Dynamic Trading Strategies

Concepts and Buzzwords

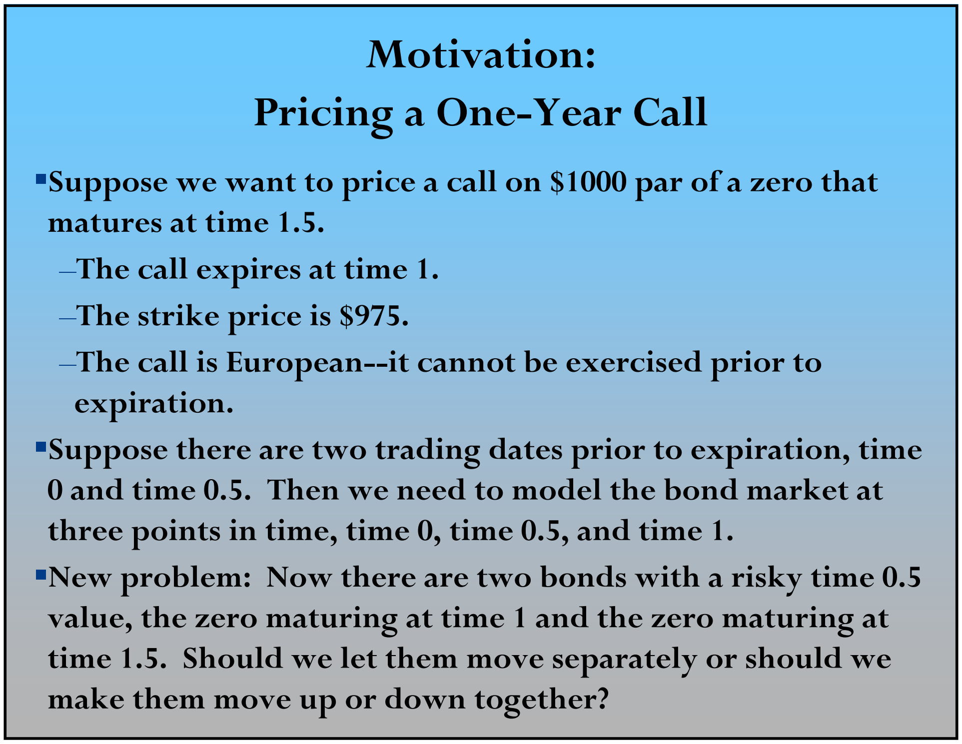
- Multi-Period Bond Model
- Replication and Pricing Using Dynamic Trading Strategies
- Pricing Using Risk-Neutral Probabilities
- One-factor model, no-arbitrage restrictions

Reading

- Tuckman, chapter 9.

No Arbitrage Pricing in a Multi-period Setting

- In the last two lectures we priced a derivative in a one-period setting:
  - payoff replication+no arbitrage => price
  - replication cost in terms of r-n probabilities
- This lecture introduces *multiple trading dates*, and shows how dynamic trading strategies with just a few basic assets can create (and thus price) more complex payoff patterns.
- For now we will treat future possible payoffs of the basic assets as given, and focus on how *to use them to price derivatives.*
- Later, after we understand how the model will be used, we can think about how best to build the model itself.
- However, one modeling issue must be addressed immediately: How many "factors " or different sources of risk do we allow for?

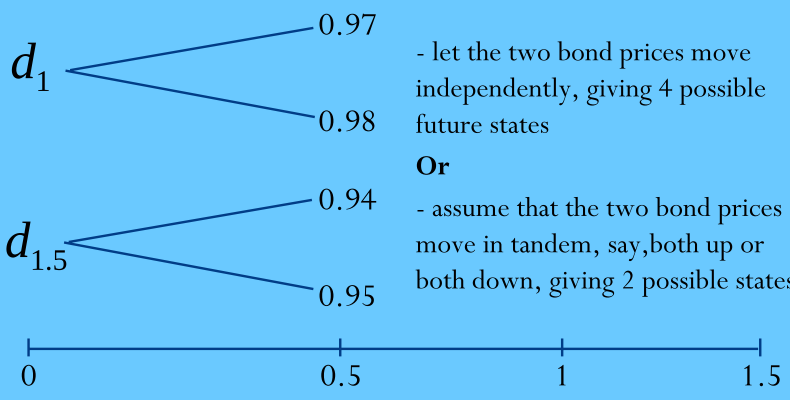


Motivation:
Pricing a One-Year Call

- Suppose we want to price a call on $1000 par of a zero that matures at time 1.5.
  - The call expires at time 1.
  - The strike price is $975.
  - The call is European--it cannot be exercised prior to expiration.
- Suppose there are two trading dates prior to expiration, time 0 and time 0.5. Then we need to model the bond market at three points in time, time 0, time 0.5, and time 1.
- New problem: Now there are two bonds with a risky time 0.5 value, the zero maturing at time 1 and the zero maturing at time 1.5. Should we let them move separately or should we make them move up or down together?

How many factors?

Say we let each zero price take on two possible values:



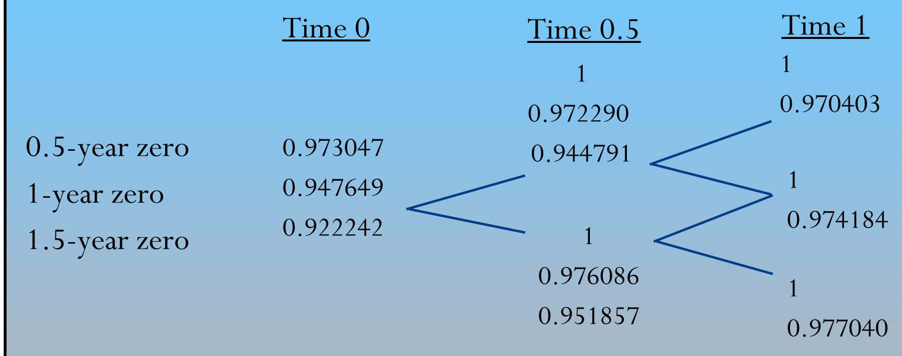
- let the two bond prices move independently, giving 4 possible future states

Or

- assume that the two bond prices move in tandem, say,both up or both down, giving 2 possible states

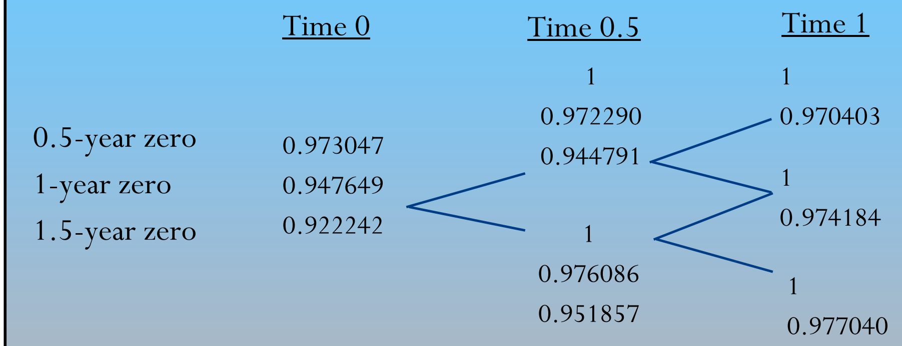
More generally, should each multi-period bond have its own tree? In principle, yes. But that's usually too hard in practice. So here we'll assume there's just one tree -- a so-called "one-factor model“.

One-Factor Model for Bond Market: Three Zeroes, Two Time Periods, One Binomial Tree



Each period the bond market can make one of two moves. Either all bond prices move up, or all bond prices move down: a single coin flip each time, and just one risk factor. For now, take the prices of the three zeros in each state as given.

One-Factor Model for Bond Market: Prices are Arbitrage-Free

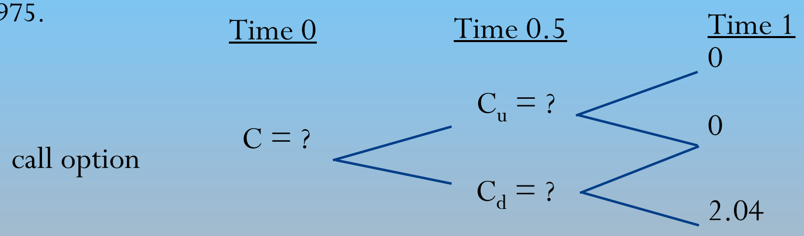


Note that at time 0, we can "synthesize" one of the bonds from the other two.

We have made sure that the prices are arbitrage-free

Pricing the Long Call: Start with its Payoffs

Recall, the call is on $1000 par of the zero maturing at time 1.5, with strike $975.



The payoff of the call option on the expiration date is:

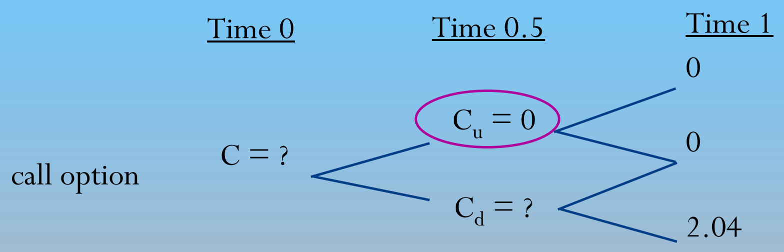
$$\max(1000 \times {}_1d_{1.5} - 975, 0)$$

What is the no arbitrage price of the call today?

Replication with a Dynamic Trading Strategy

- **Just as before, we can determine the no arbitrage value of the call by computing the value of a portfolio of priced assets (the zeroes) that replicates its payoffs.**
- **Note that *we can trade at every time and state*.**
- **Therefore, we do not need to try to construct a portfolio at time 0 that can be held for two periods and result in a payoff that matches the call payoff.**
- **Instead, we can use a *dynamic trading strategy:***
  - **➔ In each state at time 0.5, we construct a portfolio that will have the same payoff as the call at time 1 (in general: work backwards, start at time T, back to time 0).**
  - **➔ Then, at time 0, we just need a portfolio that will payoff enough to buy the required portfolio at time 0.5.**

The Call at Time 0.5, State Up



Time 0 Time 0.5 Time 1

call option

C = ?

$C_u = 0$

$C_d = ?$

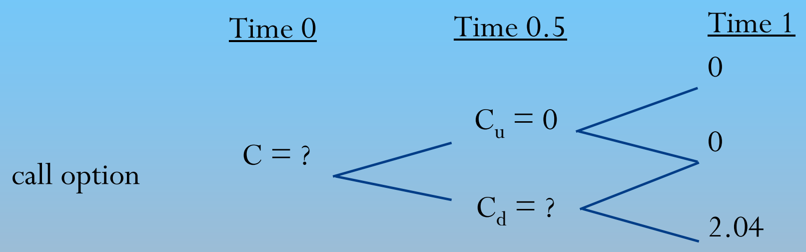
0

0

2.04

It is clear that at time 0.5, state "*up*", the replicating portfolio is empty, since its payoffs are 0. The cost of this portfolio is 0, which is the no arbitrage value of the call at that time and state.

Replicating the Call at Time 0.5, State Down



Time 0 Time 0.5 Time 1

call option C = ?

$C_u = 0$

$C_d = ?$

0

0

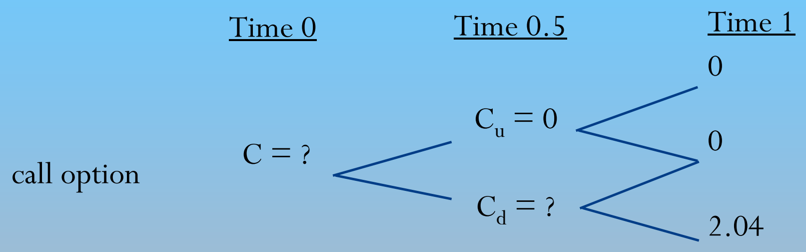
2.04

At time 0.5, state "*down*", the portfolio of the zeroes maturing at time 1 and time 1.5 that replicates the call is determined by the payoff-matching equations:

$$N_1 \times 1 + N_{1.5} \times 0.974184 = 0$$

$$N_1 \times 1 + N_{1.5} \times 0.977040 = 2.04$$

Replicating the Call at Time 0.5, State Down...



Time 0 Time 0.5 Time 1

call option C = ?

$C_u = 0$

$C_d = ?$

0

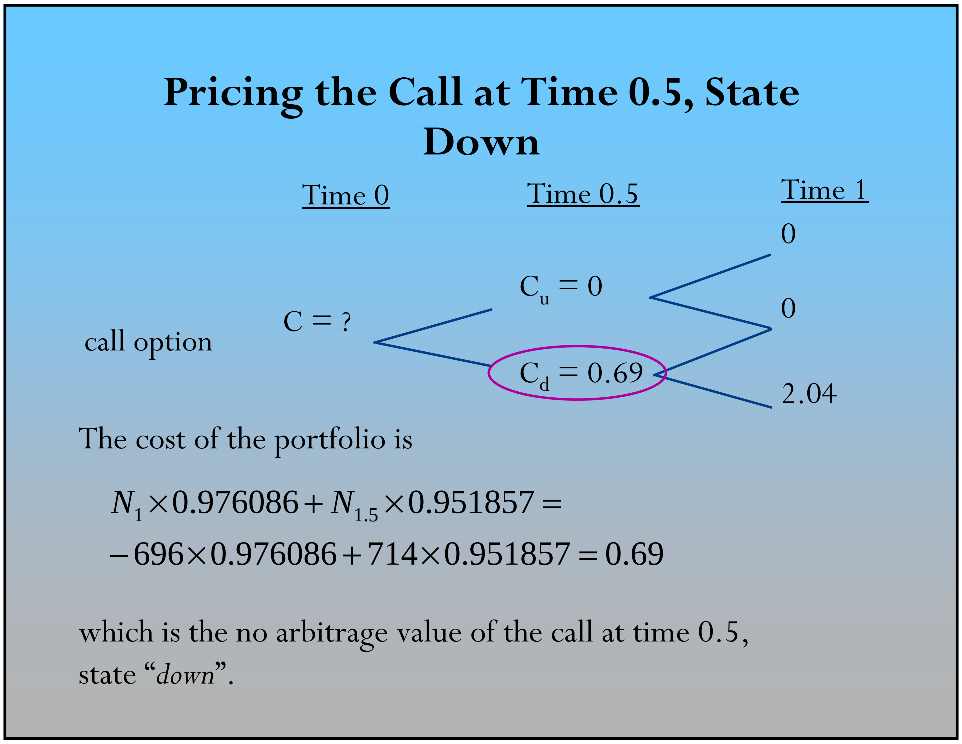
0

2.04

The solution to the payoff-matching equations is:

$$N_1 = -696 \text{ and } N_{1.5} = 714$$

In other words, at time 0.5, state down, going short 696 par of the zero maturing at time 1 and long 714 par of the zero maturing at time 1.5 will give a payoff equal to the call payoff at time 1.

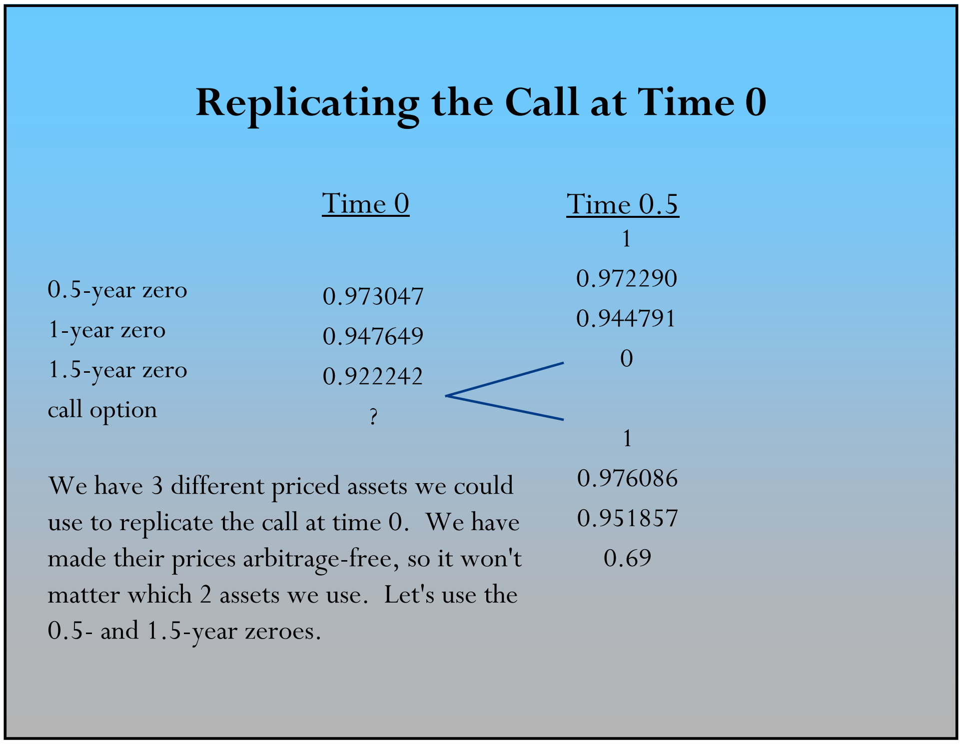


Pricing the Call at Time 0.5, State Down

The cost of the portfolio is

$$N_1 \times 0.976086 + N_{1.5} \times 0.951857 =$$
$$-696 \times 0.976086 + 714 \times 0.951857 = 0.69$$

which is the no arbitrage value of the call at time 0.5, state "*down*".



Replicating the Call at Time 0

	Time 0	Time 0.5	
0.5-year zero	0.973047	1	1
1-year zero	0.947649	0.972290	0.976086
1.5-year zero	0.922242	0.944791	0.951857
call option	?	0	0.69

We have 3 different priced assets we could use to replicate the call at time 0. We have made their prices arbitrage-free, so it won't matter which 2 assets we use. Let's use the 0.5- and 1.5-year zeroes.

Replicating the Call at Time 0…

The payoff-matching equations are:

$$N_{0.5}\times 1+N_{1.5}\times 0.944792=0$$
$$N_{0.5}\times 1+N_{1.5}\times 0.951857=0.69$$

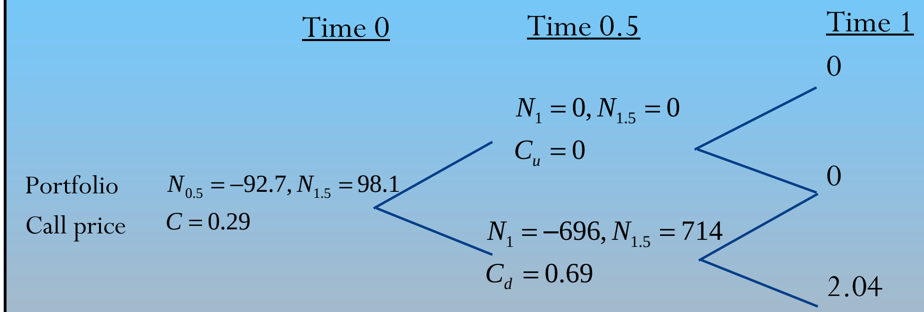
The solution is:

$$N_{0.5}=-92.7 \text{ and } N_{1.5}=98.1$$

The cost of the replicating portfolio, and hence the no arbitrage price of the call is:

$$N_{0.5}\times 0.973047+N_{1.5}\times 0.922242=$$
$$-92.7\times 0.973047+98.1\times 0.922242=0.29$$

Self-Financing Dynamic Trading Strategy for Replicating the Call



Time 0

Portfolio $N_{0.5}=-92.7, N_{1.5}=98.1$

Call price $C=0.29$

Time 0.5

$N_1=0, N_{1.5}=0$

$C_u=0$

$N_1=-696, N_{1.5}=714$

$C_d=0.69$

Time 1

0

0

2.04

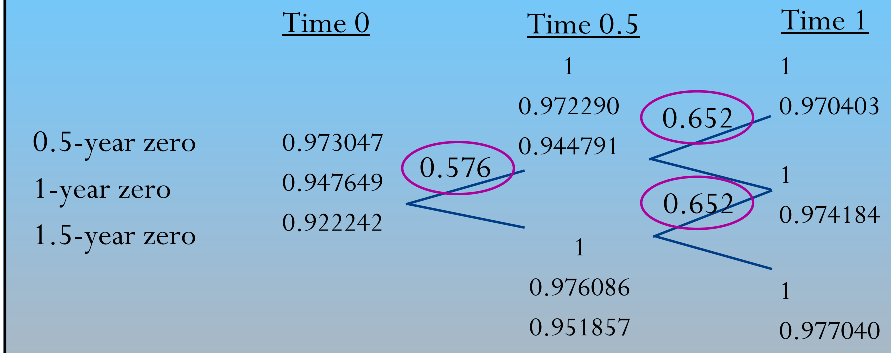
- There is no ***static*** replicating portfolio (no portfolio that replicates the call payoff which can be bought at time 0 and held until time 1).
- However, we can replicate the call payoff with a ***self-financing dynamic trading strategy*** (i.e., after it has been set up, the strategy has no cash inflows/outflows).
- The no arbitrage call price is the cost of the current replicating portfolio.

Pricing the Call with Risk-Neutral Probabilities

- We were able to price the call because we were able to find a dynamic trading strategy using the underlying bonds that replicated the call payoff.
- The no arbitrage call price is the cost of the replicating trading strategy.

- As a computational method, we could also use the risk-neutral probabilities to get the cost of the replicating trading strategy,without actually computing the portfolio holdings.
- Here's how...

Risk-Neutral Probabilities



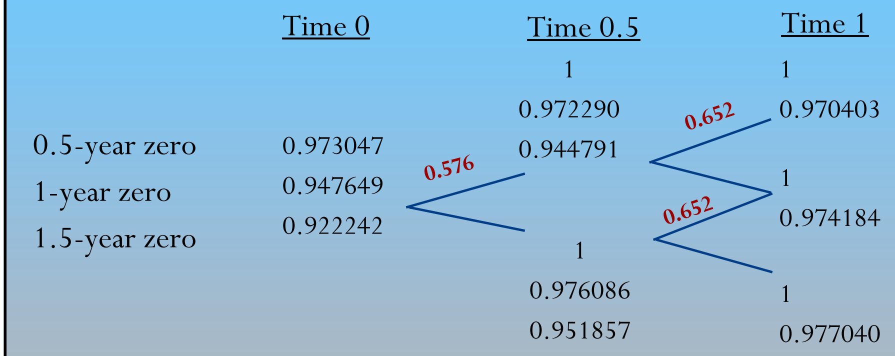
	Time 0	Time 0.5	Time 1
		1 0.972290 0.944791	1 0.970403
0.5-year zero	0.973047		
1-year zero	0.947649		1 0.974184
1.5-year zero	0.922242		
		1 0.976086 0.951857	1 0.977040

For each period, there are "risk-neutral probabilities" that make the equation

price = discounted expected payoff

hold for each asset.

Risk-Neutral Probabilities



Time 0

0.5-year zero 0.973047
1-year zero 0.947649
1.5-year zero 0.922242

Time 0.5

1
0.972290
0.944791

1
0.976086
0.951857

Time 1

1
0.970403

1
0.974184

1
0.977040

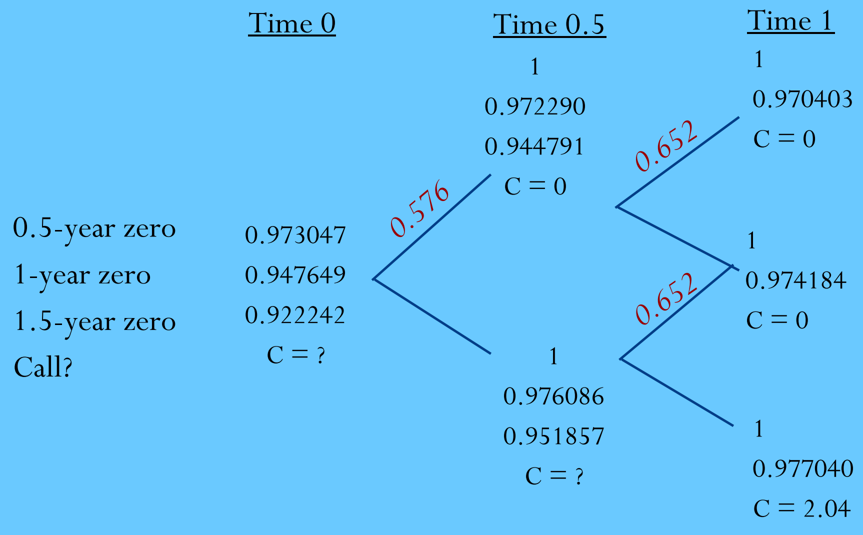
$V=(P*K_{up}+(1-p)K_d)/(1-r/2)$

Time 0 value of the 1-year zero:

*0.947649=[p*0.972290+(1-p)*0.976086]/(1+0.0554/2)*

1.5year zero?

Pricing the Call with Risk-Neutral Probabilities…



Time 0

0.5-year zero 0.973047
1-year zero 0.947649
1.5-year zero 0.922242
Call? C = ?

Time 0.5

1
0.972290
0.944791
C = 0

1
0.976086
0.951857
C = ?

Time 1

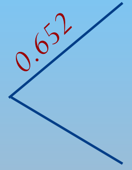
1
0.970403
C = 0

1
0.974184
C = 0

1
0.977040
C = 2.04

Pricing the Call with Risk-Neutral Probabilities at Time 0.5, State Down

We just take the expected value of the payoffs under the risk-neutral probabilities, and discount back by the riskless rate, or equivalently multiply by the price of the riskless zero:



1
0.976086
0.951857
C = 0.69

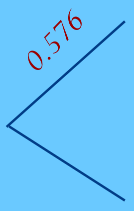
1
0.974184
C = 0

1
0.977040
C = 2.04

$$\frac{0.652\times 0+(1-0.652)\times 2.04}{1+0.0490/2}$$
$$=(0.652\times 0+(1-0.652)\times 2.04)\times 0.976086$$
$$=0.69$$

Pricing the Call with Risk-Neutral Probabilities: Time 0

	Time 0
0.5-year zero	0.973047
1-year zero	0.947649
1.5-year zero	0.922242
Call	C = 0.29



Time 0.5

1
0.972290
0.944791
C = 0

1
0.976086
0.951857
C = 0.69

$$\frac{0.576\times 0+(1-0.576)\times 0.69}{1+0.0554/2}$$
$$=(0.576\times 0+(1-0.576)\times 0.69)\times 0.973047$$
$$=0.29$$