Forward Contracts and Forward Rates

Outline and Readings

- Outline
  - Forward Contracts
  - Forward Prices
  - Forward Rates
  - Information in Forward Rates
- Buzzwords
  - settlement date, delivery, underlying asset
  - spot rate, spot price, spot market
  - forward purchase, forward sale, forward loan, forward lending, forward borrowing, synthetic forward
  - expectations theory, term premium
- Reading
  - Veronesi, Chapters 5 and 7
  - Tuckman, Chapters 2 and 16

Forward Contracts

- A *forward contract* is an agreement to buy an asset at a future *settlement date* at a *forward price* specified today.

– No money changes hands today.

– The pre-specified forward price is

exchanged for the asset at settlement date.

- By contrast, an ordinary transaction that settles immediately is called a *spot* or *cash* transaction, and the price is called the *spot* price or *cash* price.

Motivation

- Suppose today, time 0, you know you will need to do a transaction at a future date, time *t*.
- One thing you can do is wait until time *t* and then do the transaction at prevailing market prices

  - i.e., do a *spot* transaction in the *future.*
- Alternatively, you can try to lock in the terms of the transaction today

  - i.e., arrange a *forward* transaction *today*.

What is the fair forward price?

- In some cases, the forward contract can be synthesized with transaction in the current spot market.
- In that case, no arbitrage will require that the contractual forward price must be the same as the forward price that could be synthesized.

Synthetic Forward Price

- For example, if the underlying asset doesn't depreciate, make any payments, or entail any storage costs or convenience yield, the synthetic forward price of the asset is
- Spot Price + Interest to settlement date
- How to synthesize?
  – Buy the asset now for the spot price.
  – Borrow the amount of the spot price, with repayment on the settlement date
  – You pay nothing now, and you pay the spot price plus interest at the settlement date.

Synthetic Forward Contract on a Zero

Suppose $r_{0.5}$=5.54%, $d_{0.5}$=0.9730, r_1=5.45%, and d_1=0.9476.

Synthesize a forward contract to buy $1 par of the zero maturing at time 1 by

1) buying $1 par of the 1-year zero and

2) borrowing the money from time 0.5 to pay for it:

	0	0.5	1
1)	-0.9476		+1
2)	+0.9476	?	
Net:	0	-F = ?	+1

|-----------------------------|-------------------------|

0 0.5 1

Class Problem: What is the no-arbitrage forward price F?

Arbitrage Argument

- **Class Problem**: Suppose a bank quoted a forward price of 0.98. How could you make arbitrage profit?

Synthetic Forward Price for a Zero

- In general, suppose the underlying asset is $1 par of a zero maturing at time *T*.
- In the forward contract, you agree to buy this zero at time *t*.
- The forward price you could synthesize is spot price plus interest to time *t*:

$$F_t^T = d_T (1 + r_t/2)^{2t}$$

- If the quoted contractual forward price differs, there is an arbitrage opportunity.

Class Problem

Suppose the spot price of $1 par of the 1.5-year zero is 0.9222.

What is the no arbitrage forward price of this zero for settlement at time 1, $F_1^{1.5}$?

Forward Contract as a Portfolio of Zeroes

- Here's another way to view the contract:
- You agree today (t=0) to pay at t the sum *$F* to get *$1* worth of par at
- This contract is a portfolio of cash flows:

$0 -*$F* +*$1*

|-----------------|-----------------|

0 *t* *T*

- What is the PV of this contract?
- It is a portfolio:

  Long $1 par of T-year zeros

  Short $F par of t-year zeros
- So its present value is $V = -F \text{ x } d_t + 1 \text{ x } d_T$

Zero Cost Forward Price

- At t=0 the contract "costs" zero.
- The forward price is negotiated to make that true.
- What is the forward price that makes the contract worth zero?

$0 -*$F* +*$1*

|-----------------|-----------------|

0 *t* *T*

$V=-F \text{ x } d_t + 1 \text{ x } d_T = 0$

→ $F= d_T / d_t$

which is equivalent to $F=d_T\,(1+r_t/2)^{2t} = F_t^T$

Examples

Recall the spot prices of $1 par of the 0.5-, 1-, and 1.5-year zeroes are 0.9730, 0.9476, and 0.9222.

The no-arbitrage forward price of the 1-year zero for settlement at time 0.5 is

$$F_{0.5}^{1} = d_1/d_{0.5} = 0.9476/0.9730 = 0.9739$$

The no-arbitrage forward price of the 1.5-year zero for settlement at time 1 is

$$F_{1}^{1.5} = d_{1.5}/d_1 = 0.9222/0.9476 = 0.9732$$

Class Problem

- Suppose a firm has an old forward contract on its books.
- The contract commits the firm to buy, at time t=0.5, $1000 par of the zero maturing at time T=1.5 for a price of $950.
- At inception, the contract was worth zero, but now markets have moved. What is the value of this contract to the firm now?

Forward Contract on a Zero as a Forward Loan

- Just as we can think of the spot purchase of a zero as lending money, we can think of a forward purchase of a zero as a *forward loan*.
- The forward lender agrees today to lend F_t^T on the settlement date t and get back \$1 on the date T.
- Define the *forward rate*, f_t^T, as the interest rate earned from lending F_t^T for T-t years and getting back \$1:

$$F_t^T = \frac{1}{(1+f_t^T/2)^{2(T-t)}} \qquad f_t^T = 2\left(\left(\frac{1}{F_t^T}\right)^{\frac{1}{2(T-t)}} - 1\right)$$

- This is the same transaction, just described in terms of lending or borrowing at rate instead of buying or selling at a price.

Class Problem

- Recall that the no-arbitrage forward price of the 1.5-year zero for settlement at time 1 is
- What is the implied forward rate $f_1^{1.5}$ that you could lock in today for lending from time 1 to time 1.5?

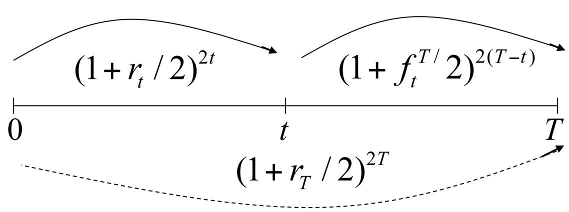
Arbitrage Argument in Terms of Rates: New Riskless Lending Possibilities

- Consider the lending possibilities when a forward contract for lending from time t to time T is available.
- Now there are two ways to lend risklessly from time 0 to time T:
  1) Lend at the current spot rate r_T (i.e., buy a T-year zero). A dollar invested at time 0 would grow risklessly to $(1+r_T/2)^{2T}$.
  2) Lend risklessly to time t (i.e., buy a t-year zero) and roll the time t payoff into the forward contract to time T. A dollar invested at time 0 would grow risklessly to $(1+r_t/2)^{2t} \text{x} (1+f_t^T/2)^{2(T-t)}$.

No Arbitrage Forward Rate

In the absence of arbitrage, the two ways of lending risklessly to time T must be equivalent:

$$(1+r_t/2)^{2t} \times (1+f_t^T/2)^{2(T-t)} = (1+r_T/2)^{2T}$$



Example: The forward rate from time t = 0.5 to time T=1 must satisfy

$$(1+0.0554/2)^1 \times (1+f_{0.5}^1/2)^1 = (1+0.0545/2)^2$$

$$\Rightarrow f_{0.5}^1 = 5.36\%$$

No Arbitrage Forward Rate…

$$(1+r_t/2)^{2t} \times (1+f_t^T/2)^{2(T-t)} = (1+r_T/2)^{2T}$$

$$\Rightarrow (1+f_t^T/2)^{2(T-t)} = \frac{(1+r_T/2)^{2T}}{(1+r_t/2)^{2t}}$$

$$\Rightarrow f_t^T = 2[(\frac{(1+r_T/2)^{2T}}{(1+r_t/2)^{2t}})^{1/[2(T-t)]} - 1]$$

Class Problem:
The 1.5-year zero rate is $r_3 = 5.47\%$. What is the forward rate from time $t = 0.5$ to time T=1.5?

Connection Between Forward Prices and Forward Rates

Of course, this is the same as the no arbitrage equations we saw before:

$$(1+f_t^T/2)^{2(T-t)} = \frac{(1+r_T/2)^{2T}}{(1+r_t/2)^{2t}} \Leftrightarrow F_t^T = \frac{d_T}{d_t}$$

Example: The implied forward rate for a loan from time 0.5 to time 1 is 5.36%. This gives a discount factor of 0.9739, which we showed before is the synthetic forward price to pay at time 0.5 for the zero maturing at time 1.

$$\frac{1}{(1+f_t^T/2)^{2(T-t)}} = \frac{(1+r_t/2)^{2t}}{(1+r_T/2)^{2T}} = \frac{d_T}{d_t} = F_t^T$$

$$\frac{1}{(1+0.0536/2)^1} = \frac{(1+0.0554/2)^1}{(1+0.0545/2)^2} = \frac{0.9476}{0.9730} = 0.9739$$

Summary: One No Arbitrage Equation, Three Economic Interpretations:

(1) Forward price = Spot price + Interest

$$F_t^T = d_T \times (1 + r_t/2)^{2t}$$

(2) Present value of forward contract cash flows at inception = 0:

$$-d_t \times F_t^T + d_T \times 1 = 0$$

(3) Lending short + Rolling into forward loan = Lending long:

$$(1 + r_t/2)^{2t} \times (1 + f_t^T/2)^{2(T-t)} = (1 + r_T/2)^{2T}$$

Using the relations between prices and rates,

$$d_t = \frac{1}{(1 + r_t/2)^{2t}} \quad \text{and} \quad F_t^T = \frac{1}{(1 + f_t^T/2)^{2(T-t)}} \quad \text{or} \quad f_t^T = 2\left(\left(\frac{1}{F_t^T}\right)^{\frac{1}{2(T-t)}} - 1\right)$$

we can verify that these equations are all the same. Other arrangements:

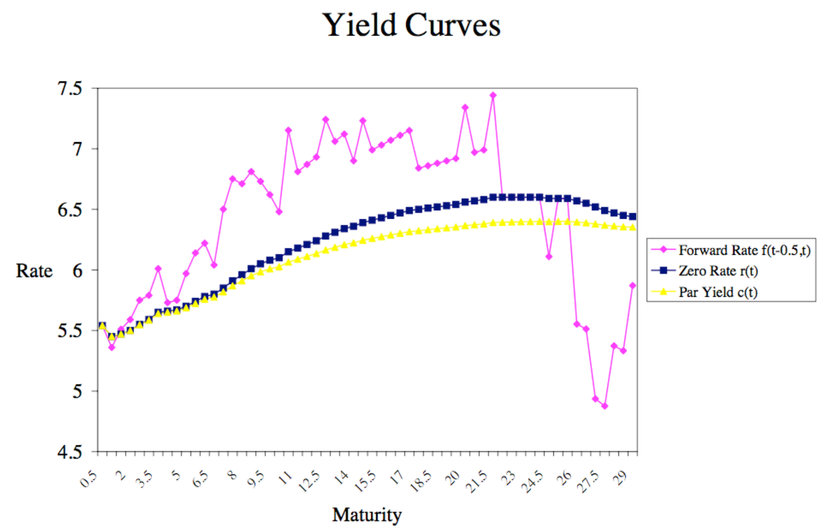
$$F_t^T = \frac{d_T}{d_t} \qquad (1 + f_t^T/2)^{2(T-t)} = \frac{(1 + r_T/2)^{2T}}{(1 + r_t/2)^{2t}}$$

Spot Rates as Averages of Forward Rates

- Rolling money through a series of short-term forward contracts is a way to lock in a long term rate and therefore synthesizes an investment in a long zero. Here are two ways to lock in a rate from time 0 to time t:

$$(1 + r_{0.5}/2) \times (1 + f_{0.5}^1/2) \times \cdots \times (1 + f_{t-0.5}^t/2) = (1 + r_t/2)^{2t}$$

- The growth factor $(1+r_t/2)$ is the geometric average of the $(1+f/2)$'s and so the interest rate r_t is approximately the average of the forward rates.
- Recall the example
  - The spot 6-month rate is 5.54% and the forward 6-month rate is 5.36%.
  - Their average is equal to the 1-year rate of 5.45%.



Zero rates are averages of the one-period forward rates up to their maturity, so while the zero curve is rising, the marginal forward rate must be above the zero rate, and while the zero curve is falling, the marginal forward rate must be below the zero rate.

Forward Rates vs. Future Spot Rates

- The forward rate is the rate you can fix today for a loan that starts at some future date.
- By contrast, you could wait around until that future date and transact at whatever is the prevailing spot rate.
- Is the *forward rate* related to the random *future spot rate*?
- For example, **is the forward rate equal to people's expectation of the future spot rate**?

The Pure Expectations Hypothesis

- The "Pure Expectations Hypothesis" says that **the forward rate is equal to the expected future spot rate**.
- It turns out that's roughly equivalent to the hypothesis that **expected returns on all bonds over a given horizon are the same**, as if people were risk-neutral.
- For example, if the forward rate from time 0.5 to time 1 equals the expected future spot rate over that time, then the expected one-year rate of return from rolling two six-month zeroes is equal to the one-year rate of return from holding a one-year zero:

$$E({}_{0.5}\tilde{r}_1) = f_{0.5}^1$$

$$\Rightarrow E\{(1 + r_{0.5}/2)(1 + {}_{0.5}\tilde{r}_1/2)\} = (1 + r_{0.5}/2)(1 + f_{0.5}^1/2)$$

$$\Rightarrow E\{(1 + r_{0.5}/2)(1 + {}_{0.5}\tilde{r}_1/2)\} = (1 + r_1/2)^2$$

Example in which the Pure Expectations Hypothesis Holds: Upward-Sloping Yield Curve

Time 0	Time 0.5: Zero rate	0.5-yr horizon: ROR on 0.5-yr z.	0.5-yr horizon: ROR on 1-yr z.	1-yr horizon: ROR on 0.5-yr z.	1-yr horizon: ROR on 1-yr z.
	${}_{0.5}r_1^u$=6.50%	5.00%	4.008%	5.749%	5.25%
${}_0r_{0.5}$=5.00% ${}_0r_1$=5.25%					
	${}_{0.5}r_1^d$=4.50%	5.00%	6.003%	4.750%	5.25%
Expected:	5.50%	5.00%	5.005%	5.249%	5.25%

Forward rate
$f_{0.5}^1$=5.50%

If the pure expectations hypothesis holds, then an upward-sloping yield curve indicates rates are expected to rise.

Example in which the Pure Expectations Hypothesis Holds: Downward-Sloping Yield Curve

Zero Rates			Rates of Return over Various Horizons			
Time 0	Time 0.5		0.5-Year ROR		1-Year ROR	
			0.5-yr zero	1-yr zero	0.5-yr zero	1-yr zero
	5.860%	(w.p. 50%)	5.540%	5.041%	5.700%	5.450%
5.540% 5.450%						
	4.860%	(w.p. 50%)	5.540%	6.042%	5.200%	5.450%
Expected:	5.360%		5.540%	5.541%	5.450%	5.450%
Forward rate: 5.360%						

If the pure expectation hypothesis holds, then the downward slope of the yield curve indicates that rates are expected to fall.

Problem with the Pure Expectations Hypothesis: Expected Rates of Return May Differ Across Bonds

- Different bonds may have different expected rates of return because their returns have different risk properties (variance, covariance with other risks, etc.)
- In that case, the pure expectations hypothesis cannot hold.
- For example, the yield curve is typically upward sloping.
  - If the pure expectations hypothesis were true, that would mean people generally expect rates to rise.
  - An alternative explanation is that investors generally require a higher expected return to be willing to hold longer bonds.

Example in which Longer Bonds Have Higher Expected Returns

Time 0	Time 0.5	0.5-yr horizon		1-yr horizon	
	Zero rate	ROR on 0.5-yr z.	ROR on 1-yr z.	ROR on 0.5-yr z.	ROR on 1-yr z.
	$_{0.5}r_1^u$=6.00%	5.00%	4.503%	5.499%	5.25%
$_0r_{0.5}$=5.00% $_0r_1$=5.25%					
	$_{0.5}r_1^d$=4.00%	5.00%	6.508%	4.499%	5.25%
Expected:	5.00%	5.00%	5.505%	4.999%	5.25%

Forward rate
$f_{0.5}^1$=5.50%

Here, the yield curve is upward-sloping, not because rates are expected to rise, but because longer bonds are priced to offer a higher expected return.

Term Premiums in Forward Rates

- Empirically, forward rates tend to be higher than the spot rate that ultimately prevails for that investment horizon, or equivalently, longer bonds appear to have higher average returns.
- The "term premium" is defined roughly by

  Forward rate = Expected future spot rate + Term Premium
- A more general version of expectations hypothesis says that term premiums are roughly constant.
- If that's true, then changes in forward rates reflect changes in expectations about future rates.
- On the other hand it could be because risk premiums have changed.

Summary of Intuition

- Conceptually:

Steepness of yield curve	=	Expected rate increase	+	Long bond risk premium

- Quantitatively:

Forward rate	=	Expected future spot rate	+	Term premium

Some Evidence

Results of regressions of

$$ {}_{t+j}r_{t+j+1} - {}_{t}r_{t+1} = a + \beta({}_{t}f_{t+j}^{\;t+j+1} - {}_{t}r_{t+1}) + \varepsilon_{t,j} $$

for j=1, 2, 3, 4 years, sample period 1980-2006.

Pure expectation hypothesis: $\alpha=0$, $\beta=1$.

Country	j	α	Std. err.	β	Std. err.	R^2
US	1	-0.30	0.33	0.11	0.26	0.21
	2	-0.70	0.82	0.25	0.42	1.16
	3	-1.45	1.12	0.72	0.37	8.39
	4	-2.25	1.09	1.22	0.25	21.17
UK	1	-0.19	0.26	0.49	0.23	9.34
	2	-0.74	0.52	1.00	0.27	26.17
	3	-1.01	0.66	1.18	0.31	34.26
	4	-1.45	0.66	1.40	0.33	46.28
Germany	1	-0.36	0.32	0.48	0.18	6.30
	2	-1.01	0.51	0.98	0.26	19.14
	3	-1.77	0.51	1.39	0.33	35.44
	4	-2.46	0.45	1.62	0.29	49.86

From Boudoukh, Richardson, Whitelaw, 2007, The information in long forward rates: Implications for exchange rates and the forward premium anomaly.