Fixed Income
Financial Engineering

Concepts and Buzzwords

- From short rates to bond prices
- The simple Black, Derman, Toy model
- Calibration to current the term structure
- Nonnegativity
- Proportional volatility
- Lognormal limiting Distribution
- Independent increments vs. Mean reversion

Readings

- Veronesi, Chapters 10-11
- Tuckman, Chapters 11-12

Implementing the No-Arbitrage Derivative Pricing Theory in Practice

1. Start with a model (tree) of one-period rates (short rates) and risk-neutral probabilities.

   For example, Black-Derman-Toy, Ho and Lee, …
2. Build the tree of bond prices from the tree of short rates using the risk-neutral pricing equation (RNPE)

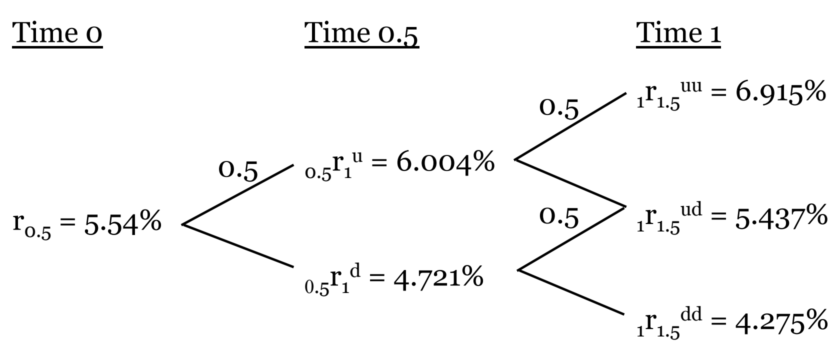
   price = discount factor x [p x up payoff + (1-p) x down payoff]
3. Build the tree of derivative prices from the tree of bond prices by pricing by replication.

   Replication cost can be also represented as

   price = discount factor x [p x up payoff + (1-p) x down payoff]
4. Calibrate the model parameters (drift, volatility) to make the model match observed bond prices and option prices.

Building the Price Tree from the Rate Tree and Risk-Neutral Probabilities (Step 2)

- Once we have a tree of one-period rates and risk-neutral probabilities, we can price any term structure asset.
- For example, suppose 0.5-year rates and risk-neutral probabilities are as follows:



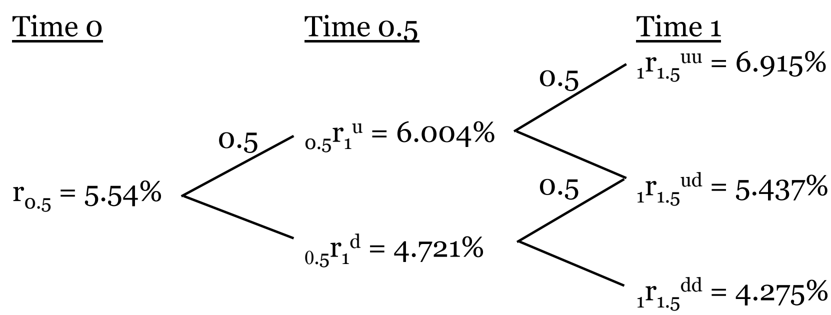
Building the Price Tree from the Rate Tree…

- Then we have the prices of bonds for maturities 0.5, 1, and 1.5:
- Time 0 price of the zero maturing at time 0.5

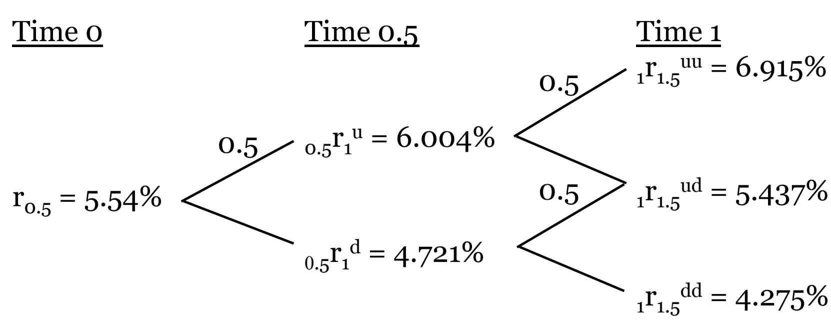
  $d_{0.5} = 1/(1+0.0554/2) = 0.973047$
- Time 0.5 possible prices of zero maturing at time 1

  ${}_{0.5}d_1^u = 1/(1+0.06004/2) = 0.9709$, ${}_{0.5}d_1^d = 1/(1+0.04721/2) = 0.9769$
- Time 0 price of the zero maturing at time 1

  $d_1 = 0.973047\,[0.5 \text{ x } 0.9709 + 0.5\text{x}0.9769] = 0.9476$



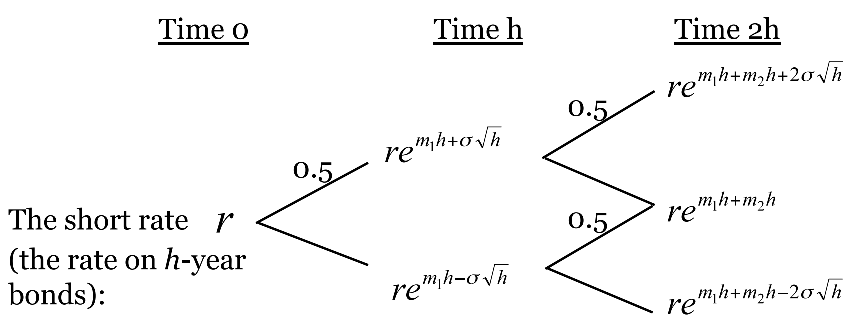
Class Problem: Fill in the tree of prices for the zero maturing at time 1.5



Modeling the Short Rates

- The goal is to build interest rate models that capture basic properties of interest rates while also fitting the current term structure (and liquid option prices).
- Some basic properties are
  - nonnegative interest rates
  - non-normal distribution
  - mean-reversion
  - stochastic volatility and the level effect.
- We will use a simple version of the Black-Derman-Toy model, which has some of these properties.

Log Model of Interest Rates (Black-Derman-Toy with Constant Volatility)



- Each date the short rate changes by a multiplicative factor:
  up factor = $e^{mh+\sigma\sqrt{h}}$,
  down factor = $e^{mh-\sigma\sqrt{h}}$
- The exponential is always positive, which guarantees that interest rates are always positive in this model.

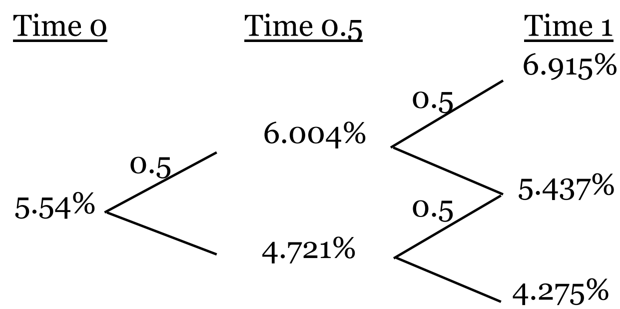
Description of the Model

- The parameter h is the amount of time between dates in the tree, in years. For example, in a semi-annual tree, h = 0.5. In a monthly tree, h = 1/12 = 0.08333.
- Each value in the tree represents the short rate or interest rate for a zero with maturity h.
- Each date the risk-neutral probability of moving up or down is 0.5.
- The drift parameters m_1, m_2,... are known (nonstochastic) but vary over time – these are calibrated to make the model bond prices match the current term structure.
- The proportional volatility σ, is constant here – this is typically calibrated to an option price.
- In the full-blown BDT model, σ also varies each period to allow the model to fit multiple option prices.
- In the limit, as h->0, the distribution of the future instantaneous short rate is lognormal, i.e., its log is normally distributed.

Example: Semi-Annual Tree Calibrated to Given Term Structure and Volatility

- Suppose
  - the time steps are 6 months, i.e., h=0.5 (typically, the choice of h is a tradeoff between speed and accuracy)
  - the current 6-month rate is 5.54%
  - the drift over the first period is m_1=-0.0797 (this sets the average level of the short rate at time 0.5—it is chosen to make the model's 1-year zero price match the actual 1-year zero price, i.e., 0.9476 in our case.)
  - the drift over the second period is m_2 =0.0422 (this sets the average level of the short rate at time 1—it is chosen to make the 1.5-year zero price in the model = 0.9222)
  - the proportional volatility σ=0.17 (could use historical volatility or volatility implied by the price of liquid option)

Resulting Tree of Short Rates

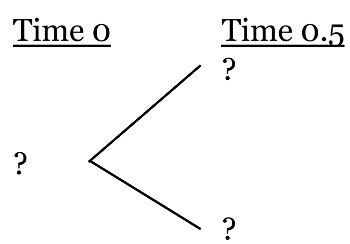


For example, at time 0.5, up, the 6-month zero rate is

$$0.0554e^{-0.0797\text{x}0.5+0.17\sqrt{0.5}} = 0.06004$$

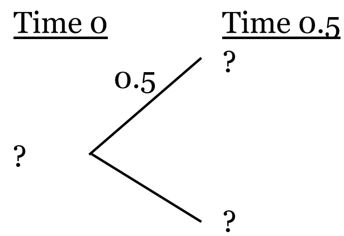
Class Problems

1) Build a tree of 0.5-year rates out to time 0.5 using h=0.5, ${}_0r_{0.5}$=2%, m_1=0.01, and σ=0.20.



Class Problems

2) What is the price of the 0.5-year zero at each node?

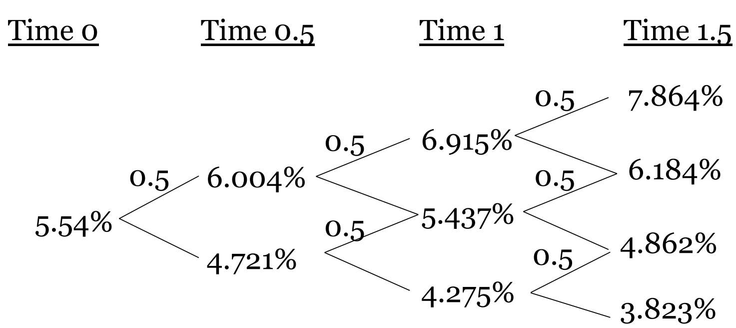


3) What is the price of the 1-year zero at time 0?

4) What is the 1-year zero rate at time 0?

Extending the Interest Rate Tree

- The tree can extended, as many periods as necessary by successively fitting drift terms to the prices of longer zeros.
- For example, to extend the tree to time 1.5, set m_3=0.01686 to make the tree correctly price the 2-year zero (${}_0d_2$=0.8972).



Resulting Zero Price Tree

- At each node, the prevailing prices of outstanding zeros are listed, in ascending order of maturity.
- For instance, the price of a 1-year zero at time 0.5. state up, is ${}_{0.5}d_{1.5}^{u} = 0.9418$.

Time 0	Time 0.5	Time 1	Time 1.5
		0.966581 0.933802	0.962167
	0.970857 0.941787 0.913180		
0.973047 0.947649 0.922242 0.897166		0.973533 0.947382	0.970009
	0.976941 0.953790 0.930855		0.976266
		0.979071 0.958270	0.981243

Resulting Tree of Term Structures

- At each node, the prevailing term structure of zero rates is listed, in ascending order of maturity.
- For instance, the 1-year zero rate at time 1, state up-down, is ${}_1r_2^{ud} = 5.479\%$.

Time 0	Time 0.5	Time 1	Time 1.5
		6.915% 6.968%	7.864%
	6.004% 6.089% 6.147%		
5.54% 5.45% 5.47% 5.50%		5.437% 5.479%	6.184%
	4.721% 4.788% 4.834%		4.862%
		4.275% 4.308%	3.823%

Limitations of This Model

- Only one volatility parameter
  - The model may not be able to fit the prices of options with different maturities simultaneously.
  - The full-blown Black-Derman-Toy model allows the proportional volatility parameter to vary over time to match prices of options with different maturities, allowing for a term structure of volatilities.
- Independent interest rate change over time.
  - Some feel that rates should be mean reverting. This would mean down moves would be more likely at higher interest rates.
  - The Black-Karasinski Model introduces mean reversion in the interest rate process.

Limitations of This Model…

- Only a One-Factor Model
  - Each period one factor (the short rate) determines the prices of all bonds.
  - This means that each period all bond prices move together. Their returns are perfectly correlated. There is no possibility that some bond yields could rise while others fall.
  - To allow for this possibility the model would require additional factors, or sources of uncertainly, which would expand the dimensions of the state-space. For example, in a two-factor model, each period you could move up or down and right or left, so there would be four possible future states.
- Large investment banks and derivatives dealers often have their own proprietary models.