Optimal Consumption and Portfolio Choice

Detailed outline

1. Standard financial market
   (a) State price density process
   (b) Equivalence of dynamic solvency constraint and static budget constraint
   (c) Utility functions
   (d) Inverse marginal utility functions
   (e) Investor's consumption and investment problems
      i. Optimal terminal wealth
      ii. Optimal consumption plan
      iii. Optimal consumption and terminal wealth
2. Deterministic coefficients
   (a) Optimal trading strategy
   (b) Merton fund separation theorem
   (c) Dynamic programming solution methods
      i. Hamilton-Jacobi-Bellman equation
      ii. Terminal condition

Readings

Karatzas and Shreve, 1998, chapter 3.

Duffie, chapter 9.

Merton, R., 1969, Lifetime portfolio selection under uncertainty: the continuous-time case, *Review of Economics and Statistics* 51, 247-257.

Merton, R., 1971, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373-413.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve, 1987, Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon, *SIAM Journal on Control and Optimization* 25, 1557-1586.

Cox, J. and C. F. Huang, 1989, Optimal consumption and portfolio choices when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33-83.

Cox, J. and C. F. Huang, 1991, A variational problem arising in financial economics, *Journal of Mathematical Economics* 20, 465-487.

Problems

- Investor A chooses an investment policy to maximize expected utility from time T wealth $\mathrm{E}U(X_T)$.
- Investor B chooses an investment and consumption policy to maximize expected utility from consumption $\mathrm{E}\int_0^T e^{-\delta t}U(c_t)\,dt$.

I. Assuming a complete, standard financial market, use martingale methods to solve for investor A's optimal payoff for each of the following utility functions.

1. HARA utility functions with decreasing absolute risk aversion:

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{a(x-\bar{x})}{1-\gamma}\right)^{\gamma}\ ,\ \ \gamma < 1\ , a > 0\ . \tag{1}$$

2. Constant absolute risk averse (CARA) utility ($\gamma = -\infty$):

$$U(x) = -e^{-ax} \tag{2}$$

3. Constant relative risk averse (CRRA) utility ($\bar{x} = 0$):

$$U(x) = \frac{x^{\gamma}}{\gamma} \tag{3}$$

4. Log utility ($\bar{x} = \gamma = 0$):

$$U(x) = \log(x) \tag{4}$$

In the HARA case, assume $\bar{x} \geq 0$ so the nonnegativity constraint on wealth is nonbinding. In the CARA case, solve the problem both with and without the nonegativity constraint on wealth.

II. Assuming the market has constant coefficients, compute the optimal trading strategy for each of the cases above.

III. Assuming the market has constant coefficients, use dynamic programming methods to solve for the optimal consumption and/or investment policy in each of the cases above, ignoring the nonnegativity constraint on wealth.

IV. Solve investor B's problem for the case of HARA utility with $\bar{x} \geq 0$ and constant coefficients. Determine both the optimal consumption plan and the optimal trading strategy.