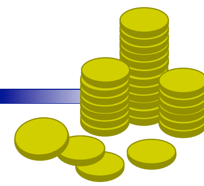
Coupon Bonds and Zeroes



Concepts and Buzzwords

- Coupon bonds
- Zero-coupon bonds
- Bond replication
- No-arbitrage price relationships
- Zero rates
- Zeroes
- STRIPS
- Dedication
- Implied zeroes
- Semi-annual compounding

Reading

- Veronesi, Chapters 1 and 2
- Tuckman, Chapters 1 and 2

Coupon Bonds

- In practice, the most common form of debt instrument is a coupon bond.
- In the U.S and in many other countries, coupon bonds pay coupons every six months and par value at maturity.
- The quoted coupon rate is annualized. That is, if the quoted coupon rate is c, and bond maturity is time T, then for each \$1 of par value, the bond cash flows are:

$c/2$	$c/2$	$c/2$	···	$1 + c/2$
0.5 years	1 year	1.5 years	···	T years

- If the par value is N, then the bond cash flows are:

$Nc/2$	$Nc/2$	$Nc/2$	···	$N(1 + c/2)$
0.5 years	1 year	1.5 years	···	T years

U.S. Treasury Notes and Bonds

- Institutionally speaking, U.S. Treasury "notes" and "bonds" form a basis for the bond markets.
- The Treasury auctions new 2-, 3-, 5-, 7-year notes monthly, and 10-year notes and 30-year bonds quarterly, as needed. See http://www.treas.gov/offices/domestic-finance/debt-management/auctions/auctions.pdf for a schedule.
- Non-competitive bidders just submit par amounts, maximum \$5 million, and are filled first. Competitive bidders submit yields and par amounts, and are filled from lowest yield to the "stop" yield. The coupon on the bond, an even eighth of a percent, is set to make the bond price close to par value at the stop yield. All bidders pay this price.
- See, for example, http://fixedincome.fidelity.com/fi/FIFrameset?page=FISearchTreasury for a listing of outstanding Treasuries.

Class Problem

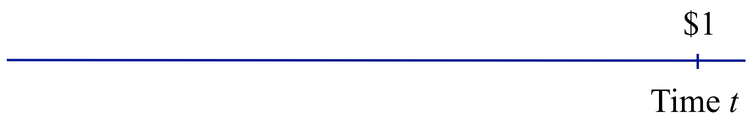
- The current "long bond," the newly issued 30-year Treasury bond, is the 3 7/8's (3.875%) of August 15, 2040.
- What are the cash flows of $1,000,000 par this bond? (Dates and amounts.)

Bond Replication and No Arbitrage Pricing

- It turns out that it is possible to construct, and thus price, all securities with fixed cash flows from coupon bonds.
- But the easiest way to see the replication and no-arbitrage price relationships is to view all securities as portfolios of "zero-coupon bonds," securities with just a single cash flow at maturity.
- We can observe the prices of zeroes in the form of Treasury STRIPS, but more typically people infer them from a set of coupon bond prices, because those markets are more active and complete.
- Then we use the prices of these zero-coupon building blocks to price everything else.

Zeroes

- Conceptually, the most basic debt instrument is a zero-coupon bond--a security with a single cash flow equal to face value at maturity.
- Cash flow of $1 par of t-year zero:



- It is easy to see that any security with fixed cash flows can be constructed, and thus priced, as a portfolio of these zeroes.

Zero Prices

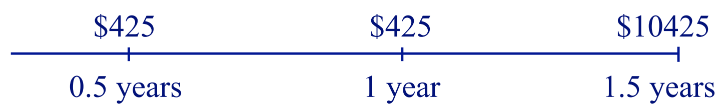
- Let d_t denote the price today of the t-year zero, the asset that pays off $1 in t years.
- I.e., d_t is the price of a t-year zero as a fraction of par value.
- This is also sometimes called the t-year "discount factor."
- Because of the time value of money, a dollar today is worth more than a dollar to be received in the future, so the price of a zero must always less than its face value:

$$d_t < 1$$

- Similarly, because of the time value of money, longer zeroes must have lower prices.

A Coupon Bond as a Portfolio of Zeroes

Consider: $10,000 par of a one and a half year, 8.5% Treasury bond makes the following payments:



Note that this is the same as a portfolio of three different zeroes:

- $425 par of a 6-month zero
- $425 par of a 1-year zero
- $10425 par of a 1 1/2-year zero

No Arbitrage and The Law of One Price

- Throughout the course we will assume:

  The Law of One Price *Two assets which offer exactly the same cash flows must sell for the same price.*

- Why? If not, then one could buy the cheaper asset and sell the more expensive, making a profit today with no cost in the future.
- This would be an *arbitrage opportunity*, which could not persist in equilibrium (in the absence of market frictions such as transaction costs and capital constraints).

Valuing a Coupon Bond Using Zero Prices

Let's value $10,000 par of a 1.5-year 8.5% coupon bond based on the zero prices (discount factors) in the table below.

These discount factors come from historical STRIPS prices (from an old WSJ). **We will use these discount factors for most examples throughout the course.**

Maturity	Discount Factor	Bond Cash Flow	Value
0.5	0.9730	$425	$414
1.0	0.9476	$425	$403
1.5	0.9222	$10425	$9614
			Total $10430

On the same day, the WSJ priced a 1.5-year 8.5%-coupon bond at 104 10/32 (=104.3125).

An Arbitrage Opportunity

- What if the 1.5-year 8.5% coupon bond were worth only 104% of par value?
- You could buy, say, $1 million par of the bond for $1,040,000 and sell the cash flows off individually as zeroes for total proceeds of $1,043,000, making $3000 of riskless profit.
- Similarly, if the bond were worth 105% of par, you could buy the portfolio of zeroes, reconstitute them, and sell the bond for riskless profit.

Class Problems

In today's market, the discount factors are:

$d_{0.5}$=0.9991 , d_1=0.9974 , and $d_{1.5}$=0.9940.

1) What would be the price of an 8.5%-coupon, 1.5-year bond today? (Say for $100 par.)

2) What would be the price of $100 par of a 2%-coupon, 1-year bond today?

Securities with Fixed Cash Flows as Portfolios of Zeroes

- More generally, if an asset pays cash flows $K_1, K_2, \ldots, K_n$, at times $t_1, t_2, \ldots, t_n$, then it is the same as:

  K_1 t_1-year zeroes + K_2 t_2-year zeroes + … + K_n t_n-year zeroes

- Therefore no arbitrage requires that the asset's value V is

$$V = K_1 \times d_{t_1} + K_2 \times d_{t_2} + ... + K_n \times d_{t_n}$$

$$\text{or} \quad V = \sum_{j=1}^{n} K_j \times d_{t_j}$$

Coupon Bond Prices in Terms of Zero Prices

For example, if a bond has coupon c and maturity T, then in terms of the zero prices d_t, its price per $1 par must be

$$P(c,T) = (c/2) \times (d_{0.5} + d_1 + d_{1.5} + \ldots + d_T) + d_T$$

or $$P(c,T) = (c/2)\sum_{s=1}^{2T} d_{s/2} + d_T$$

Constructing Zeroes from Coupon Bonds

- Often people would rather work with Treasury coupon bonds than with STRIPS, because the market is more active.
- They can imply zero prices from Treasury bond prices instead of STRIPs and use these to value more complex securities.
- In other words, not only can we construct bonds from zeroes, we can also go the other way.
- Example: Constructing a 1-year zero from 6-month and 1-year coupon bonds.
- Coupon Bonds:

Maturity	Coupon	Price in 32nds	Price in Decimal
0.5	4.250%	99-13	99.40625
1.0	4.375%	98-31	98.96875

Constructing the One-Year Zero

- Find portfolio of bonds 1 and 2 that replicates 1-year zero.
- Let $N_{0.5}$ be the par amount of the 0.5-year bond and N_1 be the par amount of the 1-year bond in the portfolio.
- At time 0.5, the portfolio will have a cash flow of
  $N_{0.5}$ x (1+0.0425/2) + N_1 x 0.04375/2
- At time 1, the portfolio will have a cash flow of
  $N_{0.5}$ x 0 + N_1 x (1+0.04375/2)
- We need $N_{0.5}$ and N_1 to solve

  (1) $N_{0.5}$ x (1+0.0425/2) + N_1 x 0.04375/2 = 0

  (2) $N_{0.5}$ x 0 + N_1 x (1+0.04375/2) =100

  => N_1 = 97.86 and $N_{0.5}$ = -2.10

Implied Zero Price

- So the replicating portfolio consists of
  - long 97.86 par value of the 1-year bond
  - short 2.10 par value of the 0.5-year bond.
- **Class Problem:** Given the prices of these bonds below,

Maturity	Coupon	Price in 32nds	Price in Decimal
0.5	4.250%	99-13	99.40625
1.0	4.375%	98-31	98.96875

what is the no-arbitrage price of $100 par of the 1-year zero?

Inferring Zero Prices from Bond Prices: Short Cut

- The last example showed how to construct a portfolio of bonds that synthesized (had the same cash flows as) a zero.
- We concluded that the zero price had to be the same as the price of the replicating portfolio (no arbitrage).
- If we don't need to know the replicating portfolio, we can solve for the implied zero prices more directly:

$$\text{Price of bond 1} = (100 + 4.25/2) \times d_{0.5} = 99.40625$$
$$\text{Price of bond 2} = (4.375/2) \times d_{0.5} + (100 + 4.375/2) \times d_1 = 98.96875$$
$$\Rightarrow d_{0.5} = 0.973,\ d_1 = 0.948$$

Class Problems

1) Suppose the price of the 4.25%-coupon, 0.5-year bond is 99.50. What is the implied price of a 0.5-year zero per $1 par?

2) Suppose the price of the 4.375%-coupon, 1-year bond is 99. What is the implied price of a 1-year zero per $1 par?

Replication Possibilities

- Since we can construct zeroes from coupon bonds, we can construct any stream of cash flows from coupon bonds.
- Uses:
  - Bond portfolio dedication--creating a bond portfolio that has a desired stream of cash flows
    - funding a liability
    - defeasing an existing bond issue
  - Taking advantage of arbitrage opportunities

Market Frictions

- In practice, prices of Treasury STRIPS and Treasury bonds don't fit the pricing relationship exactly
  - transaction costs and search costs in stripping and reconstituting
  - bid/ask spreads
- Note: The terms “bid” and “ask” are from the viewpoint of the dealer.
  - The dealer buys at the bid and sells at the ask, so the bid price is always less than the ask.
  - The customer sells at the bid and buys at the ask.

Interest Rates

- People try to summarize information about bond prices and cash flows by quoting interest rates.
- Buying a zero is lending money--you pay money now and get money later
- Selling a zero is borrowing money--you get money now and pay later
- A bond transaction can be described as
  - buying or selling at a given price, or
  - lending or borrowing at a given rate.
- The convention in U.S. bond markets is to use **semi-annually compounded interest rates**.

Annual vs. Semi-Annual Compounding

At 10% per year, *annually* compounded, \$100 grows to \$110 after 1 year, and \$121 after 2 years:

$$100 \times 1.10 = 110$$

$$100 \times (1.10)^2 = 121$$

10% per year *semi-annually* compounded really means 5% every 6 months. At 10% per year, *semi-annually* compounded, \$100 grows to \$110.25 after 1 year, and \$121.55 after 2 years:

$$100 \times (1.05)^2 = 110.25$$

$$100 \times (1.05)^4 = 121.55$$

Annual vs. Semi-Annual Compounding...

After T years, at *annually* compounded rate r_A, P grows to

$$F = P(1 + r_A)^T$$

Present value of F to be received in T years with *annually* compounded rate r_A is

$$P = \frac{F}{(1 + r_A)^T}$$

In terms of the *semi-annually* compounded rate r, the formulas become

$$F = P(1 + r/2)^{2T} \qquad P = \frac{F}{(1 + r/2)^{2T}}$$

The key: $(1 + r/2)^2 = 1 + r_A$

An (annualized) semi-annually compounded rate of r per year really means $r/2$ every six months.

Zero Rates

- If you buy a t-year zero and hold it to maturity, you lend at rate r_t where r_t is defined by

$$d_t \times (1 + r_t/2)^{2t} = 1,$$

$$\text{or } d_t = \frac{1}{(1 + r_t/2)^{2t}},$$

$$\text{or } r_t = 2 \times ((\frac{1}{d_t})^{\frac{1}{2t}} - 1)$$

- Call r_t the *t-year zero rate* or *t-year discount rate*.

Class Problems: Rate to Price

- According to market convention, zero prices are quoted using rates. Sample STRIPS rates from our historic WSJ:
  - 0.5-year rate: 5.54%
  - 1-year rate: 5.45%

1) What is the 0.5-year zero price?

2) What is the 1-year zero price?

Class Problems: Price to Rate

1) The 1-year zero price implied from coupon bond prices was 0.947665. What was the "implied zero rate?"

2) In today's market, the 5-year zero price is 0.9075. What is the 5-year zero rate?

Value of a Stream of Cash Flows in Terms of Zero Rates

- Recall that any asset with fixed cash flows can be viewed as a portfolio of zeroes.
- So its price must be the sum of its cash flows multiplied by the relevant zero prices:

$$V = \sum_{j=1}^{n} K_j \times d_{t_j}$$

- Equivalently, the price is the sum of the present values of the cash flows, discounted at the zero rates for the cash flow dates:

$$V = \sum_{j=1}^{n} \frac{K_j}{(1 + r_{t_j} / 2)^{2t_j}}$$

Example

\$10,000 par of a one and a half year, 8.5% Treasury bond makes the following payments:

\$425	\$425	\$10425
0.5 years	1 year	1.5 years

Using STRIPS rates from the WSJ to value these cash flows:

$$V = \frac{\$425}{(1+0.0554/2)^1} + \frac{\$425}{(1+0.0545/2)^2} + \frac{\$10425}{(1+0.0547/2)^3}$$

$$= \$10430$$