}\} = 1$ so $\frac{d\mathcal{P}^*}{d\mathcal{P}}$ defines a probability measure $\mathcal{P}^* \sim \mathcal{P}$. It remains to show S_k^* is a $\mathcal{P}^*$-martingale for all $k = 1, \ldots, n$. Fix $k > 0$, arbitrary times t and u in $[0, T]$ with $t < u$, and an event $B \in \mathcal{F}_t$. Consider the trading strategy given by $N_j = 0$ for $j \neq 0, k$,

$$N_k(s, \omega) = \begin{cases} 1 & \text{for } s \in (t, u] \text{ and } \omega \in B \\ 0 & \text{otherwise} \ , \end{cases} \tag{1.16}$$

and

$$N_0(s,\omega) = \begin{cases} -S_k^*(t,\omega) & \text{for } s \in (t,u] \text{ and } \omega \in B \\ S_k^*(u,\omega) - S_k^*(t,\omega) & \text{for } s \in (u,T] \text{ and } \omega \in B \\ 0 & \text{otherwise .} \end{cases} \tag{1.17}$$

I.e., in event B, buy and hold one share of security k from time t to u, using security 0 to tightly self-finance. This strategy generates payoff

$$x = N(T)S(T) = (S_k^*(u) - S_k^*(t))1_B S_0(T) \tag{1.18}$$

at time T and has price zero. So

$$\begin{aligned} 0 &= \mathrm{E}\{\zeta x\} && (1.19) \\ &= \mathrm{E}\{\frac{S_0(0)}{S_0(T)}\frac{d\mathcal{P}^*}{d\mathcal{P}}x\} && (1.20) \\ &= \mathrm{E}^*\{S_0(0)(S_k^*(u) - S_k^*(t))1_B\} \ . && (1.21) \end{aligned}$$

Thus $\mathrm{E}^*\{S_k^*(u)1_B\} = \mathrm{E}^*\{S_k^*(t)1_B\}\ \forall\ B \in \mathcal{F}_t$, i.e., $\int_B S_k^*(u)\, d\mathcal{P}^* = \int_B S_k^*(t)\, d\mathcal{P}^*\ \forall\ B \in \mathcal{F}_t$, which means $\mathrm{E}^*\{S_k^*(u)|\mathcal{F}_t\} = S_k^*(t)$.□

Definition 1.14 A consumption plan $x \in \mathcal{C}$ is *priced* if all strictly positive continuous linear extensions of p_m to $\mathcal{C}$ agree on the price of x.

Proposition 1.7 *The following conditions are equivalent.*

1. *Every consumption plan is priced.*
2. *There exists a unique strictly positive continuous linear extension of p_m to $\mathcal{C}$.*
3. *There exists a unique sdf.*
4. *There exists a unique emm.*

Proposition 1.8 *Let $x \in \mathcal{M}$ and let S^x be its implicit price process. If there exists an emm $\mathcal{P}^*$ then S^x/S_0 is a $\mathcal{P}^*$-martingale. In particular,*

$$S^x(t) = S_0(t)\mathrm{E}^*\{\frac{x}{S_0(T)}|\mathcal{F}_t\} \ a.s.\ \forall\ t \in [0,T] \ . \tag{1.22}$$

Proof Let N be an admissible trading strategy that generates x and let $0 = t_0 < t_1 < \cdots < t_J = T$ be its trading dates. Let $t \in [0, T]$ and let j be such that $t \in (t_{j-1}, t_j]$. Then by Proposition 1.6

$$\begin{aligned}
& \tfrac{x}{S_0(T)} = N(t)S^*(t) + \textstyle\sum_{i=j}^{J} N(t_{i-1})[S^*(t_i) - S^*(t_{i-1} \vee t)] \\
\Longrightarrow \quad & \mathrm{E}^*\{\tfrac{x}{S_0(T)}|\mathcal{F}_t\} = N(t)S^*(t) + \mathrm{E}^*\{\textstyle\sum_{i=j}^{J} N(t_{i-1})\mathrm{E}^*\{S^*(t_i) - S^*(t_{i-1} \vee t)|\mathcal{F}_{t_{i-1} \vee t}\}|\mathcal{F}_t\} \\
\Longrightarrow \quad & \mathrm{E}^*\{\tfrac{x}{S_0(T)}|\mathcal{F}_t\} = N(t)S^*(t) = N(t)S(t)/S_0(t) = S^x(t)/S_0(t) \\
\Longrightarrow \quad & S^x(t) = S_0(t)\mathrm{E}^*\{\tfrac{x}{S_0(T)}|\mathcal{F}_t\} \ . \ \Box
\end{aligned}$$

2 The Standard Continuous-Time Financial Market

These notes are taken primarily from Sections IV and V of Domenico Cuoco's lecture notes and Chapter 1 of Karatzas and Shreve (1998).

2.1 Probabilistic Setting

There is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and a standard Brownian motion $B = (B_1, \ldots, B_d)$ on time interval $[0, T]$. The filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t=0}^T$ is the $\mathcal{P}$-augmentation of the filtration generated by B, also called the natural filtration, i.e., the smallest filtration such that $\mathcal{F}_0$ contains all the null sets and $B_s \in \mathcal{F}_t$ for every $0 \leq s \leq t \leq T$. This filtration satisfies the usual conditions. We also assume $\mathcal{F} = \mathcal{F}_T$. We interpret $\mathcal{P}$ as the subjective probability measure believed by investors in the economy and $\mathcal{F}_t$ as the information available to investors at time t.

2.2 Consumption

The consumption space $\mathcal{C}$ is the set of pairs (c, W), where c is a progressively measurable consumption rate process with $\int_0^T |c(t)|\, dt < \infty$ a.s. and W is a random variable representing terminal wealth.

2.3 Securities Market

There are $n+1$ securities traded with ex-dividend prices $S = (S_0, \ldots, S_n)$. Security 0, the "bond," is a "locally riskless" money market account earning the *instantaneous riskless rate* r. In particular, S_0 is a strictly positive, adapted process satisfying

$$\frac{dS_0(t)}{S_0(t)} = r(t)\, dt \ , \tag{2.1}$$

or

$$S_0(t) = S_0(0)e^{\int_0^t r(u)\,du} \;, \tag{2.2}$$

where r is a progressively measurable process with

$$\int_0^T |r(t)|\,dt < \infty \text{ a.s.} \tag{2.3}$$

The other n assets are strictly positive "risky" Itô processes satisfying

$$\frac{dS_k(t)}{S_k(t)} = [\mu_k(t) - \delta_k(t)]\,dt + \underbrace{\sigma_k(t)}_{1\times d}\,dB(t) \;, \tag{2.4}$$

where the n-dimensional *instantaneous expected return* process $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$ is progressively measurable and satisfies

$$\int_0^T |\mu(t)|\,dt < \infty \text{ a.s.} \quad , \tag{2.5}$$

the n-dimensional *dividend payout rate* process $\delta = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}$ is progressively measurable and satisfies

$$\int_0^T |\delta(t)|\,dt < \infty \text{ a.s.} \quad , \tag{2.6}$$

and the $n \times d$-matrix-valued *volatility* process $\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$ is progressively measurable and satisfies

$$\int_0^T \text{trace}[\sigma(t)\sigma'(t)]\,dt < \infty \text{ a.s.} \tag{2.7}$$

This has the solution

$$S_k(t) = S_k(0)e^{\int_0^t [\mu_k(u) - \delta_k(u) - |\sigma_k(u)|^2/2]\,du + \int_0^t \sigma_k(u)\,dB(u)} \;. \tag{2.8}$$

Without loss of generality, assume $n \leq d$.

The economic effect of the dividends is essentially that if a share of security k is held in a portfolio for an instant in time, then it changes portfolio value by $dS_k + \delta_k S_k\, dt = \mu_k S_k\, dt + \sigma_k S_k\, dB$. Thus, holding μ_k constant, the effect on the portfolio is invariant to δ_k. Nevertheless, we wish to keep track of the dividend rate, because it affects the ex-dividend security price, which is the basis for many contracts written in practice.

Previously, our definition of stochastic integrals and Itô processes required the integrand to be a predictable process. However, when the integrator is a continuous, square-integrable martingale, such as Brownian motion, the stochastic integral is well-defined for integrands which are merely progressive and satisfy the almost sure square integrability condition on σ above. In fact, if the integrator is Brownian motion, it is enough that the integrand be merely adapted and square integrable (see p. 131 of Karatzas and Shreve (1991)).

Special Case: The Markov Model In order to exploit the machinery of p.d.e.'s, it is sometimes useful to specialize the model to the case in which the market coefficients r, μ, δ, and σ are functions of (S, Y, t) where Y is a vector of state variables with

$$Y(t) = Y(0) + \int_0^t \mu_Y(Y(u), u)\, du + \int_0^t \sigma_Y(Y(u), u)\, dB(u)\ . \tag{2.9}$$

Under appropriate Lipschitz and growth conditions on the coefficients, the stochastic differential equations governing S and Y have a unique solution (S, Y) which is a Markov process.

2.4 Trading Strategies

As before, we can describe a trading strategy in terms of the row-vector-valued process $N(t)$, which indicates the number of shares of each security held at time t. In the setting here, we will essentially just define the self-financing condition to be $N(t)S(t) = N(0)S(0) + \int_0^t N(u)\, dS(u)$, i.e., $d(N(t)S(t)) = N(t)dS(t)$, in the absence of dividends. However, the square integrability condition required for the stochastic trading gains integral to be well-defined is most easily expressed if we describe the trading strategy in terms of the (dollar) value of holdings process $\pi(t)$, where $\pi_k(t) = N_k(t)S_k(t)$.

Definition 2.1 A *trading strategy* is an $n+1$-dimensional adapted process $\bar{\pi} = (\pi_0, \pi_1, \ldots, \pi_n) \equiv (\pi_0, \underbrace{\pi}_{1\times n})$ satisfying

$$\int_0^T |\pi(t)\sigma(t)|^2 \, dt < \infty \text{ a.s.} \quad , \tag{2.10}$$

where $\pi_k(t)$ represents the value (dollar amount) invested in security k at time t.

Definition 2.2 A trading strategy $\bar{\pi}$ is *self-financing* if for every $t \in [0, T]$

$$\bar{\pi}(t)1 = \bar{\pi}(0)1 + \int_0^t \pi_0(u)r(u)\, du + \int_0^t \pi(u)\mu(u)\, du + \int_0^t \pi(u)\sigma(u)\, dB(u) - C(t) \ , \tag{2.11}$$

where $C(t)$ is a nondecreasing "withdrawal" process with $C(0) = 0$, and $\bar{\pi}$ is *tight* if $C = 0$.

The withdrawal process $C(t)$ is the cumulative wealth withdrawn from the portfolio by time t. It is there to allow for trading strategies that might only super-replicate, or "loosely" finance a given comsumption plan, which would be of interest when markets are incomplete. The notation 1 above indicates a vector of 1's with suitable dimension.

Connection to the simple self-financing condition For simplicity, assume no dividends. The tight self-financing condition above amounts to

$$N(t)S(t) = N(0)S(0) + \int_0^t N(u)\, dS(u) \ . \tag{2.12}$$

In the context of simple trading strategies, we called the trading strategy tight if

$$N(t_{j+1})S(t_j) = N(t_j)S(t_j) \text{ a.s.} \quad , \tag{2.13}$$

i.e., if the cost of the new portfolio always equals the proceeds of the sale of the old portfolio, and then we proved that this is equivalent to equation (2.12). In the continuous trading case we are essentially taking equation (2.12) as the definition of tight self-financing. Is there a continuous-time analog to equation (2.13)? Consider the value of the portfolio,

$$X(t) \equiv N(t)S(t) \ , \tag{2.14}$$

and suppose N is an Itô process. Then by Itô's lemma,

$$dX(t) = N(t)dS(t) + S'(t)dN(t) + d\langle N, S\rangle_t \ . \tag{2.15}$$

The self-financing condition amounts to

$$S'(t)dN(t) + d\langle N, S\rangle_t = 0 \ , \tag{2.16}$$

which is essentially the continuous-time analog of equation (2.13).

Definition 2.3 A trading strategy $\bar{\pi}$ *finances* or *generates* a consumption plan $(c, W) \in \mathcal{C}$ if $\forall\ t \in [0, T]$

$$\bar{\pi}(t)1 = \bar{\pi}(0)1 + \int_0^t \pi_0(u)r(u)\,du + \int_0^t \pi(u)\mu(u)\,du + \int_0^t \pi(u)\sigma(u)\,dB(u) - \int_0^t c(u)\,du - C(t) \ , \tag{2.17}$$

and $\bar{\pi}(T)1 = W$ for some nondecreasing process C with $C(0) = 0$. $\bar{\pi}$ is *tight* if this holds with $C = 0$.

The financing (or self-financing) constraint allows us to rewrite the evolution equation (2.17) in terms of portfolio value $X(t) = \bar{\pi}(t)1$ and eliminate π_0 using $\pi_0(t) = X(t) - \pi(t)1$:

$$\begin{aligned} X(t) &= X(0) + \int_0^t r(u)X(u)\,du + \int_0^t \pi(u)(\mu(u) - r(u)1)\,du + \int_0^t \pi(u)\sigma(u)\,dB(u) \\ &\quad - \int_0^t c(u)\,du - C(t) \ . \end{aligned} \tag{2.18}$$

Therefore, to specify a portfolio process, we need only specify the risky asset holdings π, the consumption rate c, the withdrawal process C, and the initial value $X(0)$.

Definition 2.4 A trading strategy $\bar{\pi}$ is an *arbitrage opportunity* if it finances a nonnegative consumption plan $(c, W) \in \mathcal{C}$ with $\mathcal{P}\{\int_0^T c(t)\,dt + W > 0\} > 0$ and $\bar{\pi}(0)1 \leq 0$.

2.5 Equivalent Martingale Measures

For ease of exposition, introduce the "discount factor" $\beta(t,u) = S_0(t)/S_0(u) = e^{-\int_t^u r(s)\,ds}$ for discounting cash flows from time u back to time t, $u \geq t$, and for simplicity let $\beta(t) = \beta(0,t) = e^{-\int_0^t r(s)\,ds}$. Consider the discounted price and dividend processes $S^*(t) \equiv \beta(t)S(t)$ and $D^*(t) \equiv (\int_0^t S_1^*(u)\delta_1(u)\,du, \ldots, \int_0^t S_n^*(u)\delta_n(u)\,du)$.

Definition 2.5 A probability measure $\mathcal{P}^*$ on $(\Omega, \mathcal{F}, \mathcal{P})$ is an *equivalent martingale measure (emm)* if $\mathcal{P}^* \sim \mathcal{P}$ and the discounted gains process $G^*(t) \equiv S^*(t) + D^*(t)$ is a local martingale under $\mathcal{P}^*$.

Note that this is a relaxed version of the definition in the last section.

Definition 2.6 A *market price of risk* is a progressively measurable d-dimensional process θ satisfying

$$\sigma(t)\theta(t) = \mu(t) - r(t)1 \text{ a.s.} \tag{2.19}$$

To what extent does no arbitrage imply the existence of an emm? As we show below, no arbitrage implies the existence of a market price of risk, which is the building block of an emm. However, we will need additional restrictions on security prices to complete the construction.

Proposition 2.1 *(From Theorem 4.2 of Karatzas and Shreve (1998)) If there is no arbitrage, then there exists a market price of risk.*

For the case $\text{rank}(\sigma(t)) \equiv n$, a solution is

$$\theta(t) = \sigma'(t)[\sigma(t)\sigma'(t)]^{-1}[\mu(t) - r(t)1]\ . \tag{2.20}$$

If $\text{rank}(\sigma(t)) < n$ for some (ω, t), then some of the securities are redundant at least some of the time. Nevertheless, the result holds.

Sketch of Proof If there is no arbitrage, then whenever a trading strategy is locally riskless, i.e., whenever $\pi(t)\sigma(t) = 0$, it must be the case that the resulting portfolio earns the riskless rate, i.e., $\pi(t)[\mu(t) - r(t)1] = 0$. From linear algebra, this statement is equivalent to the existence of a process $\theta(t)$ that solves equation (2.19). The rest of the proof involves establishing its progressive measurability.

The idea is that given a row vector of risk loadings σ_k associated with a given security or portfolio, $\sigma_k\theta$ indicates the risk premium the security or portfolio must offer. The absence of arbitrage guarantees that risk is priced that way.

Now for the additional restrictions on security prices that ensures the existence of an emm:

Proposition 2.2 *If there exists a market price of risk θ such that*

$$\int_0^T |\theta(t)|^2 \, dt \leq \infty \;\; a.s. \quad , \tag{2.21}$$

and the process

$$Z(t) \equiv e^{-\int_0^t \theta'(u)\, dB(u) - \frac{1}{2}\int_0^t |\theta(u)|^2 \, du} \tag{2.22}$$

is a martingale, then $\mathcal{P}^*$ defined by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z(T)$ is an emm.

Proof $\mathcal{P}^* \sim \mathcal{P}$ since $\frac{d\mathcal{P}^*}{d\mathcal{P}} > 0$. It remains to show G^* is a $\mathcal{P}^*$-local martingale. First note

$$B^*(t) = B(t) + \int_0^t \theta(u)\, du \tag{2.23}$$

is a Brownian motion under $\mathcal{P}^*$ by the Girsanov Theorem. Next,

$$dG_k^*(t) = d(\beta(t)S_k(t)) + (\beta(t)S_k(t)\delta_k(t))\, dt \tag{2.24}$$

$$= S_k^*(t)[(\mu_k(t) - r(t))\, dt + \sigma_k(t)\, dB(t)] \tag{2.25}$$

$$= S_k^*(t)\sigma_k(t)\, dB^*(t) \tag{2.26}$$

Thus G^* is a $\mathcal{P}^*$-local martingale. □

A sufficient condition for Z to be a martingale is the Novikov condition $\mathrm{E}\{e^{\frac{1}{2}\int_0^T |\theta(t)|^2\,dt}\} < \infty$.

Next, to what extent can we say that the existence of an emm implies no arbitrage? First we need to rule out the use of doubling strategies (see Harrison and Kreps (1979) or Example 2.3 on p. 8 of Karatzas and Shreve (1998)).

Definition 2.7 A self-financing trading strategy $\bar{\pi}$ is *tame* if the discounted portfolio value process $\beta\bar{\pi}1$ satisfies

$$\beta\bar{\pi}1 \geq -K \text{ a.s.} \tag{2.27}$$

for some finite constant K.

Tameness does not rule out all arbitrage opportunities (see Karatzas and Shreve (1998) p. 11 for an example of a tame arbitrage opportunity). However, we have the following:

Proposition 2.3 *If there exists an emm, then there are no tame arbitrage opportunities.*

The proof uses three results:

Lemma 2.1 *The quadratic variation of an Itô process is invariant to an equivalent change of measure.*

This is because probability limits are.

Lemma 2.2 *(From Proposition 2.24 of Karatzas and Shreve (1991) on p. 147) If M is a continuous local martingale and X is a progressively measurable process satisfying $\int_0^T X(t)\,d\langle M\rangle_t < \infty$ a.s., then $\int_0^t X(t)\,dM(t)$ is also a continuous local martingale.*

Lemma 2.3 *A local martingale bounded below is a supermartingale.*

This follows from Fatou's lemma.

Proof of Proposition 2.3 Suppose Q is an emm and suppose $\bar{\pi}$ is a tame trading strategy generating a nonnegative consumption plan (c, W) with $\mathcal{P}\{\int_0^T c(t)\,dt + W > 0\} > 0$. Then $Q\{\int_0^T c(t)\,dt + W > 0\} > 0$. Because $\bar{\pi}$ is self-financing, $X(t) \equiv \bar{\pi}(t)1$ satisfies

$$\begin{aligned} d(\beta(t)X(t)) &= \beta(t)dX(t) - \beta(t)r(t)X(t)\,dt && (2.28)\\ &= \beta(t)\pi(t)[(\mu(t) - r(t)1)\,dt + \sigma(t)\,dB(t)] && (2.29)\\ &\quad -\beta(t)c(t)\,dt - \beta(t)dC(t) && (2.30)\\ &= N(t)\,dG^*(t) - \beta(t)c(t)\,dt - \beta(t)dC(t) && (2.31)\end{aligned}$$

$$\Longrightarrow \beta(t)X(t) + \int_0^t \beta(u)c(u)\,du + \int_0^t \beta(u)dC(u) = X(0) + \int_0^t N(u)\,dG^*(u) \qquad (2.32)$$

is a Q-local martingale, because G^* is a Q-local martingale and $Q\{\int_0^T |\beta(t)\pi(t)\sigma(t)|^2\,dt < \infty\} = 1$. Furthermore, $\beta(t)X(t) + \int_0^t \beta(u)c(u)\,du + \int_0^t \beta(u)dC(u)$ is bounded below because $\bar{\pi}$ is tame, so it is a Q-supermartingale. Therefore,

$$\begin{aligned} X(0) &\geq \mathrm{E}^Q\{\beta(T)X(T) + \int_0^T \beta(t)c(t)\,dt + \int_0^T \beta(t)dC(t)\} && (2.33)\\ &\geq \mathrm{E}^Q\{\beta(T)W + \int_0^T \beta(t)c(t)\,dt\} > 0\ .\Box && (2.34)\end{aligned}$$

More generally, let $\mathcal{C}_b$ denote the subset of consumption plans $(c, W) \in \mathcal{C}$ such that the random variable $\int_0^T \beta(t)c(t)^-\,dt + \beta(T)W^-$ is essentially bounded. I.e., $\mathcal{C}_b$ is the subset of essentially lower bounded consumption plans.

Proposition 2.4 *Suppose Q is an emm and $\bar{\pi}$ is a tame trading strategy financing a consumption plan $(c, W) \in \mathcal{C}_b$. Then the process $\beta(t)\bar{\pi}(t)1 + \int_0^t \beta(u)c(u)\,du + \int_0^t \beta(u)dC(u)$ is a Q-local martingale bounded below, and thus a Q-supermartingale. It follows that*

$$\beta(t)\bar{\pi}(t)1 \geq \mathrm{E}^Q\{\int_t^T \beta(u)c(u)\,du + \beta(T)W|\mathcal{F}_t\}\ . \qquad (2.35)$$

Remark Tame trading strategies cannot turn zero wealth into positive wealth, but one could turn positive wealth into zero wealth by running a doubling strategy in reverse. Such suicidal strategies pose no problems and will be ruled out by optimality conditions.

Corollary 2.1 *If there exists a market price of risk satisfying the conditions of Proposition 2.2, then there exists an emm and there are no tame arbitrage opportunities.*

Lemma 2.4 *(From Theorem 1.4.2, Lemma 1.4.7, and Remark 1.4.11 of Karatzas and Shreve (1998) on pp. 12-16) If there is no arbitrage, then there exists a unique market price of risk* $\theta(t) \in \mathcal{K}^{\perp}(\sigma(t))$ *and thus every market price of risk* $\hat{\theta}(t)$ *can be written as* $\hat{\theta}(t) = \theta(t) + \nu(t)$ *where* $\sigma(t)\nu(t) = 0$ *a.s., for almost every* $t \in [0, T]$*. If rank*$(\sigma(t)) = n$ *then* $\theta(t) = \sigma'(t)[\sigma(t)\sigma'(t)]^{-1}[\mu(t) - r(t)1]$.

The standard market Hereafter we shall assume there exists a unique market price of risk $\theta(t) \in \mathcal{K}^{\perp}(\sigma(t))$ satisfying the conditions of Proposition 2.2, which we will call the *standard market price of risk*. Let $Z(t) \equiv e^{-\int_0^t \theta'(u)\,dB(u) - \frac{1}{2}\int_0^t |\theta(u)|^2\,du}$. We will call the measure $\mathcal{P}^*$ defined by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z(T)$ the *standard emm*. Let $B^*(t) \equiv B(t) + \int_0^t \theta(u)\,du$ be the Brownian motion under $\mathcal{P}^*$.

Note that under any tight trading strategy π, portfolio value evolves according to

$$d(\beta(t)X(t)) = \beta(t)\pi(t)\sigma(t)\,dB^*(t) - \beta(t)c(t)\,dt\ . \tag{2.36}$$

Definition 2.8 A trading strategy $\bar{\pi}$ is *martingale generating* if $\int_0^t \beta(u)\pi(u)\sigma(u)\,dB^*(u)$ is a $\mathcal{P}^*$-martingale.

Proposition 2.5 *Suppose a tame trading strategy* $\bar{\pi}$ *finances a consumption plan* $(c, W) \in \mathcal{C}_b$*. Then*

$$\beta(t)\bar{\pi}(t)1 = \mathrm{E}^*\{\int_t^T \beta(u)c(u)\,du + \beta(T)W|\mathcal{F}_t\} \quad \forall t \in [0, T] \tag{2.37}$$

if and only if $\bar{\pi}$ *is tight and martingale generating.*

Proof Homework

2.6 Market Completeness

Let $\mathcal{C}_b^1$ denote the set of consumption plans $(c, W) \in \mathcal{C}_b$ with

$$\mathrm{E}^*\{\int_0^T \beta(t)c(t)\,dt + \beta(t)W\} < \infty \tag{2.38}$$

Definition 2.9 The market is *dynamically complete* if every consumption plan in $\mathcal{C}_b^1$ can be financed by a tame, tight, martingale-generating trading strategy.

Proposition 2.6 *(From Proposition 1.6.2 of Karatzas and Shreve (1998) on p. 22) The market is dynamically complete if and only if every consumption plan* $(c, W) \in \mathcal{C}$ *satisfying*

$$\mathrm{E}^*\{\int_0^T \beta(u)|c(u)|\,du + \beta(T)|W|\} < \infty \tag{2.39}$$

can be financed by some tight, martingale-generating trading strategy.

The point is that if the market is complete in the sense of Definition 2.9, then consumption plans satisfying equation (2.39) can be financed, even if they are unbounded below.

Theorem 2.1 *(From Theorem 1.6.6 of Karatzas and Shreve (1998) on p. 24) The market is complete* $\Longleftrightarrow n = d$ *and* $\sigma(t)$ *is nonsingular a.s., a.e.* $t \in [0, T]$.

The proof uses the following result.

Lemma 2.5 *(Representation of* $\mathcal{P}^*$*-Martingales, Lemma 1.6.7 of Karatzas and Shreve (1998) on p. 24) Let* $\{M^*(t), \mathcal{F}_t; 0 \le t \le T\}$ *be a* $\mathcal{P}^*$*-martingale. Then there exists a progressively measurable process* φ *such that* $\int_0^T |\varphi(u)|^2\,du < \infty$ *a.s. and*

$$M^*(t) = M^*(0) + \int_0^t \varphi'(u)\,dB^*(u) \text{ a.s.} \tag{2.40}$$

The issue is that the filtration is generated by B, not B^*.

Proof Let $M(t) \equiv \mathrm{E}\{Z(T)M^*(T)|\mathcal{F}_t\}$. $M(t)$ is a $\mathcal{P}$-martingale, so it has representation $M(t) = M(0) + \int_0^t \psi'(u)\, dB(u)$ for some progressively measurable, almost surely square integrable process ψ. From the "Bayes Rule" for conditional expectation under an equivalent measure, $M^*(t) = \mathrm{E}^*\{M^*(T)|\mathcal{F}_t\} = M(t)/Z(t)$. By Itô's Lemma, $dM^*(t) = \frac{\psi'(t)+M(t)\theta'(t)}{Z(t)}\, dB^*(t)$. Therefore, $M^*(t) = M^*(0) + \int_0^t \varphi'(u)\, dB^*(u)$ with $\varphi(t) \equiv \frac{\psi(t)+M(t)\theta(t)}{Z(t)}$, whose square integrability follows from that of ψ, θ, and the continuity of M and Z. □

Proof of Theorem 2.1 $\Longleftarrow$: Suppose $(c, W) \in \mathcal{C}_b^1$. Let

$$M^*(t) \equiv \mathrm{E}^*\{\int_0^T \beta(u)c(u)\, du + \beta(T)W|\mathcal{F}_t\} \ . \tag{2.41}$$

This $\mathcal{P}^*$-martingale has representation $M^*(t) = M^*(0) + \int_0^t \varphi'(u)\, dB^*(u)$ for some progressively measurable process φ such that $\int_0^T |\varphi(u)|^2\, du < \infty$ a.s. Let $\pi(t) = \varphi'(t)\sigma^{-1}(t)/\beta(t)$, which is martingale-generating. Let $\pi_0(t) = [M^*(t) - \int_0^t \beta(u)c(u)\, du]/\beta(t) - \pi(t)1$. Then, since $\pi_0(t) + \pi(t)1 = \bar{\pi}(t)1$, we have $\forall\ t \in [0, T]$,

$$\beta(t)\bar{\pi}(t)1 + \int_0^t \beta(u)c(u)\, du = M^*(t) \tag{2.42}$$

$$= M^*(0) + \int_0^t \beta(u)\pi(u)\sigma(u)\, dB^*(u) \tag{2.43}$$

$$= \bar{\pi}(0)1 + \int_0^t \beta(u)\pi(u)\sigma(u)\, dB^*(u) \ , \tag{2.44}$$

so $\bar{\pi}$ tightly finances (c, W). Finally,

$$\beta(t)\bar{\pi}(t)1 = M^*(t) - \int_0^t \beta(u)c(u)\, du = \mathrm{E}^*\{\int_t^T \beta(u)c(u)\, du + \beta(T)W|\mathcal{F}_t\} \tag{2.45}$$

implies $\bar{\pi}$ is tame because $(c, W) \in \mathcal{C}_b^1$. So the market is complete.

$\Longrightarrow$: From Lemma 1.6.9 of Karatzas and Shreve (1998) on p. 26, there exists a bounded, progressively measurable, d-dimensional process $\nu(t)$ such that $\sigma(t)\nu(t) = 0\ \forall\ t \in [0, T]$ and $\nu(t) \neq 0$ whenever $\mathrm{rank}(\sigma(t)) < d$. Let $W = [1 + \int_0^T \nu'(t)\, dB^*(t)]/\beta(T)$. Note that $\mathrm{E}\{\beta(T)|W|\} < \infty$ and $\mathrm{E}^*\{\beta(T)W\} = 1$, so there exists a martingale-generating trading strategy π such that

$$\int_0^T \beta(t)\pi(t)\sigma(t)\, dB^*(t) = \beta(T)W - 1 = \int_0^T \nu'(t)\, dB^*(t) \ . \tag{2.46}$$

Now $\int_0^t \beta(u)\pi(u)\sigma(u)\, dB^*(u)$ and $\int_0^t \nu'(u)\, dB^*(u)$ are both $\mathcal{P}^*$-martingales, and they are equal because their last elements are. Therefore,

$$\beta(t)\pi(t)\sigma(t) = \nu'(t) \text{ a.s. a.e. } t \in [0, T] \; . \tag{2.47}$$

This implies $\nu(t)$ is in the range of $\sigma'(t)$ which is equal to $\mathcal{K}^{\perp}(\sigma(t))$. But $\nu(t) \in \mathcal{K}(\sigma(t))$. So $\nu(t) = 0$ a.s. a.e., which implies $\text{rank}(\sigma(t)) = d$ a.s. a.e. □

Corollary 2.2 *In a complete market there exists a unique market price of risk* $\theta(t) \equiv \sigma^{-1}(t)[\mu(t) - r(t)1]$.

3 European Contingent Claims

These notes are taken primarily from Sections IV and V of Domenico Cuoco's lecture notes and Chapter 2 of Karatzas and Shreve (1998). Suppose we are in a standard continuous-time financial market.

Definition 3.1 A *European contingent claim (ecc)* is a payoff $(c, W) \in \mathcal{C}$.

Definition 3.2 Let $x = (c, W) \in \mathcal{C}$ be a ecc. The *price* of x at time t is

$$S^x(t) = \min\{\bar{\pi}(t)1 : \bar{\pi} \text{ is a tame trading strategy that finances } x\}\ , \tag{3.1}$$

provided the minimum exists.

Theorem 3.1 *Suppose that the market is dynamically complete and let* $x = (c, W) \in \mathcal{C}_b^1$ *be a ecc. Then*

$$\beta(t)S^x(t) = \mathrm{E}^*\{\int_t^T \beta(u)c(u)\,du + \beta(T)W|\mathcal{F}_t\} \tag{3.2}$$

Proof From the definition of a complete market, there exists a tight, tame, martingale-generating strategy $\bar{\pi}$ that finances x, and by Proposition 2.4 it satisfies

$$\beta(t)\bar{\pi}(t)1 = \mathrm{E}^*\{\int_t^T \beta(u)c(u)\,du + \beta(T)W|\mathcal{F}_t\}\ . \tag{3.3}$$

For any other tame trading strategy that finance x,

$$\beta(t)\bar{\pi}(t)1 \geq \mathrm{E}^*\{\int_t^T \beta(u)c(u)\,du + \beta(T)W|\mathcal{F}_t\}\ , \tag{3.4}$$

by Proposition 2.4, and the result follows. □

Definition 3.3 A *replicating portfolio* or *hedging portfolio* for an ecc x is a tame, martingale-generating trading strategy that tightly finances x.

In what sense is S^x an equilibrium price for an ecc x? If a replicating portfolio of x exists, we would like to argue that the equilibrium price of x must be the price of its replicating portfolio. However, if we are restricted to tame trading strategies, we may not be able turn an apparent mispricing into an arbitrage. In practical terms, this is because the natural long-short position to implement might get closed out if its value gets marked below an institutionally imposed lower bound before the end of the trading horizon. Nevertheless, the concept of the price of the replicating portfolio is still the most economically natural notion of the price of x.

Note that under $\mathcal{P}^*$, the drift of the price of each security k is $r(t)S_k(t) - \delta_k(t)S_k(t)$ and the drift of $S^x(t)$ is $r(t)S^x(t) - c(t)$. In other words, gross of payout, every security price appreciates at the riskless rate. The intuition is that if we change the measure to equate the appreciation rates of all the basic securities, then all tight portfolios of those securities must appreciate at that same rate. This, together with the existence of martingale-generating replicating portfolios, produces the pricing equation (3.2).

Definition 3.4 A ecc $x = (c, W) \in \mathcal{C}_b^1$ is *path-independent* if $c(t) = \varphi_1(S_1(t), \ldots, S_n(t), t)$ and $W = \varphi_2(S_1(T), \ldots S_n(T))$ for continuous functions $\varphi_1 : \mathcal{R}_+^n \times [0, T] \to \mathcal{R}$ and $\varphi_2 : \mathcal{R}_+^n \to \mathcal{R}$.

In the Markovian model, if x is path-independent then $S^x(t) = F(S_1(t), \ldots, S_n(t), Y(t), t)$ for some real function F. If F is sufficiently smooth for an application of Itô's lemma, then it must satisfy the following p.d.e., which says the drift of F from the martingale condition must equal the drift of F from Itô's lemma:

$$\begin{aligned} rF - \varphi_1 &= F_S' I_S (r1 - \delta) + F_Y'(\mu_Y - \sigma_Y \theta) + F_t + \frac{1}{2}\mathrm{tr}(F_{SS} I_S \sigma \sigma' I_S) \\ &\quad + \frac{1}{2}\mathrm{tr}(F_{YY} \sigma_Y \sigma_Y') + \mathrm{tr}(F_{SY}' I_S \sigma \sigma_Y') \end{aligned} \tag{3.5}$$

where I_S is a diagonal matrix with diagonal elements $S_1, S_2, \ldots, S_n$ and the derivatives F_S and F_{SS} are with respect to the vector $(S_1, S_2, \ldots, S_n)$. This is subject to the boundary condition $F(S_1(T), \dots S_n(T), Y(T), T) = \varphi_2(S_1(T), \dots S_n(T))$. Furthermore, by matching the diffusion coefficient from the wealth evolution equation with the diffusion coefficient from Itô's lemma, we have that the diffusion coefficient of $S^x(T)$ must satisfy

$$\pi^x \sigma = F_S' I_S \sigma + F_Y' \sigma_Y \ , \tag{3.6}$$

which gives a way to compute the replicating trading strategy π^x.

Special Case: Complete Market with Constant Coefficients

Suppose r, μ, σ, and δ are constant.

Definition 3.5 A function $f : \mathcal{R}^d \times [0,T] \longrightarrow \mathcal{R}$ *satisfies a polynomial growth condition (pgc)* if there exist positive constants k_1 and k_2 such that

$$|f(x,t)| \leq k_1(1 + |x|^{k_2}) \ \ \forall (x,t) \in \mathcal{R}^d \times [0,T] \ . \tag{3.7}$$

Theorem 3.2 (Feynman-Kac) *Suppose the functions φ_1, φ_2, and*

$$F(S_1(t), \dots, S_n(t), t) \ \equiv \ \mathrm{E}^*\{\int_t^T e^{-r(u-t)} \varphi_1(S_1(u), \dots, S_n(u), u)\, du \tag{3.8}$$

$$+ e^{-r(T-t)} \varphi_2(S_1(T), \dots S_n(T)) | \mathcal{F}_t\} \tag{3.9}$$

each satisfy a pgc. Then F satisfies the p.d.e.

$$rF - \varphi_1 = F_S' I_s(r1 - \delta) + F_t + \frac{1}{2} tr(F_{SS} I_S \sigma \sigma' I_S) \tag{3.10}$$

subject to $F(S_1(T), \dots S_n(T), T) = \varphi_2(S_1(T), \dots S_n(T))$. There is no other solution to this p.d.e. that satisfies a pgc.

In addition, the trading strategy π^x that generates $x = (\varphi_1, \varphi_2)$ is given by $\pi_k^x = S_k \partial F / \partial S_k$, i.e., $N_k = \partial F / \partial S_k$, for $k = 1, 2, \dots, n$, and $\pi_0^x = F - \pi^x 1$.

Example 1 (Black-Scholes-Merton Call Option Model) Assume $n = d = 1$, r and σ are constant, and for ease of exposition, begin by assuming no dividends. We have

$$\beta(t,u) = e^{-r(u-t)} \tag{3.11}$$

and the price of the risky asset or "stock" follows

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma\,dB(t)\ . \tag{3.12}$$

Consider a call on the stock with time T payoff $\varphi_2(S(T)) = (S(T)-K)^+$ for some positive constant K.

- The Black-Scholes argument (assume μ is constant): The time t call price $C(t) = c(S(t),t)$ because S is Markov. Consider a portfolio short one call and long c_S shares of the stock. The portfolio value is $X = c_S S - c + \pi_0$ where π_0 adjusts to make the portfolio self-financing. On one hand, from the wealth evolution equation $dX = N\,dS$ and Itô's lemma,

$$\begin{aligned} dX &= c_S\,dS - dc + r\pi_0\,dt && (3.13)\\ &= c_S\,dS - (c_S\,dS + c_t\,dt + \frac{1}{2}c_{SS}\sigma^2S^2\,dt) + r\pi_0\,dt && (3.14)\\ &= -c_t\,dt - \frac{1}{2}c_{SS}\sigma^2S^2\,dt + r\pi_0\,dt\ . && (3.15)\end{aligned}$$

On the other hand, by no arbitrage, since the portfolio is locally riskless, i.e., has zero diffusion, it must appreciate at the riskless rate:

$$dX = rX\,dt = r(c_S S - c + \pi_0)\,dt\ . \tag{3.16}$$

Therefore, it must be that

$$\underbrace{rSc_s + \frac{1}{2}\sigma^2S^2c_{SS} + c_t}_{\text{drift of } c \text{ under } \mathcal{P}^* \text{ by Ito's lemma}} \quad = \quad \underbrace{rc}_{\text{drift of } c \text{ under p}^* \text{ by no arbitrage}}\ . \tag{3.17}$$

They recognized this as the heat equation from physics, with solution

$$c(S(t),t) = \mathrm{E}^*\{e^{-r(T-t)}(S(T)-K)^+|S(t)\} \tag{3.18}$$

which is the same as our result from equation (3.2).

- Computational trick with change of numeraire/change of measure: Now we can easily incorporate constant dividend rate δ. Write

$$\begin{aligned} C(t) &= \mathrm{E}^*\{e^{-r(T-t)}(S(T)-K)1_{\{S(T)>K\}}|\mathcal{F}_t\} && (3.19)\\ &= S(t)e^{-\delta(T-t)}\mathrm{E}^*\{e^{\sigma(B^*(T)-B^*(t))-\sigma^2(T-t)/2}1_{\{S(T)>K\}}|\mathcal{F}_t\} && (3.20)\\ &\quad -e^{-r(T-t)}K\mathcal{P}^*\{S(T)>K|\mathcal{F}_t\}\ . && (3.21)\end{aligned}$$

  Next, simplify the first term by introducing a new measure $\mathcal{P}^{(s)}$ defined by $d\mathcal{P}^{(s)}/d\mathcal{P}^* \equiv e^{\sigma B^*(T)-\sigma^2 T/2}$ (under this measure, prices are martingales if the stock is the numeraire). Then

$$C(t) = S(t)e^{-\delta(T-t)}\mathcal{P}^{(s)}\{S(T)>K|\mathcal{F}_t\} - e^{-r(T-t)}K\mathcal{P}^*\{S(T)>K|\mathcal{F}_t\}\ . \tag{3.22}$$

  Under the new measure, $B^{(s)}(t) = B^*(t) - \sigma t$ is Brownian motion, so this can be evaluated as

$$C(t) = c(S(t),t) = S(t)e^{-\delta(T-t)}N(d_1(S(t),t)) - e^{-r(T-t)}KN(d_2(S(t),t)) \tag{3.23}$$

  where N is the cumulative normal distribution function,

$$d_1(S(t),t) = \frac{\ln(S(t)e^{-\delta(T-t)}/(Ke^{-r(T-t)})) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\ , \tag{3.24}$$

  and

$$d_2(S(t),t) = d_1(S(t),t) - \sigma\sqrt{T-t}\ . \tag{3.25}$$

- The option "delta," or number of shares in the hedge portfolio or replicating trading strategy, is $c_S = e^{-\delta(T-t)}N(d_1(S(t),t))$.

Example 2 (Forward Contracts) Consider a forward contract to buy security k at time T for price F. In this case $x = (0, S_k(T) - F)$ and the time t value of the contract is

$$V^F(t) = \mathrm{E}^*\{\beta(t,T)(S_k(T)-F)|\mathcal{F}_t\} \tag{3.26}$$

The *forward price* F_t at time t is such that $V^{F_t}(t) = 0$, i.e.,

$$F_t = \frac{\mathrm{E}^*\{\beta(t,T)S_k(T)|\mathcal{F}_t\}}{\mathrm{E}^*\{\beta(t,T)|\mathcal{F}_t\}}\ . \tag{3.27}$$

If security k pays no dividends then

$$F_t = \frac{S_k(t)}{P(t,T)} = S_k(t)e^{r(t,T)(T-t)} \tag{3.28}$$

where $P(t,T)$ is the time t price of a zero-coupon bond maturing at time T and $r(t,T)$ is the continuously compounded $(T-t)$-year zero rate at time t.

Example 2 (Futures Contracts–from Duffie and Stanton, 1992) A *futures contract* on an *underlying asset* k with associated *futures price* $f(t)$, *delivery date* T, and continuous resettlement is a contract which produces a cumulative cash flow of $f(u) - f(t)$ between any two dates t and u with $0 \leq t \leq u \leq T$. In addition, at time T, the contract obliges the holder to buy one share of security k at price $f(T)$.

Given the futures price process f, the time t value of a futures contract is $V^f(t) = \mathrm{E}^*\{\int_t^T \beta(t,u)df(u) + \beta(t,T)(S_k(T) - f(T))|\mathcal{F}_t\}$. Buying and selling futures contracts is costless, so the equilibrium futures price process f must be such that $V^f \equiv 0$ and $f(T) = S_k(T)$.

Theorem 3.3 *Suppose* $\mathrm{E}^*\{S_k(T)^2\} < \infty$ *and* $\beta(t)$ *is bounded above and below away from zero. Then there exists a unique Itô process* f *with* $\mathrm{E}^*\langle f, f\rangle_T < \infty$ *satisfying* $V^f \equiv 0$ *and* $f(T) = S_k(T)$*. It is given by* $f(t) = \mathrm{E}^*\{S_k(T)|\mathcal{F}_t\}$*.*

To compare futures and forward prices, note that

$$F_t = f(t) + \frac{\mathrm{cov}^*\{S_k(T), \beta(t,T)|\mathcal{F}_t\}}{\mathrm{E}^*\{\beta(t,T)|\mathcal{F}_t\}} \ . \tag{3.29}$$

In particular, they are equal if and only if $S_k(T)$ and $\beta(t,T)$ are uncorrelated. This will be the case if interest rates are deterministic. On the other hand, if interest rates are stochastic, they may differ. For example, bond futures prices are typically lower than their forward prices.

4 American Options

These notes are taken primarily from Section V of Domenico Cuoco's lecture notes, Chapter 2 of Karatzas and Shreve (1998), and Jacka (1991). Suppose we are in a complete standard continuous-time financial market.

Definition 4.1 An *American contingent claim (acc)* associated with a continuous, adapted *payoff process* $G(t)$ and expiration date T is a claim to the payoff $G(\tau)$ at stopping time $\tau \leq T$ chosen by the holder of the claim. A stopping time $\tau \leq T$ defines an *exercise policy.*

Examples

- American call: $G(t) = (S(t) - K)^+$
- American put: $G(t) = (K - S(t))^+$
- option embedded in a nondefaultable callable bond: $G(t) = (P(t) - K(t))^+$ where $P(t)$ is the time t present value of the remaining scheduled payments to maturity and $K(t)$ is the time t call price.
- prepayment option embedded in a fixed rate mortgage: $G(t) = (P(t) - K(t))^+$ where $P(t)$ is the time t present value of the remaining scheduled payments and $K(t)$ is the remaining principal balance.
- levered equity's default option: $G(t) = (P(t) - V(t))^+$ where $P(t)$ is the time t present value of remaining scheduled debt payments and $V(t)$ is the time t value of the firm's assets.
- option embedded in a defaultable callable bond: $G(t) = (P(t) - V(t) \wedge K(t))^+$ where $P(t)$ is the time t present value of remaining scheduled debt payments, $V(t)$ is the time t value of the firm's assets, and $K(t)$ is the time t call price.

Assumption The process $\beta(t)G(t)$ is bounded below and satisfies

$$\mathrm{E}^*\{\sup_{0\le t\le T}\beta(t)G(t)\} < \infty \ . \tag{4.1}$$

For any given exercise policy τ, the claim to $G(\tau)$ at time τ can be replicated by a tame, tight, martingale-generating trading strategy because, for example, we can replicate $W = G(\tau)e^{\int_\tau^T r(u)\,du}$ to be received at time T. The replication cost of the acc under policy τ at time t is

$$V^\tau(t) \equiv \mathrm{E}^*\{\beta(t,\tau)G(\tau)|\mathcal{F}_t\} \tag{4.2}$$

Definition 4.2 An *optimal exercise policy* τ^*, if it exists, is a solution to the optimal stopping problem

$$V^*(0) = \sup_{\tau\in\mathcal{S}_{0,T}} V^\tau(0) = \sup_{\tau\in\mathcal{S}_{0,T}} \mathrm{E}^*\{\beta(\tau)G(\tau)\} \tag{4.3}$$

where $\mathcal{S}_{t,T}$ is the set of stopping times taking values in $[t,T]$.

Hedging a short position in an acc is complicated by the uncertainty about the holder's exercise policy.

Definition 4.3 A *super-replicating trading strategy* is a tight martingale-generating trading strategy $\bar\pi$ satisfying

$$\bar\pi(\tau)1 \ge G(\tau) \text{ a.s. } \ \forall\, \tau\in\mathcal{S}_{0,T} \ . \tag{4.4}$$

Definition 4.4 The *value of the acc* at time t is

$$V^{\mathrm{acc}}(t) = \inf\{\bar\pi(t)1 : \bar\pi \text{ is a super-replicating trading strategy}\} \ . \tag{4.5}$$

Definition 4.5 A *hedging trading strategy* $\bar\pi^*$ for the acc is a super-replicating trading strategy such that $\bar\pi^*(0)1 = V^{\mathrm{acc}}(0)$.

Theorem 4.1 *(Theorem 2.5.3 of Karatzas and Shreve (1998), p. 56)*

$$V^{\text{acc}}(0) = V^*(0)\ . \tag{4.6}$$

Furthermore, there exists an optimal stopping time τ^* attaining this supremum and there exists a hedging trading strategy $\bar{\pi}^*$ such that

$$\beta(\tau^*)G(\tau^*) = V^{\text{acc}}(0) + \int_0^{\tau^*} \beta(t)\pi^*(t)\sigma(t)\, dB^*(t)\ . \tag{4.7}$$

Lemma 4.1 *Let X be a continuous, adapted process bounded below. Then there exists a RCLL supermartingale ξ called the Snell envelope of X such that*

$$\xi(t) \geq X(t)\ \forall\ t \in [0, T]\ a.s. \tag{4.8}$$

and

$$\xi(t) = \text{ess} \sup_{\tau \in \mathcal{S}_{t,T}} \text{E}\{X(\tau)|\mathcal{F}_t\}\ . \tag{4.9}$$

Moreover, the stopping time

$$\tau^* \equiv \inf\{t \in [0, T] : \xi(t) = X(t)\} \tag{4.10}$$

satisfies $\xi(0) = \text{E}\{X(\tau^*)\}$ and ξ is a martingale on $[0, \tau^*]$.

Proof of Theorem Let ξ be the $\mathcal{P}^*$-Snell envelope of $\beta(t)G(t)$. ξ admits the Doob-Meyer decomposition $\xi = M - \Lambda$ where M is a $\mathcal{P}^*$-martingale and Λ is a nondecreasing process with $\Lambda(0) = \Lambda(\tau^*)$ a.s. Since the market is complete, there exists a martingale-generating trading strategy π^* satisfying $M(t) = \int_0^t \beta(u)\pi^*(u)\sigma(u)\, dB^*(u)$. Therefore,

$$\begin{aligned} \beta(t)G(t) &\leq \xi(t) = \xi(0) + M(t) - (\Lambda(t) - \Lambda(0)) && (4.11) \\ &\leq \xi(0) + \int_0^t \beta(u)\pi^*(u)\sigma(u)\, dB^*(u)\ , && (4.12) \end{aligned}$$

so $\xi(0)$ is the initial cost of a super-replicating trading strategy, which implies

$$V^{\text{acc}}(0) \leq \xi(0)\ . \tag{4.13}$$

On the other hand, if $\bar{\pi}$ is a super-replicating trading strategy, then for every $\tau \in \mathcal{S}_{0,T}$,

$$\begin{aligned}
\beta(\tau)\bar{\pi}(\tau)1 &= \bar{\pi}(0)1 + \int_0^\tau \beta(u)\pi(u)\sigma(u)\, dB^*(u) && (4.14)\\
\Longrightarrow \bar{\pi}(0)1 &= \mathrm{E}^*\{\beta(\tau)\bar{\pi}(\tau)1 - \int_0^\tau \beta(u)\pi(u)\sigma(u)\, dB^*(u)\} && (4.15)\\
&= \mathrm{E}^*\{\beta(\tau)\bar{\pi}(\tau)1\} && (4.16)\\
&\geq \mathrm{E}^*\{\beta(\tau)G(\tau)\} && (4.17)\\
\Longrightarrow V^{\mathrm{acc}}(0) &\geq \mathrm{E}^*\{\beta(\tau)G(\tau)\}\ \forall\ \tau \in \mathcal{S}_{0,T} && (4.18)\\
\Longrightarrow V^{\mathrm{acc}}(0) &\geq \xi(0)\ . && (4.19)
\end{aligned}$$

Therefore $V^{\mathrm{acc}}(0) = \xi(0)$, $\tau^* = \inf\{t \in [0,T] : \xi(t) = \beta(t)G(t)\}$ is an optimal policy, and the tightly self-financing trading strategy $\bar{\pi}^*$ defined by $\bar{\pi}_0^*(0) = \xi(0) - \pi^*(0)1$ and $\bar{\pi}_k^* = \pi_k^*$ for $k = 1, \ldots, n$ is a hedging trading strategy. □

Note that we can extend the results above to obtain $\beta(t)V^{\mathrm{acc}}(t) = \xi(t)$.

Proposition 4.1 *If $\beta(t)G(t)$ is a submartingale then $\tau^* = T$ is an optimal exercise policy.*

Corollary 4.1 *If $r \geq 0$, $\delta = 0$, and σ satisfies the Novikov condition, then leaving the American call unexercised until expiration is an optimal policy. In this case, the value of the American call is equal to the value of the European call.*

Special Case: Markovian Model with Path-Independent Payoff Process

Suppose we are in the Markovian model of security prices and $G(t) = g(S(t), t)$ for some well-behaved function g (see Krylov (1980) p. 130). Then

$$V^{\mathrm{acc}}(t) = \operatorname*{ess\ sup}_{\tau \in \mathcal{S}_{t,T}} \mathrm{E}^*\{\beta(t,\tau)G(\tau)|\mathcal{F}_t\} = h(S(t), Y(t), t) \qquad (4.20)$$

for some function $h : \mathcal{R}_+^n \times \mathcal{R}^m \times [0,T] \to \mathcal{R}$ with $h(S(t), Y(t), t) \geq g(S(t), t)$.

An optimal exercise policy is

$$\tau^* = \inf\{t \in [0,T] : h(S(t), Y(t), t) = g(S(t), t)\}\ . \qquad (4.21)$$

Thus $h(S(t), Y(t), t) > g(S(t), t)$ for all $t < \tau^*$.

Definition 4.6 The *continuation region* for this optimal stopping problem is

$$\mathcal{U} = \{(s,y,t) \in \mathcal{R}_+^n \times \mathcal{R}^m \times [0,T] : h(s,y,t) > g(s,t)\} \tag{4.22}$$

and its complement, the *exercise region* , is

$$\mathcal{E} = \{(s,y,t) \in \mathcal{R}_+^n \times \mathcal{R}^m \times [0,T] : h(s,y,t) = g(s,t)\} \ . \tag{4.23}$$

The *exercise boundary* is the boundary of $\mathcal{U}$, $\partial\mathcal{U}$.

Suppose h is smooth enough for an application of Itô's lemma. Let $\mathcal{D}h$ denote the drift of h under $\mathcal{P}^*$. Since βh is a $\mathcal{P}^*$-supermartingale, and a $\mathcal{P}^*$-martingale on $\mathcal{U}$, we have

$$\mathcal{D}h - rh \leq 0 \text{ on } \mathcal{R}_+^n \times \mathcal{R}^m \times [0,T] \tag{4.24}$$

$$\mathcal{D}h - rh = 0 \text{ on } \mathcal{U} \text{ (a Bellman equation)} \tag{4.25}$$

$$(h-g)(\mathcal{D}h - rh) = 0 \text{ on } \mathcal{R}_+^n \times \mathcal{R}^m \times [0,T] \ . \tag{4.26}$$

Under technical conditions, the *variational inequality*

$$h \geq g \tag{4.27}$$

$$\mathcal{D}h - rh \leq 0 \tag{4.28}$$

$$(h-g)(\mathcal{D}h - rh) = 0 \tag{4.29}$$

together with the boundary condition $h(s,y,T) = g(s,T)$, uniquely defines the function h. We typically also have the *smooth-pasting* condition $h_s = g_s$ on $\partial\mathcal{U}$.

Example 1: Optimal Stopping and the American Put–Jacka (1991)
Assume $n = d = 1$, $r > 0$ and $\sigma \neq 0$ are constants, $\delta = 0$, and $G(t) = g(S(t)) = (k - S(t))^+$.

Theorem 4.2

$$V^{\text{ap}}(t) \equiv \sup_{\tau \in \mathcal{S}_{t,T}} \mathrm{E}^*\{e^{-r(\tau - t)}(k - S(\tau))^+ | \mathcal{F}_t\} = f(S(t),t) \tag{4.30}$$

for some continuous function f *with* $f(s,t) \geq (k-s)^+$ *and the optimal stopping time is* $\tau^* = \inf\{t \in [0,T] : f(s(t),t) = (k - S(t))^+\}$.

For the proof, see Theorems 3.1.8 and 3.1.10 of Krylov (1980).

Lemma 4.2

$$f(s,t) > 0 \;\forall\; s \geq 0, t < T \; . \tag{4.31}$$

The continuation region for the problem is

$$\mathcal{U} = \{(s,t) \in \mathcal{R}_+ \times [0,T] : f(s,t) > (k-s)^+\} \; . \tag{4.32}$$

Proposition 4.2 *For each $t < T$, the t-section of $\mathcal{U}$ is an open ray:*

$$\mathcal{U}_t \equiv \{s \geq 0 : (s,t) \in \mathcal{U}\} = (b(t), \infty) \textit{ for some } b(t) \in (0,k) \; . \tag{4.33}$$

Proof Clearly $0 \notin \mathcal{U}_t$ since $g(0) = k \geq f(0,t)$. Also $k \in \mathcal{U}_t$ since $g(k) = 0 < f(k,t)$. It remains to show that $x \in \mathcal{U}_t \Longrightarrow y \in \mathcal{U}_t$ for every $y > x$. Suppose $x \in \mathcal{U}_t$ and $y > x$. Let τ be the optimal stopping time given $S(t) = x$, i.e., $\tau = \inf\{u \geq t : (S_x(u), u) \in \mathcal{U}^c\}$, where $S_x(u) = xe^{\sigma(B^*(u)-B^*(t))+(r-\sigma^2/2)(u-t)}$. Then, since τ is feasible starting from y,

$$\begin{aligned}
f(y,t) - f(x,t) &= f(y,t) - \mathrm{E}^*\{e^{-r(\tau-t)}(k - S_x(\tau))^+ | \mathcal{F}_t\} && (4.34)\\
&\geq \mathrm{E}^*\{e^{-r(\tau-t)}[(k - S_y(\tau))^+ - (k - S_x(\tau))^+] | \mathcal{F}_t\} && (4.35)\\
&\geq \mathrm{E}^*\{e^{-r(\tau-t)}[S_x(\tau) - S_y(\tau)] | \mathcal{F}_t\} && (4.36)\\
&\geq (x-y)\mathrm{E}^*\{e^{\sigma(B^*(\tau)-B^*(t))-\sigma^2(\tau-t)/2} | \mathcal{F}_t\} && (4.37)\\
&= x - y && (4.38)
\end{aligned}$$

Thus,

$$\begin{aligned}
f(y,t) &\geq f(x,t) + x - y && (4.39)\\
&> (k-x)^+ + x - y \quad (\text{because } x \in \mathcal{U}_t) && (4.40)\\
&\geq k - y \; . && (4.41)
\end{aligned}$$

Also, $f(y,t) > 0$, so $f(y,t) > (k-y)^+$ and thus $y \in \mathcal{U}_t$. $\square$

Lemma 4.3 *$f(\cdot, t)$ is decreasing for each t and $f(s, \cdot)$ is decreasing for each s.*

Proposition 4.3 *The boundary $b(t)$ is increasing in t.*

Proof Consider times t and u with $0 \le t \le u \le T$ and $\varepsilon > 0$.

$$
\begin{aligned}
f(b(u) + \varepsilon, t) &\ge f(b(u) + \varepsilon, u) && (4.42)\\
&> g(b(u) + \varepsilon) && (4.43)\\
\Longrightarrow b(u) + \varepsilon &\in \mathcal{U}_t \ \forall\, \varepsilon > 0 && (4.44)\\
\Longrightarrow b(u) &\ge b(t)\ .\ \square && (4.45)
\end{aligned}
$$

Example 2: The American Call–Kim (1991)

Suppose r, δ, and σ are constant. Then there exists an optimal exercise boundary $\bar{s}(t)$ below which it is optimal to continue and above which it is optimal to exercise. The boundary is increasing in r and σ and decreasing in δ and t.

- If $\delta \ge r$ then $\lim_{t \to T} \bar{s}(t) = k$.
- If $\delta < r$ then $\lim_{t \to T} \bar{s}(t) = \frac{r}{\delta} k > k$.

Note interest rk saved by waiting exceeds lost dividend δS when $S < rk/\delta$.

5 Optimal Consumption and Portfolio Choice

These notes are taken primarily from Section IV of Domenico Cuoco's lecture notes and Chapter 3 of Karatzas and Shreve (1998). Suppose we are in a complete standard continuous-time financial market.

5.1 The Investment Problem and the Static Budget Constraint

Consider the optimization problem of an investor with initial wealth $x > 0$ who chooses a trading strategy and consumption plan to maximize expected utility, i.e., a problem of the form

$$\begin{aligned} \max_{c \in \mathcal{C}, \pi} \quad & \mathrm{E}\{\int_0^T U_1(c(t),t)dt + U_2(X(T))\} \\ \text{s.t.} \quad & dX(t) = [r(t)X(t) - c(t)]\,dt + \pi(t)[\mu(t) - r(t)1]\,dt + \pi(t)\sigma(t)\,dB(t); \\ & X(0) = x; \\ & c(t) \geq 0; \\ & X(t) \geq 0, \end{aligned} \tag{5.1}$$

where U_1 and U_2 are utility functions to be described formally later. To develop a more convenient characterization of this problem, recall from Proposition 2.4 that any consumption plan (c, W) that is feasible for problem (5.1) must satisfy the "budget constraint"

$$\mathrm{E}^*\{\int_0^T \beta(t)c(t)\,dt + \beta(T)W\} \leq x\ . \tag{5.2}$$

Conversely, since the market is complete, any nonnegative consumption plan satisfying equation (5.2) can be financed by some tame, martingale generating trading strategy starting from initial wealth x. It follows that the "dynamic" problem above is equivalent to the "static" problem

$$\begin{aligned} \max_{(c,W) \in \mathcal{C}_+} \quad & \mathrm{E}\{\int_0^T U_1(c(t),t)dt + U_2(W)\} \\ \text{s.t.} \quad & \mathrm{E}^*\{\int_0^T \beta(t)c(t)\,dt + \beta(T)W\} \leq x\ . \end{aligned} \tag{5.3}$$

To solve the problem, it is more convenient to have a representation of the budget constraint (5.2) in terms of expectation under the true probability $\mathcal{P}$, to be consistent with the objective function. Note that since $d\mathcal{P}^*/d\mathcal{P} = Z(T)$,

$$\begin{aligned}
\mathrm{E}^*\{\int_0^T \beta(t)c(t)\,dt + \beta(T)W\} &= \mathrm{E}\{Z(T)\int_0^T \beta(t)c(t)\,dt + Z(T)\beta(T)W\} && (5.4)\\
&= \int_0^T \mathrm{E}\{Z(T)\beta(t)c(t)\}\,dt + \mathrm{E}\{Z(T)\beta(T)W\} && (5.5)\\
&= \int_0^T \mathrm{E}\{\mathrm{E}\{Z(T)\beta(t)c(t)|\mathcal{F}_t\}\}\,dt && (5.6)\\
&\quad + \mathrm{E}\{Z(T)\beta(T)W\} && (5.7)\\
&= \int_0^T \mathrm{E}\{Z(t)\beta(t)c(t)\}\,dt + \mathrm{E}\{Z(T)\beta(T)W\} && (5.8)\\
&= \mathrm{E}\{\int_0^T Z(t)\beta(t)c(t)\,dt + Z(T)\beta(T)W\}\ . && (5.9)
\end{aligned}$$

Define the *stochastic discount factor process* $H(t) = Z(t)\beta(t)$. Then the investor's budget constraint (5.2) can be written as

$$\mathrm{E}\{\int_0^T H(t)c(t)\,dt + H(T)W\} \le x \tag{5.10}$$

and the investor's optimization problem can be written as

$$\begin{aligned}
\max_{(c,W)\in\mathcal{C}_+} \quad & \mathrm{E}\{\int_0^T U_1(c(t),t)dt + U_2(W)\} && (5.11)\\
\text{s.t.} \quad & \mathrm{E}\{\int_0^T H(t)c(t)\,dt + H(T)W\} \le x\ .
\end{aligned}$$

Remark In this setting, Z is a martingale and the measure $\mathcal{P}^*$ is well defined. However, it is possible to arrive at the budget constraint (5.10) as a characterization of affordable consumption plans without using the measure $\mathcal{P}^*$ and using only the local martingale property of Z. The proof involves representing the $\mathcal{P}$-martingale $M(t) \equiv \mathrm{E}\{\int_0^T H(t)c(t)\,dt + H(T)W|\mathcal{F}_t\}$ in terms of a stochastic integral that can be interpreted as gains from a tight martingale-generating trading strategy. (See Remark 3.3 and Theorem 3.5 of Karatzas and Shreve (1998), pp. 92-93.) Thus, this representation of the investor's problem is valid in more general settings that we will consider, such as in the case of incomplete markets.

5.2 Utility Functions and Inverse Marginal Utility Functions

Definition 5.1 A *utility function* U is a real-valued, strictly increasing, strictly concave C^2 function defined on a domain $[\bar{x}, \infty)$ or $(\bar{x}, \infty)$, with $\bar{x} \geq 0$ and $\lim_{x\to\infty} U'(x) = 0$. Call $\bar{x}$ the *subsistence level.* Let $U'(\bar{x}+) \equiv \lim_{x\downarrow\bar{x}} U'(x)$. If $U'(\bar{x}+) < \infty$, then assume U is defined on $[\bar{x}, \infty)$.

Note that U' is a strictly positive, strictly decreasing, C^1 function mapping $(\bar{x}, \infty)$ onto $(0, U'(\bar{x}+))$.

Remark In some settings, we can allow for $\bar{x} < 0$ as well.

Definition 5.2 The *inverse marginal utility function (imuf)* associated with utility function U is the function $I : \mathcal{R}_+ \to [\bar{x}, \infty)$ given by

$$I(y) = \begin{cases} U'^{-1}(y) & \text{if} \quad 0 < y < U'(\bar{x}+) \\ \bar{x} & \text{if} \quad y \geq U'(\bar{x}+) \ . \end{cases} \tag{5.12}$$

Note that $I(U'(x)) = x$ for all $x \in (\bar{x}, \infty)$, I is continuous, and I is strictly decreasing on $(0, U'(\bar{x}+))$.

Next, let $U_1(c, t)$ be such that each $U_1(\cdot, t)$ is a utility function with imuf $I_1(y, t)$ and suppose the subsistence level $\bar{x}_1(t)$ is a continuous function of time. Let $U_2(x)$ be a utility function with subsistence level $\bar{x}_2$. Let us slightly restate the combined intermediate consumption-terminal wealth problem (5.11) above as

$$\begin{aligned} \max_{(c,W)\in\mathcal{C}, c\geq\bar{x}_1, W\geq\bar{x}_2} \quad & \mathrm{E}\{\int_0^T U_1(c(t), t)dt + U_2(W)\} \\ \text{s.t.} \quad & \mathrm{E}\{\int_0^T H(t)c(t)\,dt + H(T)W\} \leq x \ , \end{aligned} \tag{5.13}$$

and consider also two other versions of the investment problem, the pure consumption problem

$$\max_{c\in\mathcal{C}, c\geq\bar{x}_1} \mathrm{E}\{\int_0^T U_1(c(t), t)dt\} \ \text{ s.t. } \mathrm{E}\{\int_0^T H(t)c(t)\,dt\} \leq x \tag{5.14}$$

and the pure terminal wealth problem

$$\max_{W \in \mathcal{C}, W \geq \bar{x}_2} \mathrm{E}\{U_2(W)\} \;\; \text{s.t. } \mathrm{E}\{H(T)W\} \leq x \; . \tag{5.15}$$

5.3 Solutions

To gain intuition, consider the terminal wealth problem (5.15) and form the Lagrangian

$$\mathcal{L} = \mathrm{E}\{U_2(W) + \lambda(x - H(T)W) + \gamma(W - \bar{x}_2)\} \; , \tag{5.16}$$

where the constant λ is a Lagrange multiplier associated with the budget constraint and γ is a random variable with $\gamma(\omega)$ the state-dependent Lagrange multiplier associated with the constraint $W(\omega) \geq \bar{x}$. We know that since the utility function is strictly increasing, an optimal consumption plan will satisfy the budget constraint with equality. Thus, we get the following first-order conditions for $W(\omega)$, λ, and $\gamma(\omega)$:

$$U_2'(W) = \lambda H(T) - \gamma \tag{5.17}$$

$$\mathrm{E}\{H(T)W\} = x \tag{5.18}$$

$$\lambda > 0 \tag{5.19}$$

$$\gamma(W - \bar{x}_2) = 0 \tag{5.20}$$

$$\gamma \geq 0 \tag{5.21}$$

$$W \geq \bar{x}_2 \; . \tag{5.22}$$

Proposition 5.1 *Assume that* $\mathrm{E}\{H(T)\} < \infty$, $\mathrm{E}\{H(T)\bar{x}_2\} < x$, *and*

$$\mathcal{X}_2(\lambda) \equiv \mathrm{E}\{H(T)I_2(\lambda H(T))\} < \infty \;\forall\; \lambda \in (0, \infty) \; . \tag{5.23}$$

Then the solution to problem (5.15) is

$$W_2 = I_2(\lambda_2 H(T)) \tag{5.24}$$

where λ_2 *solves* $\mathrm{E}\{H(T)I_2(\lambda_2 H(T))\} = x$.

Ignoring the lower bound constraints on W, we see that the first-order conditions indicate that the investor should choose a payoff that sets his marginal utility proportional to the relative state price H and satisfies the budget constraint with equality. Proposition 5.1 says that this is essentially the solution, except that the payoff may have to be truncated to satisfy lower bound constraints. The proof uses the following lemma.

Lemma 5.1 *(Lemma 3.6.2 of Karatzas and Shreve (1998), p. 101)* $\mathcal{X}_2$ *is continuous and nonincreasing on* $(0, \infty)$ *and strictly decreasing on* $(0, \bar{\lambda})$ *where*

$$\bar{\lambda} \equiv \sup\{\lambda > 0 : \mathcal{X}_2(\lambda) > \mathrm{E}[H(T)\bar{x}_2]\} \ . \tag{5.25}$$

Moreover, $\mathcal{X}_2(\lambda) \to \infty$ *as* $\lambda \to 0$ *and* $\mathcal{X}_2(\lambda) \to \mathrm{E}\{H(T)\bar{x}\}$ *as* $\lambda \to \bar{\lambda}$*. Therefore, there exists a unique* $\lambda_2 > 0$ *such that* $\mathcal{X}_2(\lambda_2) = x$*.*

Proof of Proposition First note that $W_2 \geq \bar{x}_2$ and $\mathrm{E}\{H(T)W_2\} = x$, so W_2 is feasible for problem (5.15). Also note that W_2 satisfies the first-order conditions (5.17) through (5.22) with γ equal to $\gamma_2 \equiv \lambda_2 H(T) - U_2'(W_2)$. In particular,

$$\begin{aligned} U_2'(W_2) &= U'(I(\lambda_2 H(T))) & (5.26)\\ &= \begin{cases} \lambda_2 H(T) & \text{if} \quad \lambda_2 H(T) < U_2'(\bar{x}_2+) \\ U_2'(\bar{x}_2+) & \text{if} \quad \lambda_2 H(T) \geq U_2'(\bar{x}_2+) \ , \end{cases} & (5.27) \end{aligned}$$

so $\gamma_2 \geq 0$ and $\gamma_2(W_2 - \bar{x}_2) = 0$.

Now let W be another feasible policy with $\mathcal{P}\{W = W_2\} < 1$.

$$\begin{aligned} \mathrm{E}\{U_2(W) - U_2(W_2)\} &< \mathrm{E}\{U_2'(W_2)(W - W_2)\} & (5.28)\\ &= \mathrm{E}\{(\lambda_2 H(T) - \gamma_2)(W - W_2) & (5.29)\\ &= \mathrm{E}\{\lambda_2 H(T) W\} - \mathrm{E}\{\lambda_2 H(T) W_2\} & (5.30)\\ &\quad -\mathrm{E}\{\gamma_2(W - \bar{x}_2^+)\} + \mathrm{E}\{\gamma_2(W_2 - \bar{x}_2^+)\} & (5.31)\\ &\leq 0 \ . & (5.32) \end{aligned}$$

Therefore, W_2 is the unique solution to problem (5.15). □

The solutions to problems (5.11) and (5.14) are similar:

Proposition 5.2 *Assume that* $\mathrm{E}\{\int_0^T H(t)\,dt\} < \infty$, $\mathrm{E}\{\int_0^T H(t)\bar{x}_1(t)\,dt\} < x$, *and*

$$\mathcal{X}_1(\lambda) \equiv \mathrm{E}\{\int_0^T H(t)I_1(\lambda H(t),t)\,dt\} < \infty \;\forall\; \lambda \in (0,\infty)\ . \tag{5.33}$$

Then the solution to problem (5.14) is

$$c_1(t) = I_1(\lambda_1 H(t),t) \tag{5.34}$$

where λ_1 *solves* $\mathrm{E}\{\int_0^T H(t)I_1(\lambda_1 H(t),t)\,dt\} = x$.

Proposition 5.3 *Assume that* $\mathrm{E}\{\int_0^T H(t)\,dt + H(T)\} < \infty$, $\mathrm{E}\{\int_0^T H(t)\bar{x}_1(t)\,dt + H(T)\bar{x}_2\} < x$, *and*

$$\mathcal{X}_3(\lambda) \equiv \mathrm{E}\{\int_0^T H(t)I_1(\lambda H(t),t)\,dt + H(T)I_2(\lambda H(T))\} < \infty \;\forall\; \lambda \in (0,\infty)\ . \tag{5.35}$$

Then the solution to problem (5.11) is

$$(c_3(t), W_3) = (I_1(\lambda_3 H(t),t), I_2(\lambda_3 H(T))) \tag{5.36}$$

where λ_3 *solves* $\mathrm{E}\{\int_0^T H(t)I_1(\lambda_3 H(t),t)\,dt + H(T)I_2(\lambda_3 H(T))\} = x$.

Note that since Z is a supermartingale, the integrability condition $\mathrm{E}\{\int_0^T H(t)\,dt + H(T)\} < \infty$ will be satisfied, for example, if r is bounded below.

5.4 Optimal Trading Strategy with Deterministic Coefficients

We can solve for the optimal consumption plan, and we know there exists a trading strategy that implements it, but we cannot generally solve for the optimal trading strategy explicitly. However, in the special case that the market coefficients are deterministic, we can get an explicit expression for the optimal trading strategy.

Assume that r, σ, and θ are nonrandom, continuous functions on $[0,T]$ and r and $|\theta|$ are in fact Hölder continuous. I.e., there exist $k > 0$ and $\rho \in (0,1)$ such that $|r(t_1) - r(t_2)| \leq k|t_1 - t_2|^\rho$ and $||\theta(t_1)| - |\theta(t_2)|| \leq k|t_1 - t_2|^\rho$ for every $t_1, t_2 \in [0,T]$. This guarantees Z is a $\mathcal{P}$-martingale and allows us to use $\mathcal{P}^*$ and B^*. In addition, note that Z is Markov.

Assume I_1, I_2, $U_1 \circ I_1$, and $U_2 \circ I_2$ satisfy polynomial growth conditions, and I_1 is Hölder continuous (see Assumption 3.8.2 of Karatzas and Shreve, 1998, p. 120). Then under the optimal policy for, say, problem (5.11), investor wealth is

$$X(t) = \mathrm{E}^*\{\int_t^T \beta(t,s)c_3(s)\,ds + \beta(t,T)W_3|\mathcal{F}_t\} \tag{5.37}$$

$$= \mathrm{E}^*\{\int_t^T \beta(t,s)I_1(\lambda_3\beta(s)Z(s),s)\,ds + \beta(T)I_2(\lambda_3\beta(T)Z(T))|\mathcal{F}_t\} \tag{5.38}$$

$$= f(Z(t),t) \tag{5.39}$$

for some $C^{2,1}$ function $f : (0,\infty) \times [0,T] \to \mathcal{R}$. Therefore, we can apply Itô's lemma to write the diffusion coefficient of X as $-f_z(Z(t),t)Z(t)\theta'(t)$. On the other hand, from the general wealth evolution equation in problem (5.1), the diffusion coefficient is $\pi(t)\sigma(t)$. Equating these yields an expression for the optimal trading strategy

$$\pi^{*'}(t) = -f_Z(Z(t),t)Z(t)\sigma'^{-1}(t)\theta(t) \tag{5.40}$$

$$= -f_Z(Z(t),t)Z(t)[\sigma(t)\sigma'(t)]^{-1}[\mu(t) - r(t)1]\ . \tag{5.41}$$

This yields the following result.

Proposition 5.4 (Merton's Fund Separation Theorem) *In the deterministic coefficients setting here, all investors divide wealth between the money market account and a single fund of risky assets with weights proportional to* $(\sigma\sigma')^{-1}(\mu - r1)$.

Optimal Controls in Feedback Form For each $t \in [0,T]$, $f(\cdot,t)$ is strictly decreasing and has a strictly decreasing inverse $f^{-1}(\cdot,t)$. Thus we can write

$$\pi^{*'}(t) = -\frac{f^{-1}(X(t),t)}{f_X^{-1}(X(t),t)}[\sigma(t)\sigma'(t)]^{-1}[\mu(t) - r(t)1] \tag{5.42}$$

$$= g(X(t),t)\ , \tag{5.43}$$

where g gives the optimal trading strategy in so-called "feedback form," i.e., as a function of wealth X. Similarly, we can write optimal consumption in feedback form:

$$c^*(t) = I_1(\lambda H(t), t) = h(X(t), t) \tag{5.44}$$

for some function h.

5.5 Dynamic Programming Approach

Merton (1969, 1971) originally solved the consumption/investment problems using dynamic programming. Let p and c represent portfolio holdings and consumption flow as proportions of wealth and consider the problem

$$\begin{aligned} \sup_{p,c} \quad & \mathrm{E}\{\int_0^T U_1(c(t)X(t), t)dt + U_2(X(T))\} && (5.45) \\ \text{s.t.} \quad & \frac{dX(t)}{X(t)} = [r(t) - c(t) + p(t)(\mu(t) - r(t)1)]\,dt + p(t)\sigma(t)\,dB(t); && (5.46) \\ & X(0) = x \end{aligned}$$

where p and c are predictable controls taking values in a compact set K, r, μ, and σ are continuously differentiable functions of time, and U_1 is defined in terms of proportional consumption flow. The value function for the problem is

$$\begin{aligned} V(t) \quad &= \quad \sup_{p,c} \mathrm{E}\{\int_t^T U_1(c(s)X(s), s)\,ds + U_2(X(T))|\mathcal{F}_t\} && (5.47) \\ \text{s.t.} \quad & \qquad \frac{dX(t)}{X(t)} = [r(t) - c(t) + p(t)(\mu(t) - r(t)1)]\,dt + p(t)\sigma(t)\,dB(t)\ . \end{aligned}$$

The value of any given feasible strategy $\hat{p}, \hat{c}$ is

$$\begin{aligned} J^{\hat{p},\hat{c}}(t) \quad &= \quad \mathrm{E}\{\int_t^T U_1(\hat{c}(s)X^{\hat{p},\hat{c}}(s), s)\,ds + U_2(X(T))|\mathcal{F}_t\} && (5.48) \\ \text{s.t.} \quad & \qquad \frac{dX^{\hat{p},\hat{c}}(t)}{X^{\hat{p},\hat{c}}(t)} = [r(t) - \hat{c}(t) + \hat{p}(t)(\mu(t) - r(t)1)]\,dt + \hat{p}(t)\sigma(t)\,dB(t) && (5.49) \end{aligned}$$

where the hat and superscript notation above are just to emphasize the use of a particular (though not necessarily optimal) set of controls, and will be omitted below when the meaning is clear.

To develop the intuition for the Bellman equation informally, note that because of the Markov properties of Brownian motion, we may restrict attention to controls that are "Markov" or "feedback," i.e., of the form $p(t) = g(X(t), t)$ and $c(t) = h(X(t), t)$, and assume that the value function is of the form $V(t) = v(X(t), t)$.

The Bellman Principle For $t < u < T$,

$$v(X(t), t) = \sup_{p,c} \mathrm{E}\{\int_t^u U_1(c(s)X(s), s)\, ds + v(X^{p,c}(u), u)|X(t)\} \tag{5.50}$$

where $X^{p,c}$ indicates the wealth process under controls p, c. Now suppose v is $C^{2,1}$. Then by Itô's lemma,

$$\begin{aligned} v(X^{p,c}(u), u) &= v(X(t), t) + \int_t^u \mathcal{D}^{p,c} v(X^{p,c}(s), s)\, ds & (5.51) \\ &\quad + \int_t^u v_X(X^{p,c}(s), s) p(s)\sigma(s) X^{p,c}(s)\, dB(s) & (5.52) \end{aligned}$$

where

$$\mathcal{D}^{p,c} v \equiv v_X[r - c + p(\mu - r1)]X^{p,c} + v_t + \frac{1}{2} v_{XX} p\sigma\sigma' p' (X^{p,c})^2 \tag{5.53}$$

is the drift of v under the controls p, c. Thus, assuming the stochastic integral has mean zero,

$$\begin{aligned} 0 &= \sup_{p,c} \mathrm{E}\{\int_t^u U_1(c(s)X(s), s)\, ds + v(X^{p,c}(u), u) - v(X(t), t)|X(t)\} & (5.54) \\ &= \sup_{p,c} \mathrm{E}\{\int_t^u U_1(c(s)X(s), s)\, ds + \int_t^u \mathcal{D}^{p,c} v(X^{p,c}(s), s)\, ds|X(t)\}\ . & (5.55) \end{aligned}$$

Dividing by $u - t$ and letting $u \downarrow t$ gives the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} 0 &= \sup_{p,c} U_1(cX, t) + \mathcal{D}^{p,c} v(X, t) & (5.56) \\ &= \sup_{p,c} U_1(cX, t) + v_X(X, t)[r - c + p(\mu - r1)]X + v_t(X, t) & (5.57) \\ &\quad + \frac{1}{2} v_{XX}(X, t) p\sigma\sigma' p' X^2\ . & (5.58) \end{aligned}$$

The first-order conditions for optimal p and c imply

$$\begin{aligned} U_1'(cX, t)X &= v_X(X, t)X\ , & (5.59) \\ p'X &= -\frac{v_X(X, t)}{v_{XX}(X, t)} (\sigma\sigma')^{-1}(\mu - r1)\ , & (5.60) \end{aligned}$$

which yields the fund separation result. In addition, v must satisfy the terminal condition

$$v(X,T) = U_2(X) \ . \tag{5.61}$$

Proposition 5.5 *Suppose U_1 and U_2 satisfy a pgc. If $Q \in C^{2,1}$ satisfies the HJB equation, the terminal condition, and a pgc, then*

$$Q(x,t) \geq J^{p,c}(x,t) \tag{5.62}$$

for any predictable K-valued controls (p,c). If, in addition, there exist predictable K-valued controls (p^*,c^*) s.t.

$$0 = U_1(c^*(t)X(t),t) + \mathcal{D}^{p*,c*}Q(X(t),t) \ \text{a.s. a.e.} \ , \tag{5.63}$$

then $Q(x,t) = J^{p*,c*}(x,t) = v(x,t)$ and (p^*,c^*) are optimal controls.

This is sometimes called a "verification" theorem.

Sketch of proof Suppose Q satisfies the HJB equation and the terminal condition, and let (p,c) be feasible controls. Then

$$0 \geq U_1(cX(t),t) + \mathcal{D}^{p,c}Q(X(t),t) \ . \tag{5.64}$$

Note that by Itô's lemma,

$$Q(X(T),T) - Q(X(t),t) = \int_t^T \mathcal{D}^{p,c}Q(X(s),s)\,ds + \int_t^T Q_X(X(s),s)X(s)p(s)\sigma(s)\,dB(s) \ , \tag{5.65}$$

where we have dropped the superscripts on X for ease of exposition. The conditions imposed on σ, p, and Q ensure that the stochastic integral above has mean zero, so

$$\begin{aligned}
\mathrm{E}\{Q(X(T),T) - Q(X(t),t)|\mathcal{F}_t\} &= \mathrm{E}\{\int_t^T \mathcal{D}^{p,c}Q(X(s),s)\,ds|\mathcal{F}_t\} && (5.66)\\
&\leq -\mathrm{E}\{\int_t^T U_1(c(s)X(s),s)\,ds|\mathcal{F}_t\} && (5.67)\\
\Longrightarrow Q(X(t),t) &\geq \mathrm{E}\{\int_t^T U_1(c(s)X(s),s)\,ds + U_2(X(T))|\mathcal{F}_t\} && (5.68)\\
&= J^{p,c}(X(t),t) && (5.69)\\
\Longrightarrow Q(X(t),t) &\geq v(X(t),t) \ . && (5.70)
\end{aligned}$$

On the other hand, if there exist controls (p^*, c^*) such that

$$0 = U_1(c^*(t)X(t), t) + \mathcal{D}^{p*,c*}Q(X(t), t) \ , \tag{5.71}$$

then, by the same reasoning, it follows that

$$\begin{aligned} Q(X(t), t) &= J^{p*,c*}(X(t), t) & (5.72) \\ \Longrightarrow Q(X(t), t) &\leq v(X(t), t) & (5.73) \\ \Longrightarrow Q(X(t), t) &= v(X(t), t) \ , & (5.74) \end{aligned}$$

and (p^*, c^*) are optimal controls. □

More generally (see Appendix to Domenico Cuoco's lecture notes, part IV), consider the problem

$$\begin{aligned} &\max_{\alpha \in \mathcal{A}} \quad \mathrm{E}\{\int_0^T f(\alpha(t), X^\alpha(t), t)\, dt + g(X^\alpha(T))\} & (5.75) \\ &\text{s.t.} \quad dX^\alpha(t) = \mu(\alpha(t), X^\alpha(t), t)\, dt + \sigma(\alpha(t), X^\alpha(t), t)\, dB(t) & (5.76) \end{aligned}$$

where B is an n-dimensional Brownian motion, $\mathcal{A}$ is a set of predictable controls taking values in a closed set $K \subset \mathcal{R}^m$, X^α is a d-dimensional vector of state variables, and f, g, μ, and σ are given continuous functions. Assume that for every $\alpha \in \mathcal{A}$ there exists a unique solution to the stochastic differential equation above and the objective function is finite.

Theorem 5.1 *Suppose that K is compact, the functions μ and σ are continuously differentiable, μ_X, σ_X, μ_t, and σ_t are bounded, there exists a constant M such that $|\mu(\alpha, X, t)| + |\sigma(\alpha, X, t)| \leq M(1 + |\alpha| + |X|)$ for every $(\alpha, X, t) \in K \times \mathcal{R}^d \times [0, T]$, and f and g satisfy a pgc. Suppose $Q \in C^{2,1}$ satisfies the HJB equation*

$$0 = \sup_{\alpha \in \mathcal{A}} f(\alpha, X^\alpha, t) + \mathcal{D}^\alpha Q(\alpha, X^\alpha, t) \tag{5.77}$$

and terminal condition $Q(x, T) = g(x)$. If Q satisfies a pgc, then

$$Q(X(t), t) \geq J^\alpha(X(t), t) \equiv \mathrm{E}\{\int_t^T f(\alpha(s), X^\alpha(s), t)\, ds + g(X^\alpha(T))|\mathcal{F}_t\} \ \forall\ \alpha \in \mathcal{A} \ . \tag{5.78}$$

If, in addition, there exists an $\alpha^* \in \mathcal{A}$ *such that*

$$0 = f(\alpha^*(t), X^{\alpha*}(t), t) + \mathcal{D}^{\alpha*} Q(X^{\alpha*}(t), t) \tag{5.79}$$

for $(\lambda \times \mathcal{P})$*-almost every* $(t, \omega) \in [0, T] \times \Omega$ *then* $Q(x, t) = J^{\alpha*}(x, t)$ *and* α^* *is optimal.*

6 Incomplete Markets and Portfolio Constraints

These notes are taken from Chapters 5 and 6 of Karatzas and Shreve (1998).

6.1 The Financial Market

Assume the market is the standard market with the following modifications:

1. The number of securities n equals the number of Brownian motions d.
2. The volatility process $\sigma(t)$ is bounded and nonsingular, with $\sigma^{-1}(t)$ bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$.
3. There exists a positive constant s_0 such that $S_0(T) \geq s_0$ a.s.
4. $Z(t) = e^{-\int_0^t \theta'(s)\, dB(s) - \frac{1}{2}\int_0^t |\theta(s)|^2\, ds}$ may be a martingale but need not be.

6.2 Consumption and Portfolio Value

Let $C(t)$ denote an investor's cumulative consumption, an adapted process with non-decreasing, right-continuous paths, C(0)=0, and $C(T) < \infty$ a.s. Given initial wealth x, consumption plan C, and trading strategy π, the investor's portfolio value or wealth process $X^{x,C,\pi}(t)$ satisfies

$$M(t) \equiv H(t)X^{x,C,\pi}(t) + \int_{(0,t]} H(s)\, dC(s) = x + \int_0^t H(s)[\pi(s)\sigma(s) - X^{x,C,\pi}(s)\theta'(s)]\, dB(s)\ . \tag{6.1}$$

We can also write

$$\beta(t)X^{x,C,\pi}(t) + \int_{(0,t]} \beta(s)dC(s) = x + \int_0^t \beta(s)\pi(s)\sigma(s)\, dB^*(s)\ . \tag{6.2}$$

6.3 Portfolio Constraint

Suppose the investor is required to keep proportional portfolio holdings $p(t) \equiv \pi(t)/X^{x,C,\pi}(t)$ in a nonempty, closed, convex set $K \subseteq \mathcal{R}^n$. Define $p(t) \equiv p_*$ for some fixed but arbitrary vector $p_* \in K$ whenever $X^{x,C,\pi}(t) = 0$.

Definition 6.1 The consumption-portfolio pair (C, π) is *admissible for initial wealth* $x \geq 0$ *and constraint set* K, or $(C, \pi) \in \mathcal{A}(x; K)$, if $X^{x,C,\pi}(t) \geq 0$ for every $t \in [0, T]$ a.s. and $p(t) \in K$ a.s. for almost every $t \in [0, T]$.

Let $\mathcal{M}(K)$ denote the financial market in which all investors are constrained to hold portfolios in K. For $x \geq 0$ and $(C, \pi) \in \mathcal{A}(x; K)$, $M(t)$ defined in equation (6.1) is a supermartingale, and thus satisfies the budget constraint

$$\mathrm{E}\{H(T)X^{x,C,\pi}(T) + \int_{(0,T]} H(s)\, dC(s)\} \leq x \ . \tag{6.3}$$

Conversely, extending previous results to cases where cumulative consumption $C(t)$ need not be of the form $\int_0^t c(s)\, ds$ for some consumption flow process, we have the following.

Theorem 6.1 *Let* $x \geq 0$ *and suppose* $K = \mathcal{R}^n$. *Let* C *be a cumulative consumption plan and* W *a nonnegative random variable such that*

$$\mathrm{E}\{H(T)W + \int_{(0,T]} H(s)\, dC(s)\} = x \ . \tag{6.4}$$

Then there exists a trading strategy π *such that* $(C, \pi) \in \mathcal{A}(x; \mathcal{R}^n)$ *and the corresponding wealth process is*

$$X^{x,C,\pi}(t) = \frac{1}{H(t)}\mathrm{E}\{H(T)W + \int_{(t,T]} H(s)\, dC(s) | \mathcal{F}_t\} \ , \ 0 \leq t \leq T \ . \tag{6.5}$$

In particular, $H(t)X^{x,C,\pi}(t) + \int_{(0,t]} H(s)\, dC(s)$ *is a martingale and* $X^{x,C,\pi}(T) = W$ *a.s.*

6.4 Convex Sets and Support Functions

In order to characterize contingent claim prices and optimal consumption and portfolio choice in the presence of portfolio constraints, we make use of the mathematical constructions described in this section and the next.

The *support function* of the convex set $-K$, $\zeta : \mathcal{R}^n \to \mathcal{R} \bigcup \{+\infty\}$, is defined as

$$\zeta(\nu) \equiv \sup_{p \in K}(-p\nu), \nu \in \mathcal{R}^n \ . \tag{6.6}$$

Its *effective domain* is $\tilde{K} \equiv \{\nu \in \mathcal{R}^n : \zeta(\nu) < \infty\}$, a convex cone (called the barrier cone of $-K$). $0 \in \tilde{K}$ and $\zeta(0) = 0$. The function ζ is *positively homogeneous*, i.e.,

$$\zeta(\alpha\nu) = \alpha\zeta(\nu) \ \forall \nu \in \mathcal{R}^n, \alpha \geq 0 \ , \tag{6.7}$$

and *subadditive*, i.e.,

$$\zeta(\nu + \mu) \leq \zeta(\nu) + \zeta(\mu) \ \forall \nu, \mu \in \mathcal{R}^n \ . \tag{6.8}$$

In addition, note that

$$p \in K \Longleftrightarrow \zeta(\nu) + p\nu \geq 0 \ \forall \nu \in \tilde{K} \ . \tag{6.9}$$

Assume that ζ is bounded below on $\mathcal{R}^n$. This will be true, for example, if $0 \in K$.

Examples

1. No constraint: $K = \mathcal{R}^n$. Then $\tilde{K} = \{0\}$ and $\zeta \equiv 0$ on $\tilde{K}$.

2. No short sales: $K = [0, \infty)^n$. Then $\tilde{K} = K$ and $\zeta \equiv 0$ on $\tilde{K}$.

3. Incomplete market: $K = \{p \in \mathcal{R}^n : p_{m+1} = \cdots = p_n = 0\}$ for some $m \in \{1, \ldots, n-1\}$. Then $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 = \cdots = \nu_m = 0\}$ and $\zeta \equiv 0$ on $\tilde{K}$.

4. Incomplete market with no short sales: $K = \{p \in \mathcal{R}^n : p_1 \geq 0, \ldots, p_m \geq 0, p_{m+1} = \cdots = p_n = 0\}$ for some $m \in \{1, \ldots, n-1\}$. Then $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 \geq 0, \ldots, \nu_m \geq 0\}$ and $\zeta \equiv 0$ on $\tilde{K}$.

5. No borrowing: $K = \{p \in \mathcal{R}^n : p1 \leq 1\}$. Then $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 = \cdots = \nu_n \leq 0\}$ and $\zeta(\nu) = -\nu_1$ on $\tilde{K}$.

6. Constraints on short sales: $K = [-\kappa, \infty)^n$ for some $\kappa > 0$. Then $\tilde{K} = [0, \infty)^n$ and $\zeta(\nu) = \kappa\nu'1$ on $\tilde{K}$.

7. Constraints on borrowing: $K = \{p \in \mathcal{R}^n : p1 \leq \kappa\}$ for some $\kappa > 1$. Then $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 = \cdots = \nu_n \leq 0\}$ and $\zeta(\nu) = -\kappa\nu_1$ on $\tilde{K}$.

8. Rectangular constraints: $K = I_1 \times \cdots \times I_n$ with $I_j = [\alpha_j, \beta_j], -\infty \leq \alpha_j \leq 0 \leq \beta_j \leq +\infty$, with the understanding that I_j is open on the left if $\alpha_j = -\infty$ and I_j is open on the right if $\beta_j = \infty$. If all the α_j and β_j are finite, then

$$\tilde{K} = \mathcal{R}^n \text{ and } \zeta(\nu) = -\sum_{j=1}^{n}(\alpha_j \nu_j^+ - \beta_j \nu_j^-) \ . \tag{6.10}$$

More generally,

$$\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_j \geq 0 \ \forall j \text{ s.t. } \beta_j = \infty \text{ and } \nu_j \leq 0 \ \forall j \text{ s.t. } \alpha_j = -\infty\} \tag{6.11}$$

and the above formula for ζ remains valid.

6.5 Family of Auxiliary Markets

Definition 6.2 Let $\mathcal{H}$ denote the Hilbert space of progressively measurable $\mathcal{R}^n$-valued processes $\nu(t)$ with norm $[[\nu]]^2 = \mathrm{E}\int_0^T ||\nu(t)||^2\,dt$ and inner product $\langle\nu_1, \nu_2\rangle = \mathrm{E}\int_0^T \nu_1'(t)\nu_2(t)\,dt$. Let $\mathcal{D} = \{\nu \in \mathcal{H} : \nu(t) \in \tilde{K}$ and $\mathrm{E}\int_0^T \zeta(\nu(t))\,dt < \infty\}$. Let $\mathcal{D}^{(b)}$ be the set of bounded processes in $\mathcal{D}$.

For any given $\nu \in \mathcal{D}$, define

$$r_\nu(t) = r(t) + \zeta(\nu(t)) \ , \tag{6.12}$$

$$\mu_\nu(t) = \mu(t) + \nu(t) + \zeta(\nu(t))1 \ , \tag{6.13}$$

and construct a new market $\mathcal{M}_\nu$ with coefficients $(r_\nu, \mu_\nu, \delta, \sigma, S(0))$. In this new market,

$$\begin{aligned} \theta_\nu(t) &= \sigma^{-1}(t)[\mu_\nu(t) - r_\nu(t)1] = \theta(t) + \sigma^{-1}(t)\nu(t)\ , && (6.14)\\ Z_\nu(t) &= e^{-\int_0^t \theta'_\nu(s)\,dB(s) - \frac{1}{2}\int_0^t |\theta_\nu(s)|^2\,ds}\ , && (6.15)\\ B_\nu(t) &= B(t) + \int_0^t \theta_\nu(s)\,ds\ , && (6.16)\\ \beta_\nu(t) &= e^{-\int_0^t r_\nu(s)\,ds}\ , && (6.17)\\ H_\nu(t) &= \beta_\nu(t) Z_\nu(t)\ . && (6.18) \end{aligned}$$

If Z from the original market is a martingale then we may define the standard martingale measure $\mathcal{P}^*$ by $d\mathcal{P}^*/d\mathcal{P} = Z(T)$. If Z_ν is a martingale, then we say $\nu \in \mathcal{D}^{(m)}$ and we can define a standard martingale measure $\mathcal{P}_\nu$ for $\mathcal{M}_\nu$ by $d\mathcal{P}_\nu/d\mathcal{P} = Z_\nu(T)$. Note that if θ is bounded then $\mathcal{D}^{(b)} \subseteq \mathcal{D}^{(m)}$.

In the market $\mathcal{M}_\nu$, the wealth process $X_\nu^{x,C,\pi}(t)$ corresponding to initial wealth $x \geq 0$, consumption plan C, and trading strategy π satisfies

$$\begin{aligned} M_\nu(t) &\equiv H_\nu(t) X_\nu^{x,C,\pi}(t) + \int_{(0,t]} H_\nu(s)\,dC(s) \\ &= x + \int_0^t H_\nu(s) X_\nu^{x,C,\pi}(s)[p(s)\sigma(s) - \theta'_\nu(s)]\,dB(s)\ , && (6.19) \end{aligned}$$

or equivalently,

$$\beta_\nu(t) X_\nu^{x,C,\pi}(t) + \int_{(0,t]} \beta_\nu(s) dC(s) = x + \int_0^t \beta_\nu(s) X_\nu^{x,C,\pi}(s) p(s)\sigma(s)\,dB_\nu(s)\ , \qquad (6.20)$$

or equivalently,

$$\begin{aligned} \beta(t) X_\nu^{x,C,\pi}(t) + \int_{(0,t]} \beta(s) dC(s) &= x + \int_0^t \beta(s) X_\nu^{x,C,\pi}(s)[\zeta(\nu(s)) + p(s)\nu(s)]\,ds \\ &\quad + \int_0^t \beta(s) X_\nu^{x,C,\pi}(s) p(s)\sigma(s)\,dB^*(s)\ . && (6.21) \end{aligned}$$

Definition 6.3 Let $\nu \in \mathcal{D}$ be given. A consumption-portfolio pair (C, π) is *admissible in* $\mathcal{M}_\nu$ *for initial wealth* $x \geq 0$, $(C, \pi) \in \mathcal{A}_\nu(x)$ if the process $X_\nu^{x,C,\pi}(t) \geq 0$ for all $t \in [0, T]$ a.s.

Note that for $x \geq 0$ and $(C, \pi) \in \mathcal{A}_\nu(x)$, $M_\nu(t)$ is a nonnegative local martingale, hence a supermartingale, so we have the budget constraint

$$\mathrm{E}\{H_\nu(T) X_\nu^{x,C,\pi}(T) + \int_{(0,T]} H_\nu(s)\, dC(s)\} \leq x \ . \tag{6.22}$$

Theorem 6.2 *Let $\nu \in \mathcal{D}$ and $x \geq 0$ be given. Let C be a consumption plan and W a nonnegative random variable such that*

$$\mathrm{E}\{H_\nu(T) W + \int_{(0,T]} H_\nu(s)\, dC(s)\} = x \ . \tag{6.23}$$

Then there exists a trading strategy π such that $(C, \pi) \in \mathcal{A}_\nu(x)$ and the corresponding wealth process is

$$X_\nu^{x,C,\pi}(t) = \frac{1}{H_\nu(t)} \mathrm{E}\{H_\nu(T) W + \int_{(t,T]} H_\nu(s)\, dC(s) | \mathcal{F}_t\} \ , 0 \leq t \leq T \ . \tag{6.24}$$

In particular, $H_\nu(t) X_\nu^{x,C,\pi}(t) + \int_{(0,t]} H_\nu(s)\, dC(s)$ is a martingale and $X_\nu^{x,C,\pi}(T) = W$ a.s.

6.6 Upper Hedging Price

In the presence of constraints, we can no longer replicate every contingent claim exactly, so we can no longer get a unique no arbitrage price. However we can define no arbitrage bounds on the equilibrium claim price in an intuitive and useful way. The upper hedging price is the infimal cost of hedging a short position in a given contingent claim. The lower hedging price is essentially the supremal price of the saleable component of the claim payoff–the most the holder of a long position could sell off and still be hedged. We will focus on the upper hedging price.

Definition 6.4 A *contingent claim W* is a nonnegative random variable. The *unconstrained hedging price of W in market $\mathcal{M}$* is

$$u_0 \equiv \mathrm{E}\{H(T) W\} \ . \tag{6.25}$$

The *upper hedging price of* W *in* $\mathcal{M}(K)$ is

$$h_{up}(K) \equiv \inf\{x \geq 0 : \exists (C, \pi) \in \mathcal{A}(x; K) \text{ with } X^{x,C,\pi}(T) \geq W \text{ a.s.}\} \ . \tag{6.26}$$

W is K-*attainable* if $h_{up}(K) < \infty$ and there exists π such that $(0, \pi) \in \mathcal{A}(h_{up}(K); K)$ and $X^{h_{up}(K),0,\pi}(T) = W$ a.s.

Note that $h_{up}(\mathcal{R}^n) = u_0$ and there exists $\pi \in \mathcal{A}(u_0; \mathcal{R}^n)$ such that $X^{u_0,0,\pi}(t) = \frac{1}{H(t)}\mathrm{E}\{H(T)W|\mathcal{F}_t\}$ and $X^{u_0,0,\pi}(T) = W$ a.s. Moreover, π is uniquely defined.

Definition 6.5 Let W be a contingent claim. The *unconstrained hedging price of* W *in market* $\mathcal{M}_\nu$ is

$$u_\nu \equiv \mathrm{E}\{H_\nu(T)W\} \geq 0 \ . \tag{6.27}$$

If $u_\nu < \infty$, then an *unconstrained hedging strategy* is any trading strategy π_ν satisfying

$$X_\nu^{u_\nu,0,\pi_\nu}(t) = \frac{1}{H_\nu(t)}\mathrm{E}\{H_\nu(T)W|\mathcal{F}_t\} \ , 0 \leq t \leq T \ . \tag{6.28}$$

Note that π_ν is uniquely defined.

Now for the main result about upper hedging:

Theorem 6.3 (Characterization of the Upper Hedging Price) *For any contingent claim* W*, we have*

$$h_{up}(K) = \sup_{\nu \in \mathcal{D}} u_\nu \ . \tag{6.29}$$

Furthermore, if $\hat{u} \equiv \sup_{\nu \in \mathcal{D}} u_\nu$ *is finite, then there exists a pair* $(\hat{C}, \hat{\pi}) \in \mathcal{A}(\hat{u}, K)$ *such that*

$$X^{\hat{u},\hat{C},\hat{\pi}}(t) = \hat{X}(t) \equiv \operatorname*{ess\,sup}_{\nu \in \mathcal{D}} \frac{\mathrm{E}\{H_\nu(T)W|\mathcal{F}_t\}}{H_\nu(t)}, \ 0 \leq t \leq T \ , \tag{6.30}$$

and in particular, $X^{\hat{u},\hat{C},\hat{\pi}}(T) = W$ *a.s. Call* $\hat{X}(t)$ *the* upper hedging process *for claim* W.

Sketch of Proof First we establish $h_{up}(K) \geq \hat{u}$. Suppose $h_{up}(K) < \infty$. Let $x \in [0,\infty)$ be such that $\exists (C,\pi) \in \mathcal{A}(x;K)$ with $X^{x,C,\pi}(T) \geq W$. Let $\nu \in \mathcal{D}$ be given. Define $C_\nu(t)$ by

$$\int_{(0,t]} \beta(s)\,dC_\nu(s) = \int_{(0,t]} \beta(s)\,dC(s) + \int_0^t \beta(s)[X^{x,C,\pi}(s)\zeta(\nu(s)) + \pi(s)\nu(s)]\,ds \tag{6.31}$$

$$= \int_{(0,t]} \beta(s)\,dC(s) + \int_0^t \beta(s)X^{x,C,\pi}(s)[\zeta(\nu(s)) + p(s)\nu(s)]\,ds \tag{6.32}$$

where $p(t) \in K$. Since $\zeta(\nu) + p\nu \geq 0$, it follows that $C_\nu(t)$ is nondecreasing, and hence a consumption process. From equation (6.21), using (C_ν, π) in market $\mathcal{M}_\nu$ generates wealth process $X_\nu^{x,C,\pi}(t)$ such that

$$\begin{aligned}\beta(t)X_\nu^{x,C_\nu,\pi}(t) &+ \int_{(0,t]} \beta(s)dC(s) + \int_0^t \beta(s)[X^{x,C,\pi}(s)\zeta(\nu(s)) + \pi(s)\nu(s)]\,ds \\ &= x + \int_0^t \beta(s)[X_\nu^{x,C_\nu,\pi}(s)\zeta(\nu(s)) + \pi(s)\nu(s)]\,ds \\ &\quad + \int_0^t \beta(s)\pi(s)\sigma(s)\,dB^*(s)\ .\end{aligned} \tag{6.33}$$

By comparison with equation (6.2), we see that $X_\nu^{x,C_\nu,\pi}(t) = X^{x,C,\pi}(t)$. Because of the budget constraint,

$$x \geq \mathrm{E}\{H_\nu(T)X_\nu^{x,C_\nu,\pi}(T)\} = \mathrm{E}\{H_\nu(T)X^{x,C,\pi}(T)\} \geq \mathrm{E}\{H_\nu(T)W\} = u_\nu \tag{6.34}$$

Therefore,

$$\inf\{x \geq 0 : \exists (C,\pi) \in \mathcal{A}(x;K) \text{ with } X^{x,C,\pi}(T) \geq W \text{ a.s.}\} \geq \sup_{\nu\in\mathcal{D}} u_\nu \tag{6.35}$$

$$\Longrightarrow h_{up}(K) \geq \hat{u}\ . \tag{6.36}$$

Next, we establish $h_{up}(K) \leq \hat{u}$. Suppose $\hat{u} < \infty$. We use the following result:

Lemma 6.1 *(Proposition 5.6.5, Karatzas and Shreve (1998), p. 213) If $\hat{u} < \infty$, then $\hat{X}(t)$ is finite and satisfies the dynamic programming equation*

$$\hat{X}(s) = \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \frac{\mathrm{E}\{H_\nu(t)\hat{X}(t)|\mathcal{F}_s\}}{H_\nu(s)},\ 0 \leq s \leq t \leq T\ . \tag{6.37}$$

Furthermore, $\hat{X}(t)$ has an RCLL modification and choosing this modification, we have that $H_\nu(t)\hat{X}(t)$ is an RCLL supermartingale for every $\nu \in \mathcal{D}$.

Now fix $\nu \in \mathcal{D}$. The supermartingale $H_\nu(t)\hat{X}(t)$ has a unique Doob-Meyer decomposition

$$H_\nu(t)\hat{X}(t) = \hat{u} + \int_0^t \psi'_\nu(s)\, dB(s) - A_\nu(t), 0 \leq t \leq T \ , \tag{6.38}$$

where ψ_ν is progressively measurable and almost surely square-integrable, and $A_\nu(t)$ is an adapted, nondecreasing, RC process with $\mathrm{E}\{A_\nu(T)\} < \infty$ and $A_\nu(0) = 0$.

Next, let μ be another process on $\mathcal{D}$ and note that

$$d(\frac{H_\mu(t)}{H_\nu(t)}) = \frac{H_\mu(t)}{H_\nu(t)}[(\theta_\nu(t) - \theta_\mu(t))'\, dB(t) + ((\theta_\nu(t) - \theta_\mu(t))'\theta_\nu(t) + \zeta(\nu(t)) - \zeta(\mu(t)))\, dt] \tag{6.39}$$

By equating the drift (i.e, finite variation) and diffusion terms of $d(H_\mu(t)\hat{X}(t))$ with those obtained by applying the Itô product rule to $d[(\frac{H_\mu(t)}{H_\nu(t)})(H_\nu(t)\hat{X}(t))]$, it can be shown that

$$\varphi(t) \equiv \frac{\psi_\nu(t)}{H_\nu(t)} + \hat{X}(t)\theta_\nu(t) \tag{6.40}$$

and

$$\hat{C}(t) \equiv \int_{(0,t]} \frac{dA_\nu(s)}{H_\nu(s)} - \int_0^t [\hat{X}(s)\zeta(\nu(s)) + \varphi'(s)\sigma^{-1}(s)\nu(s)]\, ds \tag{6.41}$$

are the same for all $\nu \in \mathcal{D}$ (see pp. 217-218 of Karatzas and Shreve (1998)). Setting $\nu \equiv 0$ gives

$$\hat{C}(t) \equiv \int_{(0,t]} \frac{dA_0(s)}{H(s)} \tag{6.42}$$

which shows that $\hat{C}$ is nondecreasing. Let $\hat{\pi}(t) \equiv \varphi'(t)\sigma^{-1}(t)$. Then we have

$$H(t)\hat{X}(t) = \hat{u} + \int_0^t \psi'_0(s)\, dB(s) - A_0(t) \tag{6.43}$$

$$= \hat{u} + \int_0^t H(s)[\hat{\pi}(s)\sigma(s) - \hat{X}(s)\theta'(s)]\, dB(s) - \int_0^t H(s)\, d\hat{C}(s)\ . \tag{6.44}$$

Thus, by comparison with equation (6.1), $\hat{X}(t) = X^{\hat{u},\hat{C},\hat{\pi}}(t)$.

Finally, let $\hat{p}(t) = \frac{\hat{\pi}(t)}{\hat{X}(t)}$, if $\hat{X}(t) \neq 0$, and $\hat{p}(t) = p_*$, otherwise. It remains show that $\hat{p}(t) \in K$, because then $\hat{u}$ will be a superreplication cost, and thus greater than or equal to $h_{up}(K)$.

Lemma 6.2 *(Lemma 5.4.2, Karatzas and Shreve (1998), p. 207) For any progressively measurable $\mathcal{R}^n$-valued process p, there exists a progressively measurable $\mathcal{R}^n$-valued process ν such that $|\nu(t)| \leq 1$, $|\zeta(\nu(t))| \leq 1$ for $0 \leq t \leq T$ a.s., and, for every $t \in [0,T]$, $p(t) \in K \iff \nu(t) = 0$, $p(t) \notin K \iff \zeta(\nu) + p\nu < 0$ a.s.*

Now let ν be as above and note that $k\nu \in \mathcal{D}\,\forall k > 0$. Let $\hat{\tau} = \inf\{t \in [0,T) : \hat{X}(t) = 0\} \wedge T$.

$$0 \leq \int_{(0,\hat{\tau}]} \frac{dA_{k\nu}(s)}{H_{k\nu}(s)} \tag{6.45}$$

$$= \hat{C}(\hat{\tau}) + k \int_0^{\hat{\tau}} \hat{X}(s)[\zeta(\nu(s)) + \hat{p}(s)\nu(s)]\, ds\ . \tag{6.46}$$

The integrand on the right-hand side is nonpositive by choice of ν. But the right-hand side is nonnegative for all k, so the integrand must be zero, which implies $\hat{p} \in K$. □

Remark In the proof above, and thus in the theorem, one can replace $\mathcal{D}$ with $\mathcal{D}^{(b)}$, the set of bounded processes in $\mathcal{D}$.

Upper Hedging with Constant Coefficients and Path-Independent Claims

Assume r and σ are constant, $\delta = 0$, and θ is bounded, so that Z is a martingale and we may use $\mathcal{P}^*$. Consider a path-independent claim $W = \varphi(S(T))$. In the original market we can write

$$u(T-t, x; \varphi) = \mathrm{E}^*\{\beta(t,T)\varphi(S(T))|\mathcal{F}_t\} \tag{6.47}$$

$$= e^{-r(T-t)}\mathrm{E}^*\{\varphi(x_1 Y_1^*(t,T), \ldots, x_n Y_n^*(t,T))|\mathcal{F}_t\}\ , \tag{6.48}$$

where

$$Y_k^*(t,T) \equiv e^{r(T-t)+\sigma_k(B^*(T)-B^*(t))-\frac{1}{2}|\sigma_k|^2(T-t)}\ . \tag{6.49}$$

The unconstrained value of W in market $\mathcal{M}_\nu$ is

$$u_\nu(t) \equiv \frac{\mathrm{E}\{H_\nu(T)\varphi(S(T))|\mathcal{F}_t\}}{H_\nu(t)}\ . \tag{6.50}$$

If Z_ν is a martingale, we can use $\mathcal{P}_\nu$, the emm for market $\mathcal{M}_\nu$, and write

$$u_\nu(t) = \mathrm{E}_\nu\{\beta_\nu(t,T)\varphi(S(T))|\mathcal{F}_t\}\ . \tag{6.51}$$

Note that since $dB_\nu(t) = dB(t) + (\theta + \sigma^{-1}\nu(t))\,dt$,

$$S_k(T) = S_k(t)e^{\int_t^T \mu_k(u)\,du + \sigma_k(B(T)-B(t)) - \frac{1}{2}|\sigma_k|^2(T-t)} \quad (6.52)$$

$$= S_k(t)e^{\int_t^T [r-\nu_k(u)]\,du + \sigma_k(B_\nu(T)-B_\nu(t)) - \frac{1}{2}|\sigma_k|^2(T-t)} . \quad (6.53)$$

Therefore,

$$u_\nu(t) = e^{-r(T-t)}\mathrm{E}_\nu\{e^{-\int_t^T \zeta(\nu(s))\,ds}\varphi(x_1 e^{-\int_t^T \nu_1(s)\,ds}Y_1^{(\nu)}(t,T), \ldots, \quad (6.54)$$

$$x_n e^{-\int_t^T \nu_n(s)\,ds}Y_n^{(\nu)}(t,T))|\mathcal{F}_t\} , \quad (6.55)$$

where

$$Y_k^{(\nu)}(t,T) \equiv e^{r(T-t)+\sigma_k(B_\nu(T)-B_\nu(t))-\frac{1}{2}|\sigma_k|^2(T-t)} . \quad (6.56)$$

We are interested in the upper hedging value process for $\varphi(S(T))$, which from Theorem 6.3, is

$$\hat{X}(t) = \sup_{\nu\in\mathcal{D}^{(b)}} u_\nu(t), 0 \leq t \leq T . \quad (6.57)$$

It turns out that one can achieve the supremum in equation (6.57) using just the constant ν processes because of the convexity of ζ and $\tilde{K}$ (see pp. 222-223 of Karatzas and Shreve, 1998). Restricting attention to constant ν, and we can write,

$$u_\nu(t) = u_\nu(T-t, x; \varphi) = e^{-r(T-t)}\mathrm{E}_\nu\{e^{-\zeta(\nu)}\varphi(x_1 e^{-\nu_1}Y_1^{(\nu)}(t,T), \ldots, \quad (6.58)$$

$$x_n e^{-\nu_n}Y_n^{(\nu)}(t,T))|\mathcal{F}_t\} , \quad (6.59)$$

Note that the distribution of $Y^{(\nu)}$ under $\mathcal{P}_\nu$ is the same as the distribution of Y^* under $\mathcal{P}^*$. So we can write

$$u_\nu(T-t, x; \varphi) = e^{-r(T-t)}\mathrm{E}^*\{e^{-\zeta(\nu)}\varphi(x_1 e^{-\nu_1}Y_1^*(t,T), \ldots, x_n e^{-\nu_n}Y_n^*(t,T))|\mathcal{F}_t\} . \quad (6.60)$$

The following theorem essentially says that we can switch the order of sup and E^* in equation (6.57):

Theorem 6.4 *Assume* $\varphi : (0,\infty)^n \to [0,\infty)$ *is lower semi-continuous and satisfies a pgc. Define* $\hat{\varphi} : (0,\infty)^n \to [0,\infty)$ *by*

$$\hat{\varphi}(x) \equiv \sup_{\nu \in \tilde{K}} [e^{-\zeta(\nu)} \varphi(x_1 e^{-\nu_1}, \ldots, x_n e^{-\nu_n})] \; . \tag{6.61}$$

Then the upper hedging process is

$$\begin{aligned} \hat{X}(t) &= e^{-r(T-t)} \mathrm{E}^* \{ \hat{\varphi}(S(T)) | \mathcal{F}_t \} & (6.62) \\ &= u(T-t, S(t); \hat{\varphi}), 0 \le t \le T, \; a.s. & (6.63) \end{aligned}$$

6.7 Optimal Consumption and Investment with Constraints

Now assume $\mathrm{E} \int_0^T |\theta(s)|^2 \, ds < \infty$, Z is a martingale, and the investor chooses a consumption rate $c(t)$, i.e., $dC(t) = c(t)\,dt$, where $c(t)$ is progressively measurable and nonnegative, with $\int_0^T c(t)\,dt < \infty$ a.s. Specifying initial wealth x, consumption rate plan $c(t)$, and proportional trading strategy $p(t)$ determines the investor's wealth process $X^{x,c,p}$, which satisfies

$$\beta(t) X^{x,c,p}(t) + \int_0^t \beta(s) c(s)\, ds = x + \int_0^t \beta(s) X^{x,c,p}(s) p(s) \sigma(s)\, dB^*(s) \; . \tag{6.64}$$

Let

$$\tau_0 \equiv \inf\{ t \in [0,T] : X^{x,c,p}(t) = 0 \} \; . \tag{6.65}$$

Definition 6.6 Given $x \ge 0$, the pair (c,p) is *admissible at* x *in the unconstrained market* $\mathcal{M}$, $(c,p) \in \mathcal{A}(x)$, if $c(t) = 0$ for a.e. $t \in [\tau_0, T]$, and take $X^{x,c,p}(t) \equiv 0$ on $[\tau_0, T]$. If, in addition, $p(t) \in K$ for a.e. $t \in [0,T]$, a.s., then (c,p) is *admissible at* x *in the constrained market* $\mathcal{M}(K)$, $(c,p) \in \mathcal{A}(x;K)$.

Preferences Suppose the investor derives utility from consumption and terminal wealth $\mathrm{E}\{\int_0^T U_1(c(t),t)\,dt + U_2(X^{x,c,p}(T))\}$ where $U_1(\cdot,t)$ and $U_2(\cdot)$ are utility functions with subsistence level zero and $\lim_{c \downarrow 0} U_1'(c,t) = \lim_{x \downarrow 0} U_2'(x) = +\infty$. Then $U_1'(\cdot,t)$ and $U_2'(\cdot)$ have strictly decreasing C^1 inverse functions mapping $(0,\infty)$ onto $(0,\infty)$.

Remark The theory goes through without change if one sets $U_1 \equiv 0$ and $c \equiv 0$.

Let $\mathcal{A}_3(x;K) = \{(c,p) \in \mathcal{A}(x;K) : -\mathrm{E}\int_0^T U_1(c(t),t)^- \, dt \wedge -\mathrm{E}U_2(X^{x,c,p}(T))^- > -\infty\}$.

Investor's Constrained Optimization Problem Find a pair $(\hat{c},\hat{p})$ to solve

$$V(x;K) \equiv \sup_{(c,p)\in\mathcal{A}_3(x;K)} \mathrm{E}\{\int_0^T U_1(c(t),t)\,dt + U_2(X^{x,c,p}(T))\} \; . \tag{6.66}$$

Family of Unconstrained Problems In market $\mathcal{M}_\nu$, the wealth process $X_\nu^{x,c,p}$ evolves according to

$$H_\nu(t)X_\nu^{x,c,p}(t) + \int_0^t H_\nu(s)c(s)\,ds = x + \int_0^t H_\nu(s)X_\nu^{x,c,p}(s)[p(s)\sigma(s) - \theta_\nu'(s)]\,dB(s) \; , \tag{6.67}$$

or, equivalently,

$$\beta_\nu(t)X_\nu^{x,c,p}(t) + \int_0^t \beta_\nu(s)c(s)\,ds = x + \int_0^t \beta_\nu(s)X_\nu^{x,c,p}(s)p(s)\sigma(s)\,dB_\nu(s) \; , \tag{6.68}$$

or, equivalently,

$$\begin{aligned} \beta(t)X_\nu^{x,c,p}(t) + \int_0^t \beta(s)c(s)\,ds \;\; &= \;\; x + \int_0^t \beta(s)X_\nu^{x,c,p}(s)p(s)\sigma(s)\,dB^*(s) + \\ &\int_0^t \beta(s)X_\nu^{x,c,p}(s)[\zeta(\nu(s)) + p(s)\nu(s)]\,ds \; . \end{aligned} \tag{6.69}$$

Definition 6.7 Given $x \geq 0$, the pair (c,p) is *admissible at* x *in market* $\mathcal{M}_\nu$, $(c,p) \in \mathcal{A}_\nu(x)$, if $c(t) = 0$ for a.e. $t \in [\tau_\nu, T]$, where $\tau_\nu \equiv \inf\{t \in [0,T] : X_\nu^{x,c,p}(t) = 0\}$, and take $X^{x,c,p}(t) \equiv 0$ on $[\tau_\nu, T]$.

Let $\nu \in \mathcal{D}$ and let $\mathcal{A}_3^\nu(x) \equiv \{(c,p) \in \mathcal{A}_\nu(x) : -\mathrm{E}\int_0^T U_1(c(t),t)^- \, dt \wedge -\mathrm{E}U_2(X_\nu^{x,c,p}(T))^- > -\infty\}$.

Unconstrained $\mathcal{M}_\nu$-Problem Given $\nu \in \mathcal{D}$ and $x \geq 0$, find $(\hat{c},\hat{p})$ to solve

$$V_\nu(x) \equiv \sup_{(c,p)\in\mathcal{A}_3^\nu(x)} \mathrm{E}\{\int_0^T U_1(c(t),t)\,dt + U_2(X_\nu^{x,c,p}(T))\} \; . \tag{6.70}$$

The solution to the unconstrained $\mathcal{M}_\nu$-problem is a straightforward extension of Proposition 5.3 in Section 5 on unconstrained optimization: Given $\nu \in \mathcal{D}$, let

$$\mathcal{X}_\nu(\lambda) \equiv \mathrm{E}\{\int_0^T H_\nu(t) I_1(\lambda H_\nu(t), t)\, dt + H_\nu(T) I_2(\lambda H_\nu(T))\} \ . \tag{6.71}$$

For every $\nu \in \mathcal{D}$ such that $\mathcal{X}_\nu(\lambda) < \infty$ for every $\lambda > 0$, $\mathcal{X}_\nu$ maps $(0,\infty)$ onto $(0,\infty)$. Let $\lambda_\nu(x)$ denote the inverse of $\mathcal{X}_\nu(\lambda)$ which maps $(0,\infty)$ onto $(0,\infty)$. Define

$$W_\nu \equiv I_2(\lambda_\nu(x) H_\nu(T)) \ , \tag{6.72}$$

$$c_\nu(t) \equiv I_1(\lambda_\nu(x) H_\nu(t), t) \ , \tag{6.73}$$

$$X_\nu(t) \equiv \frac{1}{H_\nu(t)} \mathrm{E}\{\int_t^T H_\nu(s) c_\nu(s)\, ds + H_\nu(T) W_\nu | \mathcal{F}_t\} \ , \tag{6.74}$$

$$M_\nu(t) \equiv H_\nu(t) X_\nu(t) + \int_0^t H_\nu(s) c_\nu(s)\, ds = \mathrm{E}\{\int_0^T H_\nu(s) c_\nu(s)\, ds + H_\nu(T) W_\nu | \mathcal{F}_t\} \ .$$

Note that $M_\nu(t)$ is a $\mathcal{P}$-martingale with $M_\nu(0) = \mathcal{X}(\mathcal{Y}(x)) = x$. By the Martingale Representation Theorem, we have

$$M_\nu(t) = x + \int_0^t \psi'(s)\, dB(s), 0 \le t \le T, \text{ a.s.} \tag{6.75}$$

for some predictable, square-integrable process ψ. Let

$$p_\nu(t) \equiv [\frac{\psi'(t)}{H_\nu(t) X_\nu(t)} + \theta'_\nu(t)] \sigma^{-1}(t) \ . \tag{6.76}$$

Then $X_\nu(t) = X_\nu^{x,c_\nu,p_\nu}(t)$ and (c_ν, p_ν) solves problem 6.70.

Proposition 6.1 *If $(c,p) \in \mathcal{A}(x;K)$ and $\nu \in \mathcal{D}$, then*

$$X_\nu^{x,c,p}(t) \ge X^{x,c,p}(t), \ 0 \le t \le T, \ a.s. \quad , \tag{6.77}$$

which implies $\mathcal{A}_3(x;K) \subseteq \mathcal{A}_3^\nu(x)$ and $V(x;K) \le V_\nu(x)$ for every $x > 0$ and $\nu \in \mathcal{D}$. Moreover, if $\zeta(\nu(t)) + p(t)\nu(t) = 0$ for a.e. $t \in [0,T]$ a.s., then $X_\nu^{x,c,p}(t) = X^{x,c,p}(t)$.

The intuition for the proof comes from noting that the wealth evolution equation (6.69) for X_ν differs only from that for X by the extra drift term on the right-hand side, which is nonnegative if $p \in K$ (see pp. 267-268 of Karatzas and Shreve, 1998).

Let $\mathcal{D}_0 = \{\nu \in \mathcal{D} : \mathcal{X}_\nu(\lambda) < \infty \ \forall \lambda > 0 \text{ and } V_\nu(x) < \infty \ \forall x > 0\}$.

Proposition 6.2 *Let $x > 0$ be given and suppose there exists $\hat{\nu} \in \mathcal{D}_0$ such that $p_{\hat{\nu}}(t) \in K$ and $\zeta(\hat{\nu}(t)) + p_{\hat{\nu}}(t)\hat{\nu}(t) = 0$ for a.e. $t \in [0,T]$ a.s. Then $(c_{\hat{\nu}}, p_{\hat{\nu}})$ is optimal for the original constrained problem 6.66, $V(x;K) = V_{\hat{\nu}}(x)$, and $\hat{\nu}$ minimizes $V_\nu(x)$.*

Proof On one hand, $(c_{\hat{\nu}}, p_{\hat{\nu}})$ is feasible for the original constrained problem, so

$$\begin{aligned} V(x;K) &\geq \mathrm{E}\{\int_0^T U_1(c_{\hat{\nu}}(t),t)\,dt + U_2(X^{x,c_{\hat{\nu}},p_{\hat{\nu}}}(T))\} && (6.78)\\ &= \mathrm{E}\{\int_0^T U_1(c_{\hat{\nu}}(t),t)\,dt + U_2(X_{\hat{\nu}}^{x,c_{\hat{\nu}},p_{\hat{\nu}}}(T))\} && (6.79)\\ &= V_{\hat{\nu}}(x)\ . && (6.80)\end{aligned}$$

On the other hand, $V(x;K) \leq V_\nu(x)\ \forall \nu \in \mathcal{D}$. Therefore, $V(x;K) = V_{\hat{\nu}}(x)$ and $\hat{\nu}$ minimizes $V_\nu(x)$. □

Note that $V_\nu(x)$ is like a Lagrangian for the constrained optimization problem, $\nu(t)$ is like a Lagrange multiplier for the constraint on $p(t)$, and $\zeta(\nu(t)) + p(t)\nu(t) = 0$ is like a complementary slackness condition.

Equivalent Optimality Conditions Let $x > 0$, let $(\hat{c},\hat{p}) \in \mathcal{A}_3(x;K)$, and let $\hat{X}(t) \equiv X^{x,\hat{c},\hat{p}}(t)$. Consider first the statement

(A) Optimality of $(\hat{c},\hat{p})$ for the original constrained problem:

$$V(x;K) = \mathrm{E}\{\int_0^T U_1(\hat{c}(t),t)\,dt + U_2(\hat{X}(T))\}\ . \tag{6.81}$$

Next, consider the following statements in terms of a process $\hat{\nu} \in \mathcal{D}_0$.

(B) Financeability of $(c_{\hat{\nu}}, p_{\hat{\nu}})$: There exists $p_{\hat{\nu}}(t)$ such that $(c_{\hat{\nu}}(t), p_{\hat{\nu}}(t)) \in \mathcal{A}_3(x;K)$ and $\zeta(\hat{\nu}(t)) + p_{\hat{\nu}}(t)\hat{\nu}(t) = 0$ for a.e. $t \in [0,T]$ a.s.

(C) Minimality of $\hat{\nu}$: $V_{\hat{\nu}}(x) \leq V_\nu(x)$ for every $\nu \in \mathcal{D}$.

(D) Dual Optimality of $\hat{\nu}$: With $y = \mathcal{Y}_{\hat{\nu}}(x)$, $\tilde{V}_{\hat{\nu}}(y) \leq \tilde{V}_\nu(y)$ for every $\nu \in \mathcal{D}$, where $\tilde{V}_\nu(y) \equiv \sup_{x>0}[V_\nu(x) - xy]$ is the convex dual of V_ν.

(E) Parsimony of $\hat{\nu}$: $\mathrm{E}\{\int_0^T H_\nu(t)c_{\hat{\nu}}(t)\,dt + H_\nu(T)W_{\hat{\nu}}\} \leq x$ for every $\nu \in \mathcal{D}$.

Theorem 6.5 *Conditions (B)-(E) are equivalent and imply (A) with* $(\hat{c}, \hat{p}) = (c_{\hat{\nu}}(t), p_{\hat{\nu}}(t))$. *Conversely, (A) implies the existence of a process* $\hat{\nu} \in \mathcal{D}_0$ *that satisfies (B)-(E) with* $p_{\hat{\nu}} = \hat{p}$, *provided that* U_1 *and* U_2 *satisfy the following conditions:*
(a) the map $x \mapsto xU_i'(x)$ *is nondecreasing on* $(0, \infty)$, $i = 1, 2$, *and*
(b) there exists $\beta \in (0, 1), \gamma \in (0, \infty)$ *such that* $\beta U_i'(x) \geq U_i'(\gamma x)$ $\forall x \in (0, \infty)$, $i = 1, 2$.

For the proof, see Karatzas and Shreve (1998), pp. 276-282, 335-347. That (B) implies (A) and (C) follows from Proposition 6.2. It is also intuitive that (B) implies (E). That (C) implies (D) is straightforward, and that (E) implies (D) is not hard, but proving (D) implies (B) (and, thus, (A)) takes several pages, and the proof that (A) implies (B) is relegated to Appendix C, 13 pages.

7 Equilibrium in a Pure Exchange Economy

These notes are taken from Chapter 4 of Karatzas and Shreve (1998). Suppose there are m investors in the economy.

7.1 Endowments

Each investor j is endowed with an exogenous flow $e_j(t)$ of the single, nonstorable, consumption good. Each $e_j(t)$ is nonnegative and progressively measurable on $[0, T]$. The *aggregate endowment*

$$e(t) \equiv \sum_{j=1}^{m} e_j(t) \tag{7.1}$$

is an Itô process described by

$$\frac{de(t)}{e(t)} = \mu_e(t)\,dt + \sigma_e(t)\,dB(t) \ , \tag{7.2}$$

where μ_e and σ_e are progressively measurable and bounded. Assume there exist constants $0 < \gamma_1 < \gamma_2 < \infty$ such that $\gamma_1 \leq e(t) \leq \gamma_2$.

7.2 Preferences

Each investor wants to maximize $\mathrm{E}\int_0^T U_j(c(t), t)\,dt$ where $c(t)$ is a nonnegative, progressively measurable consumption process satisfying $\mathrm{E}\int_0^T c(t)\,dt < \infty$ and U_j is a utility function satisfying

$$U_j(c, t) = e^{-\alpha(t)} u_j(c) \ , \tag{7.3}$$

where $u_j : (0, \infty) \to \mathcal{R}$ is C^3 and satisfies $u_j'(c) > 0$, $u_j''(c) < 0$, $\lim_{c\downarrow 0} u_j'(c) = \infty$, $\lim_{c\to\infty} u_j'(c) = 0$, and $\lim_{c\downarrow 0} \frac{u_j'''(c)}{(u_j''(c))^2}$ exists and is finite. Thus, the nonnegativity constraint on consumption is nonbinding and there exists an imuf $I_j : (0, \infty) \times [0, T] \to (0, \infty)$ such that $I_j(U_j'(c, t), t) = c$ for every $c \in (0, \infty)$, where $U_j'(c, t) \equiv \frac{\partial U_j(c,t)}{\partial c}$. The imuf $I_j(y, t)$ is C^2 and strictly decreasing in y.

7.3 Arrow-Debreu Equilibrium

Suppose investors can trade consumption plans at prices described by a state-price density $H(t)$ (thus, the market is complete). Then each investor trades so as to solve

$$\max_c \mathrm{E} \int_0^T U_j(c(t),t))\, dt \text{ s.t. } \mathrm{E} \int_0^T H(t)c(t)\, dt \leq \mathrm{E} \int_0^T H(t)e_j(t)\, dt \ . \tag{7.4}$$

From Proposition 5.2, the solution is

$$c_j^*(t) = I_j(\frac{1}{\lambda_j} H(t), t) \ , \tag{7.5}$$

where the Lagrange multiplier $\frac{1}{\lambda_j}$ solves

$$\mathrm{E} \int_0^T H(t) I_j(\frac{1}{\lambda_j} H(t), t)\, dt = \mathrm{E} \int_0^T H(t)e_j(t)\, dt \ . \tag{7.6}$$

Definition 7.1 An *Arrow-Debreu equilibrium* is a sdf $H(t)$ and allocations $c_j^*(t)$, $j = 1, \ldots, m$, such that each $c_j^*(t)$ solves investor j's optimization problem 7.4, and markets clear, i.e.,

$$\sum_{j=1}^m c_j^*(t) = e(t) \ , 0 \leq t \leq T \ . \tag{7.7}$$

Proposition 7.1 *In an Arrow-Debreu equilibrium, there exists* $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{R}^m_{++}$ *satisfying*

$$e(t) = \sum_{j=1}^m I_j(\frac{1}{\lambda_j} H(t), t) \ , 0 \leq t \leq T \tag{7.8}$$

and

$$\mathrm{E} \int_0^T H(t)[I_j(\frac{1}{\lambda_j} H(t), t) - e_j(t)]\, dt = 0 \quad , \ \forall j = 1, \ldots, m \ . \tag{7.9}$$

Conversely, if there exist $H(t)$ *and* $\Lambda \in \mathcal{R}^m_{++}$ *satisfying equations (7.8) and (7.9) then* H *is an equilibrium sdf. In either case,*

$$c_j^*(t) = I_j(\frac{1}{\lambda_j} H(t), t) \tag{7.10}$$

are the optimal consumption plans for investors $j = 1, \ldots, m$.

Sketch of Proof The c_j^* are optimal and markets clear iff equations (7.8)-(7.10) hold.

So the search for an Arrow-Debreu equilibrium reduces to a search for H and Λ satisfying equations (7.8) and (7.9). First use equation (7.8) to define H as a function of e and Λ as follows. Let $\Lambda \in \mathcal{R}^m_{++}$ be given. Then for each (ω, t) there exists a unique $H(\omega, t) \in (0, \infty)$ solving equation (7.8) (because the right-hand side is continuous and strictly decreasing in H, goes to ∞ as H goes to zero, and goes to 0 as H goes to ∞). Thus, define $\mathcal{H}(e(t), t; \Lambda)$ by

$$\sum_{j=1}^{m} I_j(\frac{1}{\lambda_j}\mathcal{H}(e(t), t; \Lambda), t) = e(t) \ . \tag{7.11}$$

Note that

$$\mathcal{H}(e(t), t; a\Lambda) = a\mathcal{H}(e(t), t; \Lambda) \ \forall a > 0 \ . \tag{7.12}$$

Corollary 7.1 *The process $H(t)$ is an equilibrium sdf if and only if $H(t) = \mathcal{H}(e(t), t; \Lambda)$ where $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{R}^m_{++}$ solves the system*

$$\mathrm{E} \int_0^T \mathcal{H}(e(t), t; \Lambda)[I_j(\frac{1}{\lambda_j}\mathcal{H}(e(t), t; \Lambda), t) - e_j(t)]\, dt = 0, \ \ j = 1, \ldots, m \ . \tag{7.13}$$

So now we are left with a search for $\Lambda \in \mathcal{R}^m_{++}$ that solves equation (7.13).

Theorem 7.1 (Existence and Uniqueness of Λ) *(Theorem 4.6.1 on p. 179 of Karatzas and Shreve, 1998) There exists $\Lambda \in \mathcal{R}^m_{++}$ that solves equation (7.13). Moreover, if*

$$-\frac{cU_j''(c,t)}{U_j'(c,t)} \leq 1 \ \forall j = 1, \ldots, m \tag{7.14}$$

and if $M = (\mu_1, \ldots, \mu_m)$ is any other solution of equation (7.13), then for some positive constant η,

$$\eta\mathcal{H}(e(t), t; \Lambda) = \mathcal{H}(e(t), t; M) \quad , 0 \leq t \leq T \ , \tag{7.15}$$

and

$$c_j^*(t) = I_j(\frac{1}{\lambda_j}\mathcal{H}(e(t), t; \Lambda), t) = I_j(\frac{1}{\mu_j}\mathcal{H}(e(t), t; M), t) \ , 0 \leq t \leq T \ , j = 1, \ldots, m \ . \tag{7.16}$$

7.4 Construction of the "Representative Agent"

Define

$$U(c,t;\Lambda) \equiv \max_{(c_1,\ldots,c_m)\in\mathcal{R}^m} \sum_{j=1}^{m} \lambda_j U_j(c_j,t) \text{ s.t. } \sum_{j=1}^{m} c_j \le c \ . \tag{7.17}$$

Let $(\hat{c}_1(c),\ldots,\hat{c}_m(c))$ denote the solution to the maximization problem above, where the dependence of $\hat{c}$ on t and Λ is suppressed for brevity.

Proposition 7.2 *The function $U(\cdot,t;\Lambda):(0,\infty)\to\mathcal{R}$ has the following properties.*

$$U(\cdot,t;\Lambda) \in C^2 \ , \tag{7.18}$$

$$U'(c,t;\Lambda) = \lambda_j U_j'(\hat{c}_j(c),t) \ \forall j=1,\ldots,m \ , \tag{7.19}$$

$$\lim_{c\downarrow 0} U'(c,t;\Lambda) = \infty \ , \tag{7.20}$$

$$\lim_{c\to\infty} U'(c,t;\Lambda) = 0 \ , \tag{7.21}$$

$$U''(c,t;\Lambda) < 0 \ . \tag{7.22}$$

Proof Homework

Theorem 7.2 *(Theorem 4.5.6 on p. 175 of Karatzas and Shreve, 1998)*

$$U'(c,t;\Lambda) = \mathcal{H}(c,t;\Lambda) \tag{7.23}$$

and

$$\hat{c}_j(c) = I_j(\frac{1}{\lambda_j}\mathcal{H}(c,t;\Lambda),t) \ . \tag{7.24}$$

Proof Let $c_j^* = I_j(\frac{1}{\lambda_j}\mathcal{H}(c,t;\Lambda),t)$. Note that $\sum_{j=1}^m c_j^* = c$, by construction of $\mathcal{H}(c,t;\Lambda)$. Now suppose $(c_1,\ldots,c_m)$ is a different feasible choice for the maximization problem in equation (7.17). So, for example, $\sum_{j=1}^m c_j \le c$. Then

$$\sum_{j=1}^{m} \lambda_j U_j(c_j,t) < \sum_{j=1}^{m} \lambda_j [U_j(c_j^*,t) + (c_j - c_j^*)U_j'(c_j^*,t)] \tag{7.25}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \lambda_j [U_j(c_j^*, t) + (c_j - c_j^*) U_j'(I_j(\frac{1}{\lambda_j}\mathcal{H}(c, t; \Lambda), t), t)] && (7.26)\\
&= \sum_{j=1}^{m} \lambda_j U_j(c_j^*, t) + \mathcal{H}(c, t; \Lambda) \sum_{j=1}^{m} (c_j - c_j^*) && (7.27)\\
&= \sum_{j=1}^{m} \lambda_j U_j(c_j^*, t) + \mathcal{H}(c, t; \Lambda)[(\sum_{j=1}^{m} c_j) - c] && (7.28)\\
&\leq \sum_{j=1}^{m} \lambda_j U_j(c_j^*, t) \ . && (7.29)
\end{aligned}
$$

Thus $\hat{c}_j(c) = c_j^* = I_j(\frac{1}{\lambda_j}\mathcal{H}(c, t; \Lambda), t)$, $j = 1, \ldots, m$, so

$$
\begin{aligned}
U'(c, t; \Lambda) &= \lambda_j U_j'(I_j(\frac{1}{\lambda_j}\mathcal{H}(c, t; \Lambda), t), t) && (7.30)\\
&= \mathcal{H}(c, t; \Lambda) \ . \ \square && (7.31)
\end{aligned}
$$

Now fix Λ at an equilibrium value. Then we have $U'(e(t), t) = \mathcal{H}(e(t), t; \Lambda) = H(t)$, i.e., we can interpret the equilibrium sdf as the marginal utility of a "representative agent" consuming the aggregate endowment.

7.5 Security Market Equilibrium

The Arrow-Debreu equilibrium can be implemented in a market where investors trade securities dynamically, rather than making one-time trades of consumption bundles. To construct a complete, correctly priced securities market, we first deduce $r(t)$ and $\theta(t)$ from $H(t)$, then specify a nonsingular $d \times d$-matrix-valued volatility process $\sigma(t)$ and set $\mu(t) = r(t)1 + \sigma(t)\theta(t)$, and finally, define and verify admissibility of trading strategies that generate affordable plans.

To deduce $r(t)$ and $\theta(t)$ from $H(t)$, note that on one hand, $H(t) = U'(e(t), t)$, so

$$
\begin{aligned}
dH(t) &= [U''(e(t), t)e(t)\mu_e(t) + \frac{\partial U'(e(t), t)}{\partial t} + \frac{1}{2}U'''(e(t), t)(e(t))^2|\sigma_e(t)|^2]\, dt\\
&\quad + U''(e(t), t)e(t)\sigma_e(t)\, dB(t) \ . && (7.32)
\end{aligned}
$$

On the other hand, in a securities market, we have $H(t) = \beta(t)Z(t)$ so

$$\frac{dH(t)}{H(t)} = -r(t)\,dt - \theta'(t)\,dB(t)\ . \tag{7.33}$$

Equating drift and diffusion terms in the two expressions for $dH(t)$ above yields

$$r(t) = -\frac{1}{U'(e(t),t)}[U''(e(t),t)e(t)\mu_e(t) + \frac{\partial U'(e(t),t)}{\partial t} + \frac{1}{2}U'''(e(t),t)(e(t))^2|\sigma_e(t)|^2] \tag{7.34}$$

and

$$\theta(t) = -\frac{U''(e(t),t)e(t)}{U'(e(t),t)}\sigma_e'(t) = R(t)\sigma_e'(t)\ , \tag{7.35}$$

where

$$R(t) \equiv -\frac{U''(e(t),t)e(t)}{U'(e(t),t)} \tag{7.36}$$

is the relative risk aversion of the representative agent. Thus, the equilibrium mpr is increasing in the risk aversion of the economy and the aggregate volatility that must be borne.

To construct security prices, let $\sigma(t)$ be a nonsingular, $d \times d$-matrix-valued volatility process, let $\mu(t) = r(t)1 + \sigma(t)\theta(t)$, and let $S_k(0)$, $k = 0, 1, \ldots, d$, be given. Let $S_0(t) = S_0(0)e^{\int_0^t r(u)\,du}$ and let $S_k(t)$ be given by

$$\frac{dS_k(t)}{S_k(t)} = \mu_k(t)\,dt + \sigma_k(t)\,dB(t)\ ,\ k = 1, \ldots, d\ . \tag{7.37}$$

These securities are in zero net supply ("side bets").

Finally, suppose π is a trading strategy satisfying the standard integrability conditions. The wealth process generated by π is $X_j(t)$ such that

$$\beta(t)X_j(t) = \int_0^t \beta(u)[e_j(u) - c_j(u)]\,du + \int_0^t \beta(u)\pi(u)\sigma(u)\,dB^*(u)\ . \tag{7.38}$$

Definition 7.2 The consumption/portfolio pair (c_j, π^j) is *admissible* for investor j if $X_j(t)$ satisfies

$$\beta(t)X_j(t) + \mathrm{E}^*\{\int_t^T \beta(u)e_j(u)\,du|\mathcal{F}_t\} \geq 0\ ,\ 0 \leq t \leq T\ . \tag{7.39}$$

Theorem 7.3 *(Theorem 4.3.6 on p. 167 of Karatzas and Shreve, 1998) Let c_j be a consumption plan that satisfies*

$$\mathrm{E}\int_0^T H(t)c_j(t)\,dt = \mathrm{E}\int_0^T H(t)e_j(t)\,dt \ . \tag{7.40}$$

Then there exists a trading strategy π^j such that (c_j, π^j) is admissible for investor j, and the corresponding wealth process is

$$X_j(t) = \frac{1}{H(t)}\mathrm{E}\{\int_t^T H(u)[c_j(u) - e_j(u)]\,du|\mathcal{F}_t\} \tag{7.41}$$

$$= \mathrm{E}^*\{\int_t^T \beta(t,u)[c_j(u) - e_j(u)]\,du|\mathcal{F}_t\}\ ,\ 0 \le t \le T\ . \tag{7.42}$$

Definition 7.3 The securities market is in *equilibrium* if

1. investors' consumption plans and trading strategies are optimal,
2. the commodities market clears, i.e., $\sum_{j=1}^m c_j^*(t) = e(t)$,
3. security markets clear, i.e.,

$$\sum_{j=1}^m \pi^{j*}(t) = 0 \text{ and } \sum_{j=1}^m \pi_0^{j*}(t) \equiv \sum_{j=1}^m (X_j^*(t) - \pi^{j*}(t)1) = 0\ ,\ 0 \le t \le T\ . \tag{7.43}$$

Theorem 7.4 (Existence of a Securities Market Equilibrium) *(Theorem 4.6.3 on p. 183 of Karatzas and Shreve, 1998) Let Λ and H be as in an Arrow-Debreu equilibrium and let $\sigma(t)$ be a given nonsingular, $d \times d$-matrix-valued volatility process. Define $r(t), \theta(t), \mu(t)$ and security prices as above. Then the market is in equilibrium.*

Proof For homework, prove that security markets clear.

7.6 Consumption CAPM

Note that in equilibrium,

$$\mu_k(t) - r(t) = \sigma_k(t)\theta(t) \tag{7.44}$$

$$= -\frac{U''(e(t),t)e(t)}{U'(e(t),t)}\sigma_k(t)\sigma_e'(t) \quad (7.45)$$

$$\equiv R(t)\mathrm{cov}_{ke}(t) \ . \quad (7.46)$$

In other words, the instantaneous excess expected return on each security is equal to the representative agent coefficient of relative risk aversion times the security's instantaneous covariance with aggregate endowment (consumption).

Let $\sigma^*(t) = a(t)\sigma_e(t)$, where $a(t)$ is any strictly positive one-dimensional adapted process and let

$$\mu^*(t) = r(t) + \sigma^*(t)\theta(t) = r(t) + R(t)a(t)\sigma_e(t)\sigma_e'(t) \quad (7.47)$$

be the volatility vector and equilibrium appreciation rate on a security that is perfectly instantaneously correlated with aggregate consumption. Then we have

$$\mu_k(t) - r(t) = \beta_k(t)(\mu^*(t) - r(t)) \ , \quad (7.48)$$

where $\beta_k(t) = \frac{\sigma_k(t)\sigma^{*'}(t)}{\sigma^*(t)\sigma^{*'}(t)}$ is security k's "consumption beta."