The Standard Continuous-Time Financial Market

Detailed outline

1. Brownian probability space
2. Payoff space – flow rates and terminal payoffs
3. Security prices – Itô processes
4. Trading strategies – self-financing, square-integrable, tame
5. Equivalent martingale measures – characterization of Radon-Nikodym derivatives
6. Portfolio value - martingale properties, lower bounds
7. Market completeness

Readings

Domenico Cuoco's lecture notes, parts IV and V.

Karatzas and Shreve, 1998, chapter 1.

Problem

1. Suppose $\text{rank}(\sigma(t)) = n$. Let ν be an d-dimensional process with $\sigma(t)\nu(t) = 0$ and let

$$\begin{aligned} \theta(t) &= \sigma(t)'(\sigma(t)\sigma(t)')^{-1}[\mu(t) - r(t)1] \ , & (1) \\ \hat{\theta}(t) &= \theta(t) + \nu(t) \ , & (2) \\ \hat{Z}(t) &\equiv e^{-\int_0^t \hat{\theta}(s)' \, dB(s) - \frac{1}{2} \int_0^t |\hat{\theta}(s)|^2 \, ds} \ , & (3) \\ \beta(t) &\equiv e^{-\int_0^t r(s) \, ds} \ , \text{and} & (4) \\ m(t) &\equiv \beta(t)\hat{Z}(t) \ . & (5) \end{aligned}$$

Finally, let

$$\begin{aligned} Z(t) &\equiv e^{-\int_0^t \theta(s)' \, dB(s) - \frac{1}{2} \int_0^t |\theta(s)|^2 \, ds} \ , & (6) \\ m^*(t) &\equiv \beta(t)Z(t) \ , & (7) \\ m^* &\equiv m^*(T) \ . & (8) \end{aligned}$$

(a) Show that m^* is the only m of the form above in the payoff space, that is, it is the only such m for which there exists a trading strategy that strictly finances a consumption plan $(c, W) \in \mathcal{C}$ with $W = m$.

(b) Show that $m^*(t)$ is the m process with the smallest instantaneous volatility, or in other words, that its log has the smallest quadratic variation.