Convexity

Concepts and Buzzwords

- Dollar Convexity
- Convexity
- Curvature
- Taylor series
- Barbell, Bullet

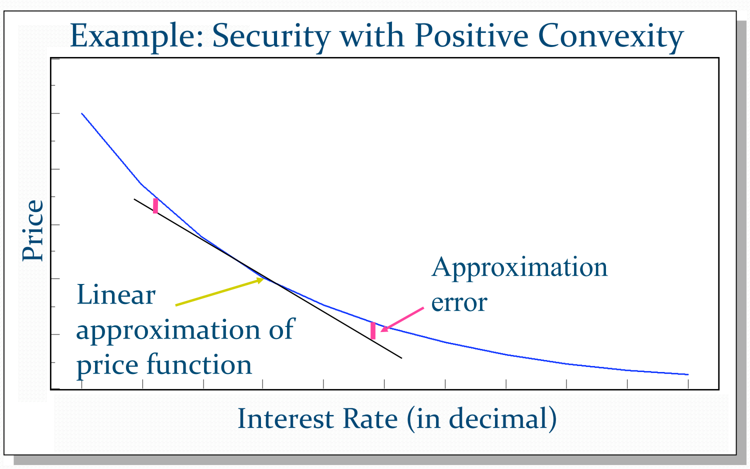
Readings

- Veronesi, Chapter 4
- Tuckman, Chapters 5 and 6

Convexity

- Convexity is a measure of the curvature of the value of a security or portfolio as a function of interest rates.
- Duration is related to the slope, i.e., the first derivative.
- Convexity is related to the curvature, i.e. the second derivative of the price function.
- Using convexity together with duration gives a better approximation of the change in value given a change in interest rates than using duration alone.

Price-Rate Function



Correcting the Duration Error

- The price-rate function is nonlinear.
- Duration and dollar duration use a linear approximation to the price rate function to measure the change in price given a change in rates.
- The error in the approximation can be substantially reduced by making a convexity correction.

Taylor Series

- The Taylor Theorem from calculus says that the value of a function can be approximated near a given point using its "Taylor series" around that point.
- Using only the first two derivatives, the Taylor series approximation is:

$$f(x) \approx f(x_0) + f'(x_0) \times (x - x_0) + \frac{1}{2} f''(x_0) \times (x - x_0)^2$$

Or, $f(x) - f(x_0) \approx f'(x_0) \times (x - x_0) + \frac{1}{2} f''(x_0) \times (x - x_0)^2$

Dollar Convexity

- Think of bond prices, or bond portfolio values, as functions of interest rates.
- The Taylor Theorem says that if we know the first and second derivatives of the price function (at current rates), then we can approximate the price impact of a given change in rates.

$$f(x) - f(x_0) \approx f'(x_0) \times (x - x_0) + 0.5 \times f''(x_0) \times (x - x_0)^2$$

★The first derivative is minus dollar duration.

★Call the second derivative *dollar convexity*.

- Then change in price ≈ -$duration x change in rates

  + 0.5 x $convexity x change in rates squared

Dollar Convexity of a Portfolio

If we assume all rates change by the same amount, then

the dollar convexity of a portfolio is the sum of the dollar convexities of its securities.

Sketch of proof:

$$\sum f_i(x) - f_i(x_0) \approx \left(\sum f_i'(x_0)\right) \times (x - x_0) + 0.5 \times \left(\sum f_i''(x_0)\right) \times (x - x_0)^2$$

I.e., Δ portfolio value ≈ - (sum of dollar durations) x Δr

+ 0.5 x (sum of dollar convexities) x $(\Delta \text{rates})^2$

⇒Portfolio dollar duration = sum of dollar durations

⇒Portfolio dollar convexity = sum of dollar convexities

Convexity

- Just as dollar duration describes dollar price sensitivity, dollar convexity describes curvature in dollar performance.
- To get a scale-free curvature measure, i.e., curvature per dollar invested, we define

$$\text{convexity} = \frac{\text{dollar convexity}}{\text{price}}$$

⇒The convexity of a portfolio is the average convexity of its securities, weighted by present value:

$$\text{convexity} = \frac{\sum \text{price}_i \times \text{convexity}_i}{\sum \text{price}_i} = \text{pv wtd average convexity}$$

- Just like dollar duration and duration, dollar convexities add, convexities average.

Dollar Formulas for $1 Par of a Zero

For $1 par of a t-year zero-coupon bond

$$\text{price} = d_t(r_t) = \frac{1}{(1 + r_t/2)^{2t}}$$

$$\text{dollar duration} = -d_t'(r_t) = \frac{t}{(1 + r_t/2)^{2t+1}}$$

$$\text{dollar convexity} = d_t''(r_t) = \frac{t^2 + t/2}{(1 + r_t/2)^{2t+2}}$$

For $$N$ par, these would be multiplied by N.

Percent Formulas for Any Amount of a Zero

$$\text{duration} = \frac{\text{dollar duration}}{\text{price}} = \frac{N \times t/(1+r_t/2)^{2t+1}}{N \times 1/(1+r_t/2)^{2t}} = \frac{t}{1+r_t/2}$$

$$\text{convexity} = \frac{\text{dollar convexity}}{\text{price}} = \frac{N \times (t^2+t/2)/(1+r_t/2)^{2t+2}}{N \times 1/(1+r_t/2)^{2t}} = \frac{t^2+t/2}{(1+r_t/2)^2}$$

- These formulas hold for any par amount of the zero – they are scale-free.
- The duration of the t-year zero is approximately t.
- The convexity of the t-year zero is approximately t^2.
- (If we defined price as $d_t = e^{-rt}$, and differentiated w.r.t. this r, then the duration of the t-year zero would be exactly t and the convexity of the t-year zero would be exactly t^2.)

Class Problems

Calculate the price, dollar duration, and dollar convexity of $1 par of the 20-year zero if $r_{20} = 6.50\%$.

Class Problems

Suppose r_{20} rises to 7.50%.

1) Approximate the price change of $1,000,000 par using only dollar duration.

2) Approximate the price change of $1,000,000 using both dollar duration and dollar convexity.

3) What is the exact price change?

Class Problems

Suppose r_{20} falls to 5.50%.

1) Approximate the price change of $1,000,000 par using only dollar duration.

2) Approximate the price change of $1,000,000 using both dollar duration and dollar convexity.

3) What is the exact price change?

Sample Risk Measures

Duration and convexity for $1 par of a 10-year, 20-year, and 30-year zero.

Maturity	Rate	Price	$Duration	Duration	$Convexity	Convexity
10	6.00%	0.553676	5.375493	9.70874	54.7987	98.9726
20	6.50%	0.278226	5.389364	19.37046	107.0043	384.5951
30	6.40%	0.151084	4.391974	29.06977	129.8015	859.1356

For zeroes,

- duration is roughly equal to maturity,
- convexity is roughly equal to maturity squared.

Dollar Convexity of a Portfolio of Zeroes

- Consider a portfolio with fixed cash flows at different points in time ($K_1, K_2, \ldots, K_n$ at times $t_1, t_2, \ldots, t_n$).
- Just as with dollar duration, the dollar convexity of the portfolio is the sum of the dollar convexities of the component zeroes.
- The dollar convexity of the portfolio gives the correction to make to the duration approximation of the change in portfolio value given a change in zero rates, assuming all zero rates change by the same amount.

$$\text{portfolio dollar convexity} = \sum_{j=1}^{n} K_j \times \frac{t_j^2 + t_j/2}{(1 + r_{t_j}/2)^{2t_j+2}}$$

Class Problem:
Dollar Convexity of a Portfolio of Zeroes

Consider a portfolio consisting of

- $25,174 par value of the 10-year zero,
- $91,898 par value of the 30-year zero.

Maturity	Rate	Price	$Duration	Duration	$Convexity	Convexity
10	6.00%	0.553676	5.375493	9.70874	54.7987	98.9726
20	6.50%	0.278226	5.389364	19.37046	107.0043	384.5951
30	6.40%	0.151084	4.391974	29.06977	129.8015	859.1356

What is the dollar convexity of the portfolio?

Convexity of a Portfolio of Zeroes

$$\text{convexity} = \frac{\$\text{convexity}}{\text{present value}} = \frac{\sum_{j=1}^{n} K_j \times \frac{t_j^2 + t_j/2}{(1+r_{t_j}/2)^{2t_j+2}}}{\sum_{j=1}^{n} \frac{K_j}{(1+r_{t_j}/2)^{2t_j}}}$$

$$= \frac{\sum_{j=1}^{n} \frac{K_j}{(1+r_{t_j}/2)^{2t_j}} \times \frac{t_j^2 + t_j/2}{(1+r_{t_j}/2)^2}}{\sum_{j=1}^{n} \frac{K_j}{(1+r_{t_j}/2)^{2t_j}}}$$

$=$ average convexity weighted by present value

$\approx$ average maturity^2 weighted by present value

Class Problem

Consider the portfolio of 10- and 30-year zeroes.

- The 10-year zeroes have market value
  $25,174 x 0.553676 = $13,938.
- The 30-year zeroes have market value
  $91,898 x 0.151084 = $13,884.
- The market value of the portfolio is $27,822.
- What is the convexity of the portfolio?

Maturity	Rate	Price	$Duration	Duration	$Convexity	Convexity
10	6.00%	0.553676	5.375493	9.70874	54.7987	98.9726
20	6.50%	0.278226	5.389364	19.37046	107.0043	384.5951
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Barbells and Bullets

- Consider two portfolios with the same duration:
  - A *barbell* consisting of a long-term zero and a short-term zero
  - A *bullet* consisting of an intermediate-term zero
- The barbell will have more convexity.

Example

•Bullet portfolio: \$100,000 par of 20-year zeroes

market value = \$100,000 x 0.27822 = 27,822

duration = 19.37

•Barbell portfolio: from previous example

\$25,174 par value of the 10-year zero

\$91,898 par value of the 30-year zero.

market value = 27,822

$$\text{duration} = \frac{(13{,}938 \times 9.70874) + (13{,}884 \times 29.06977)}{13{,}938 + 13{,}884} = 19.37$$

•The convexity of the bullet is 385.

•The convexity of the barbell is 478.

Securities with Fixed Cash Flows: More disperse cash flows, more convexity

•In the previous example,

the duration of the bullet is about 20 and

the convexity of the bullet is about 20^2=400.

•The duration of the barbell is about

0.5 x 10 + 0.5 x 30 = 20

but the convexity is about

$0.5 \times 10^2 + 0.5 \times 30^2 = 500 > 400 = (0.5 \times 10 + 0.5 \times 30)^2$

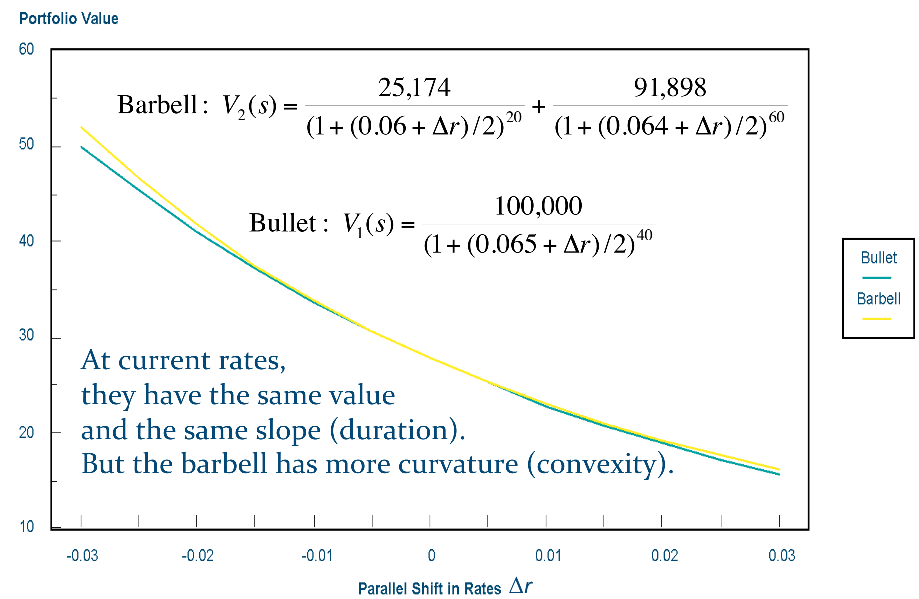
•I.e., the average squared maturity is greater than the average maturity squared.

•Indeed, recall $\text{Var}(X) = \text{E}(X^2)-(\text{E}(X))^2$

•Think of cash flow maturity t as the variable and pv weights as probabilities. Duration is like $\text{E}(t)$ and convexity is like $\text{E}(t^2)$.

•So convexity ≈ duration^2 + dispersion (variance) of maturity.

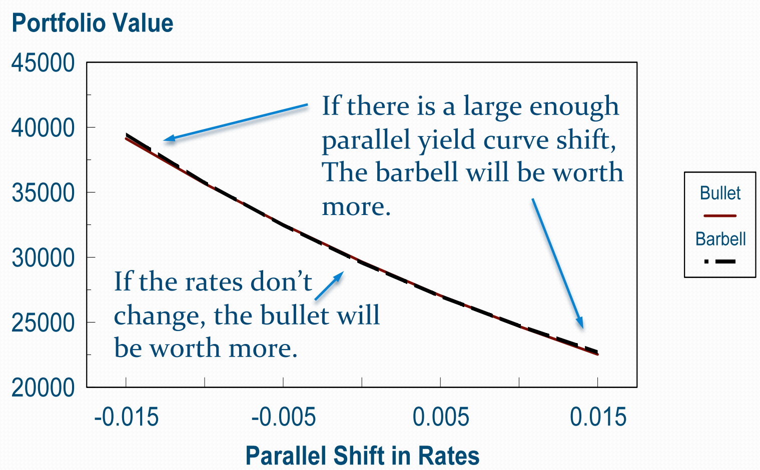
Value of Barbell and Bullet



Does the Barbell Always Outperform the Bullet?

- If there is an immediate parallel shift in interest rates, either up or down, then the barbell will outperform the bullet.
- If the shift is not parallel, anything could happen.
- If the rates on the bonds stay exactly the same, then as time passes the bullet will actually outperform the barbell:
  - the bullet will return 6.5%
  - the barbell will return about 6.2%, the market value-weighted average of the 6% and 6.4% on the 10- and 30-year zeroes.

Value of Barbell and Bullet: One Year Later



Convexity and the Shape of the Yield Curve?

- If the yield curve were flat and made parallel shifts, more convex portfolios would always outperform less convex portfolios, and there would be arbitrage.
- So to the extent that market movement is described by parallel shifts, bullets must have higher yield to start with, to compensate for lower convexity.
- This would explain why the term structure is often hump-shaped, dipping down at very long maturities where convexity is greatest relative to duration—investors may give yield to buy convexity.
- Some evidence suggests that the yield curve is more curved when volatility is higher and convexity is worth more.