



No Arbitrage Pricing of Derivatives



Concepts and Buzzwords

- Replicating Payoffs
- No Arbitrage Pricing

Derivative, contingent claim, redundant asset, underlying asset, riskless asset, call, put, expiration date, strike price, binomial tree

Reading

- Veronesi, Chapter 9
- Tuckman, Chapter 9



Derivative Debt Instruments

- So far, the course has focused mainly on assets with fixed future cash flows.
- Now we will begin to study assets with random future cash flows that depend on future bond prices or interest rates.
- These are sometimes called derivatives or contingent claims. A derivative is an asset whose payoff depends on the future prices of some other underlying assets.



Redundant Securities

- In some cases, the payoff of the derivative can be replicated by the payoff of a portfolio containing underlying assets whose prices are already known.
- In that case the derivative is called a redundant security.
- It is possible to price a redundant security by no arbitrage.



No Arbitrage Pricing

The *no arbitrage pricing* approach for valuing a derivative proceeds as follows:

1. Start with a description (model) of the future payoff or price of the underlying assets across different possible states of the world.
2. Construct a portfolio of underlying assets that has the same random payoff as the derivative.
3. Set the price of the derivative equal to the value of the replicating portfolio.

Prices and Payoffs of Underlying Assets?



- Nonredundant assets (that can't be synthesized with other assets) cannot be priced by arbitrage.
- What determines their prices?
- We can think of their current and future prices as the outcome of market clearing in a general equilibrium.
- Bond prices (interest rates) are influenced by
 - the size of the deficit, international trade balances, the productivity of real investment (real rates)
 - monetary policy (inflation rates)
 - people's investment horizons, risk preferences, and endowments, correlations with other assets (term premia)



No Arbitrage Pricing Approach

- The no arbitrage pricing approach picks up where equilibrium theory leaves off.
- It takes the prices and payoffs of the underlying (non-redundant) assets as given.
- Current prices of underlying assets are in fact observable.
- Future price or payoff distributions aren't really known in practice, although the theory treats them as known.
- Modeling future payoffs for no arbitrage pricing in practice is a problem of forecasting and financial engineering.



Motivation: A One-Period Model for Pricing A Call Option

- Before constructing an elaborate interest rate model, let's see how no-arbitrage pricing works in a one-period model.
- To motivate the model, consider a call option on a \$1000 par of a zero maturing at time 1.
- The call gives the owner the right but not the obligation to buy the underlying asset for the strike price at the expiration date.
- Suppose the expiration date is time 0.5 and the strike price \$975.



Call Option Payoff

- Let ${}_{0.5}d_1$ represent the price at time 0.5 of \$1 par of the zero maturing at time 1.

- The payoff of the call is

$$\text{Max}[1000 \times {}_{0.5}d_1 - 975, 0]$$

- To value the option, we need to form an idea about what the possible values of the underlying asset will be on the option expiration date.
- Note that the underlying asset, a bond maturing at time 1, will be a 6-month bond at time 0.5, the option expiration date.

Future Payoffs of the Underlying



- Suppose that at the option expiration date, the underlying bond can take on only two possible prices (per \$1 par):

$${}_{0.5}d_1 = 0.972290 \text{ or } {}_{0.5}d_1 = 0.976086$$

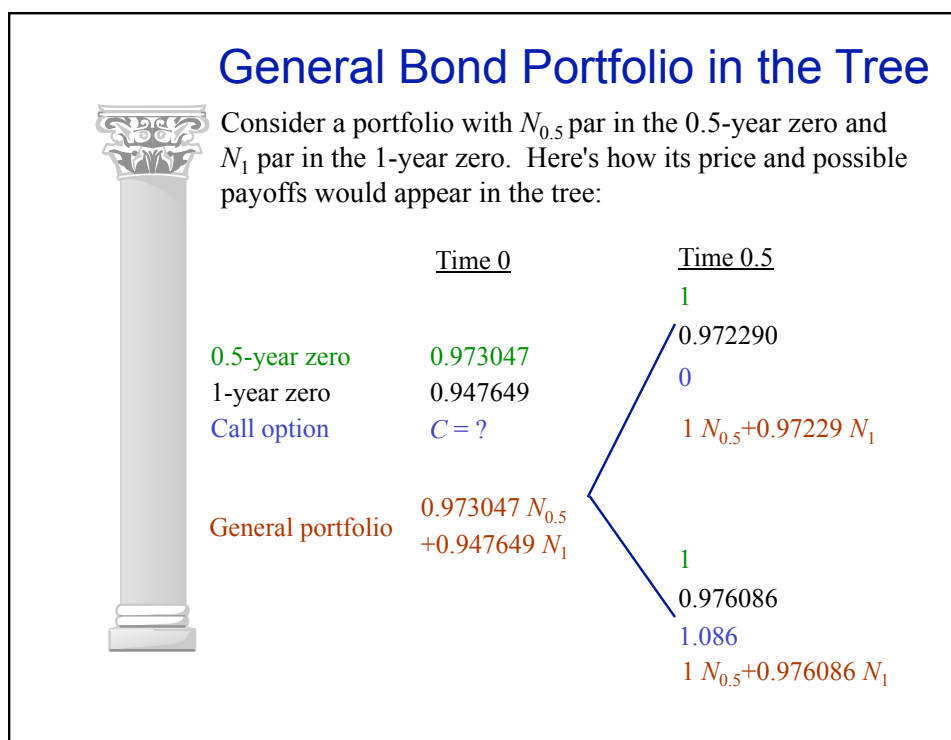
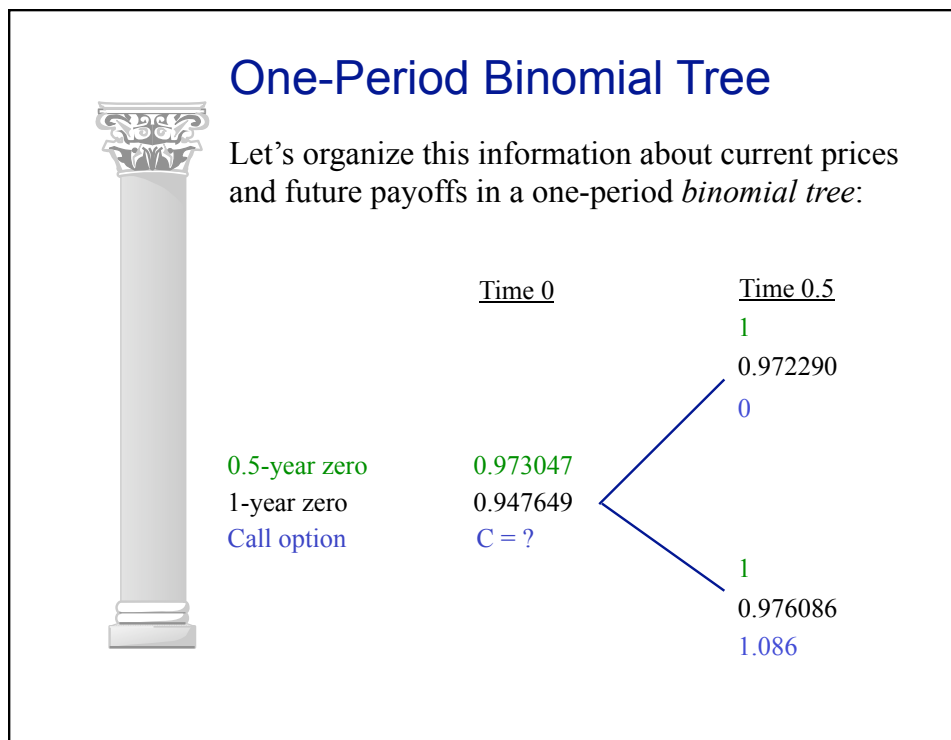
- This means that the option has only two possible payoffs:

“up (high interest rate) state”: \$0

“down state”: $\$976.086 - \$975 = \$1.086$

- Recall that today (time 0) the prices of 6-month and 1-year zeroes are:

$${}_0d_{0.5} = 0.973047 \text{ and } {}_0d_1 = 0.947649.$$





Replicating the Call Payoff

If there are only two possible future values of the underlying asset, then the call can be replicated with a portfolio of the 0.5-year and 1-year zeroes (the riskless asset and the underlying asset).

A portfolio of the underlying and riskless asset *replicates* the call if it satisfies two equations:

- ① Up state portfolio payoff = Up state call payoff
- ② Down state portfolio payoff = Down state call payoff



Replicating the Call Payoff...

- The portfolio and its payoff are described by specifying the number of securities, or in other words, the face value of the bonds, in the portfolio.
- Let $N_{0.5}$ and N_1 denote the number of the 0.5-year bonds and the number of the 1-year bonds in the portfolio.
- Then the payoff-matching equations are:

Up state portfolio payoff = Up state call payoff

$$N_{0.5} \times 1 + N_1 \times 0.97229 = 0$$

Down state portfolio payoff = Down state call payoff

$$N_{0.5} \times 1 + N_1 \times 0.976086 = 1.086$$



The Replicating Portfolio

- The replicating portfolio (the solution to the simultaneous equations) is

$$N_{0.5} = -278.163 \text{ and } N_1 = 286.091$$

- In other words, a portfolio that is
 - long 286.091 par of the 1-year zero and
 - short 278.163 par of the 0.5-year zero
- will have exactly the same payoff as the call at time 0.5, regardless of which state is actually realized.



Class Problem

What is the no-arbitrage call price?

General Bond Portfolio in the Tree



Consider a portfolio with $N_{0.5}$ par in the 0.5-year zero and N_1 par in the 1-year zero. Here's how its price and possible payoffs would appear in the tree:

	<u>Time 0</u>	<u>Time 0.5</u>
0.5-year zero	0.973047	1 0.972290
1-year zero	0.947649	0 0.976086
Call option	$C = ?$	1 $N_{0.5} + 0.97229 N_1$ 1.086
General portfolio	$0.973047 N_{0.5} + 0.947649 N_1$	1 $N_{0.5} + 0.976086 N_1$

Summary



- To price the call, we assumed two possible future values of the underlying at the option expiration date,
- determined the two possible future payoffs of the call,
- constructed a portfolio that replicates the call using two assets that are already priced:
 - the underlying asset and
 - the zero maturing on the expiration date,
- And then set the price of the call equal to the cost of the replicating portfolio.
- The call price is only as accurate as the future underlying payoffs we assumed.

Pricing a Put Option



- Let's price another derivative -- say, a put option.
- A put gives the owner the right but not the obligation to *sell* the underlying asset for the strike price at the expiration date.
- Suppose that, again,
 - the underlying is \$1000 par of the zero maturing at time 1,
 - expiration date is time 0.5, and
 - the strike price \$975.
- The put payoff is $\max[975 - 1000 \times_{0.5} d_1, 0]$

Put Option in the Tree



	<u>Time 0</u>	<u>Time 0.5</u>
0.5-year zero	0.973047	1 0.972290
1-year zero	0.947649	0
Call option	?	$1N_{0.5} + 0.97229N_1$ 2.71
General portfolio	$0.973047N_{0.5} + 0.947649N_1$	1 0.976086
Put option	?	1.086 $1N_{0.5} + 0.976086N_1$ 0

Class Problem: Replicating and Pricing the Put



- Again, the put can be replicated with a portfolio of the 0.5-year and 1-year zeroes.
 - Let $N_{0.5}$ and N_1 denote the par amounts of the 0.5-year bonds and 1-year bonds in the portfolio.
- 1) What are the payoff-matching equations that determine the replicating portfolio par amounts?
 - 2) What are the par amounts?
 - 3) What is the no-arbitrage price of the put?

Put Option in the Tree



	<u>Time 0</u>	<u>Time 0.5</u>
0.5-year zero	0.973047	1
1-year zero	0.947649	0.972290
Call option	?	0
General portfolio	$0.973047N_{0.5} + 0.947649N_1$	$1N_{0.5} + 0.97229N_1$
Put option	?	2.71
		1
		0.976086
		1.086
		$1N_{0.5} + 0.976086N_1$
		0

General Bond Derivative



Any security whose time 0.5 payoff is a function of the time 0.5 price of the zero maturing at time 1 can be priced by no arbitrage. Suppose its payoff is K_u in the up state, and K_d in the down state.

	<u>Time 0</u>	<u>Time 0.5</u>
		1
0.5-year zero	0.973047	0.972290
1-year zero	0.947649	
General portfolio	$0.973047 N_{0.5}$ $+ 0.947649 N_1$	$1N_{0.5} + 0.97229N_1$ K_u
General derivative ?		1 0.976086 $1N_{0.5} + 0.976086N_1$ K_d

Replicating and Pricing the General Derivative



1) Determine the replicating portfolio by solving the equations

$$1N_{0.5} + 0.97229N_1 = K_u$$

$$1N_{0.5} + 0.96086N_1 = K_d$$

for the unknown N 's. (The two possible K 's are known.)

2) Price the replicating portfolio as

$$0.973047N_{0.5} + 0.947649N_1$$

This is the no arbitrage price of the derivative.