Model Calibration and Hedging

Concepts and Buzzwords

- Choosing the Model Parameters
- Choosing the Drift Terms to Match the Current Term Structure
- Hedging the Rate Risk in the Binomial Model

–Term structure of volatilities, Black-Derman-Toy Model, Black-Karasinski model, hedge ratio, interest rate delta

Readings

- Tuckman, Chapter 8, The art of term structure modeling.
- Black, Derman, and Toy, 1990, A one-factor model of interest rates and its application to Treasury bond options, *Financial Analysts Journal*

Interest Rate Model

Time 0	Time h	Time 2h
		$re^{m_1h+m_2h+2\sigma\sqrt{h}}$
	$re^{m_1h+\sigma\sqrt{h}}$	
The short rate (the rate on h-year bonds) r		$re^{m_1h+m_2h}$
	$re^{m_1h-\sigma\sqrt{h}}$	
		$re^{m_1h+m_2h-2\sigma\sqrt{h}}$

(each branch: 0.5)

How do we choose values for the time step *h*, the drift terms, *m*1, *m*2, …, and the proportional volatility σ?

Choosing the Step Size *h*

- Analysis of a particular debt instrument usually dictates the amount of calendar time that the interest rate model must cover (3 months, 1 year, 30 years, etc.)
- The smaller the step size,
  - The richer the set of possible states in the tree,
  - The more computer time required to run the model.
- In practice, choosing *h* is a tradeoff between speed and accuracy.
- In our stylized examples, we will generally use semi-annual steps (h=1/2).

Choosing the Proportional Volatility σ

- Historical Volatility
  - Estimate volatility from actual interest rate data.
  - For example, based on historical data, an estimate of the standard deviation of the weekly change in the log of the six-month rate is 0.0237.
  - Assuming weekly changes are independent, the volatility of the annual change in the log of the six-month rate is about σ = 0.17 (0.0237 multiplied by the square root of 52).
- Implied Volatility
  - Pick the volatility that best fits the price of liquid options similar to the instrument being analyzed.

Justification for Historical Volatility

- Recall that the parameter σ in the model gives the volatility of interest rates under the risk-neutral distribution.
- Historical data on interest rates only provides an estimate of volatility under the true probability distribution.
- Result from Asset Pricing Theory
  - As the time steps grow small, the discrete-time binomial interest rate process approaches a *continuous-time* process.
  - For the continuous time process, the volatility over an instant of time (a small time step) is the same under the two distributions.

Choosing the Drift Parameters $m1$, $m2$, $m3$, … to Fit the Term Structure.

- Suppose we know the current term structure, i.e. the current prices of zeros maturing at every date in the tree:
  ${}_0d_h$, ${}_0d_{2h}$, ${}_0d_{3h}$,…
  - If h=0.5, we have ${}_0d_{0.5}$=0.9730, ${}_0d_1$=0.9476, ${}_0d_{1.5}$=.9222, ${}_0d_2$=0.8972,…
  - If the dates in the tree are very frequent, we may have to use estimate a spline discount function to get zero prices first.
- Then we have to choose the drift parameters so that the zero prices implied by the interest rate model match the actual zero prices.

Choosing the Drift Parameters…

Recall that the current price of a 0.5-year zero is ${}_0d_{0.5}$=0.9730. The current 0.5-year rate is 5.54%.

With h=0.5 and σ =0.17, the interest rate tree must start over as follows:

Time 0		Time h=0.5
	0.5	$r_u = 0.0554e^{m_1\times 0.5+0.17\sqrt{0.5}}$
5.54%		
	0.5	$r_d = 0.0554e^{m_1\times 0.5-0.17\sqrt{0.5}}$

Choosing the Drift Parameter $m1$

Picking a value for $m1$ will determine the values of the two future possible 6-month rates, r_u and r_d.

Values for r_u and r_d determine the two possible time 0.5 prices for the zero maturing at time 1:

	Time h=0.5		Time 1
0.5	$\frac{1}{(1+r_u/2)} = \frac{1}{1+0.0554e^{m_1\times 0.5+0.17\sqrt{0.5}}/2}$		1
			1
0.5	$\frac{1}{(1+r_d/2)} = \frac{1}{1+0.0554e^{m_1\times 0.5-0.17\sqrt{0.5}}/2}$		1

Choosing the Drift Parameter $m1$

The two possible time 0.5 prices of the zero maturing at time 1 determine its time 0 price in the model: the risk-neutral expected future value of the zero, discounted back to time 0 at the riskless rate.

$$ {}_0d_1 = \left(0.5 \times \frac{1}{(1+r_u/2)} + 0.5 \times \frac{1}{(1+r_d/2)}\right) \times \; 0.9730 $$

We want the model price to match the observed price of 0.9476:

$$ 0.5\left(\frac{1}{(1+r_u/2)} + \frac{1}{(1+r_d/2)}\right)0.9730 = 0.9476 $$

This yields an equation that determines $m1$:

$$ 0.5\left(\frac{1}{1+0.0554e^{m_1\times 0.5+0.17\sqrt{0.5}}/2} + \frac{1}{1+0.0554e^{m_1\times 0.5-0.17\sqrt{0.5}}/2}\right) = \frac{0.9476}{0.9730} $$

$$ \Rightarrow m_1 = -0.07968 $$

Partially Calibrated Interest Rate Tree

With m_1= -0.07968, the interest rate tree is now as follows:

Time 0		Time 0.5		Time 1
			0.5	$r_{uu} = 0.06004e^{m_2\times 0.5+0.17\sqrt{0.5}}$
	0.5	r_u = 6.004%		
			0.5	$r_{ud} = 0.06004e^{m_2\times 0.5-0.17\sqrt{0.5}}$
5.54%			0.5	$= 0.04721e^{m_2\times 0.5+0.17\sqrt{0.5}}$
	0.5	r_d = 4.721%		
			0.5	$r_{dd} = 0.04721e^{m_2\times 0.5-0.17\sqrt{0.5}}$

Choosing the Drift Parameter *m*2

We choose *m*2 to correctly price the 1.5–year zero. Recall that its price is ${}_0d_{1.5}$ = 0.9222. Consider a price tree for this zero maturing at time 1.5.

Time 0	Time 0.5	Time 1	Time 1.5
		$P_{uu} = \frac{1}{(1+r_{uu}/2)}$	1
	P_u		
0.9222		$P_{ud} = \frac{1}{(1+r_{ud}/2)}$	1
	P_u		1
		$P_{dd} = \frac{1}{(1+r_{dd}/2)}$	1

(All branch probabilities: 0.5)

Choosing the Drift Parameter *m*2

The current and future prices of the zero maturing at time 1.5 fit together through the following equations:

$$P_u = \frac{0.5 \times P_{uu} + 0.5 \times P_{ud}}{1+0.06004/2}$$

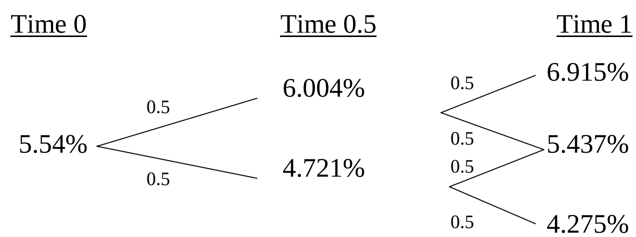
$$P_d = \frac{0.5 \times P_{ud} + 0.5 \times P_{dd}}{1+0.04721/2}$$

$$0.9222 = \frac{0.5 \times P_u + 0.5 \times P_d}{1+0.0554/2}$$

$$m_2 = 0.04222$$

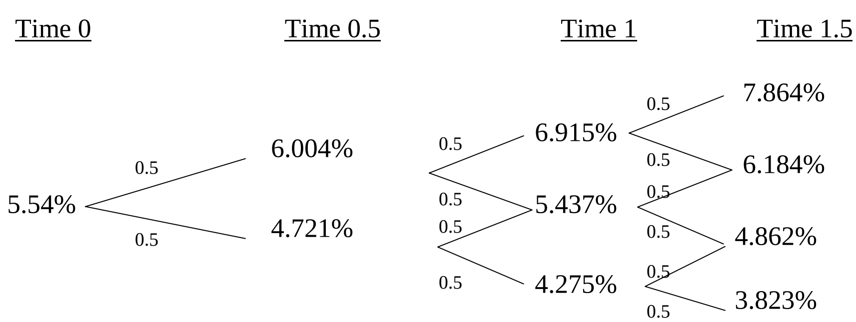
Resulting Interest Rate Tree

With $m2$=0.04222, the interest rate tree is now as follows:



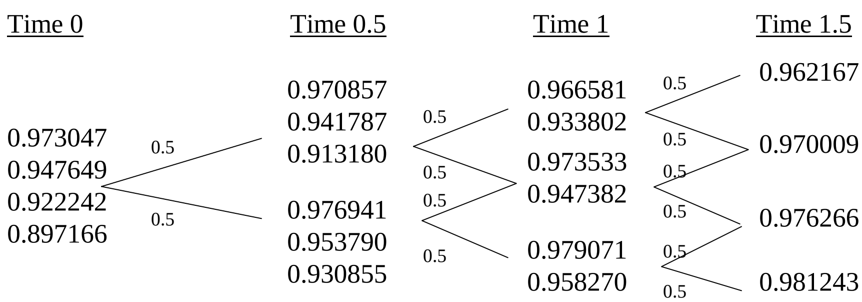
Extending the Interest Rate Tree

The tree can extended, as many periods as necessary by successively fitting drift terms to the prices of longer zeros. For example, to extend the tree to time 1.5, set $m3$=0.01686 to make the tree correctly price the 2-year zero (${}_0d_2$=0.8972).



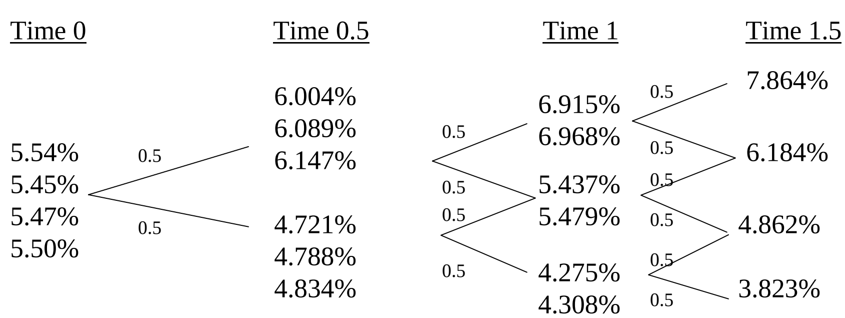
Resulting Zero Price Tree

At each node, the prevailing prices of outstanding zeros are listed, in ascending order of maturity. For instance, the price of a 1-year zero at time 0.5. state up, is 0.9418.



Resulting Tree of Term Structures

At each node, the prevailing discount rates of outstanding zeros are listed, in ascending order of maturity. For instance, the 1-year zero rate at time 1, state up-down, is 5.479%.



Limitations of This Model

- Only one volatility parameter
  - The model may not be able to fit the prices of options with different maturities simultaneously.
  - The Black-Derman-Toy model allows the proportional volatility parameter to vary over time to match prices of options with different maturities (allowing for a term structure of volatilities).
- Independent interest rate change over time.
  - Some feel that rates should be mean reverting. This would mean down moves would be more likely at higher interest rates..
  - The Black-Karasinski Model introduces mean reversion in the interest rate process.

Limitations of This Model…

- Only a One-Factor Model
  - Each period one factor (the short rate) determines the prices of all bonds.
  - This means that each period all bond prices move together. Their returns are perfectly correlated. There is no possibility that some bond yields could rise while others fall.
  - To allow for this possibility the model would require additional factors, or sources of uncertainly, which would expand the dimensions of the state-space. For example, in a two-factor model, each period you could move up or down and right or left, so there would be four possible future states.
  - Large investment banks and derivatives deals often have their own proprietary models.

Hedging and Measuring Interest Rate Risk

- The traditional measures of interest rate risk, duration and dollar duration, tell how sensitive a bond is to changes in interest and can be used to calibrate hedges between different bonds.
- We can use the interest rate model to develop a measure of interest rate risk similar to dollar duration that applies to all fixed income instruments, not just bonds with fixed cash flows.
- The remainder of this lecture introduces and relates two concepts:
  - Hedge ratios
  - Interest rate deltas

Hedge Ratios

- A net position is <u>hedged</u> (over one period) if its value next period is known with certainty (non-random).
- The <u>hedge ratio</u> between two instruments is the number of units of one asset necessary to hold to hedge a short position in one unit of the other asset.
- In particular, suppose you have a short position in asset 1 (say, a derivative) and you want to hedge it by going long in asset 2 (say, a plain bond). The hedge ratio is the number of units of asset 2 to buy to make your net position value riskless next period.

Hedge Ratios…

- Suppose the possible future value of asset 1 are B_{1u} and B_{1d}, and the possible future values of asset 2 are B_{2u} and B_{2d}.
- Let the <u>hedge ratio</u>, N_2, be the number of units of asset 2 to hold to hedge a short position in asset 1.

	Time 0.5 (up)	Time 0.5 (down)
Short position in asset 1	$-B_{1,u}$	$-B_{1,d}$
Long position in asset 2	$B_{2,u}$	$B_{2,d}$
Net position with N_2 units of asset 2.	$N_2B_{2,u} - B_{1,u}$	$N_2B_{2,d} - B_{1,d}$

To make the net position riskless at time 0.5, what should N_2 be?

It must make the up value equal the down value:

$$N_2B_{2,u} - B_{1,u} \equiv N_2B_{2,d} - B_{1,d}$$

$$\Rightarrow N_2 = \frac{B_{1,u} - B_{1,d}}{B_{2,u} - B_{2,d}}$$

Example

<u>Zeroes</u>

Time 0	Time 0.5 (up)	Time 0.5 (down)
	1	1
0.973047	0.970857	0.976941
0.947649	0.941787	0.953790
0.922242	0.913180	0.930855
0.897166		

The hedge ratio of the 1-year zero with respect to the 2-year zero is 0.344215:

$$\frac{0.970857 - 0.976941}{0.913180 - 0.930855} = 0.344215$$

To hedge a short position in \$1,000,000 par of 1-year zeroes using 2-year zeroes, how many 2-year zeroes must be bought? \$344,215 par.

In other words, for each \$1 par short in the 1-year zero, you must buy \$0.344215 par of the 2-year zero to make your net position hedged (at time 0.5).

Interest Rate Risk

- We can see from the hedge ratio formula that a measure of how risky an asset is how much variation there is in its value next period, or how much spread there is between its up and down values, B_u-B_d,:

Time 0.5

$$\text{Asset 1} \begin{cases} B_{1,u} \\ B_{1,d} \end{cases}$$

$$N_2 = \frac{B_{1,u} - B_{1,d}}{B_{2,u} - B_{2,d}}$$

Twice as much spread ($B_{1,u} - B_{1,d}$) would require twice as much hedging (N_2)

Interest Rate Delta

Instead of measuring an asset's interest rate risk by its spread, $B_u - B_d$, we introduce a measure which is more reminiscent of dollar duration, <u>interest rate delta</u>. The interest rate delta of an asset is just its spread divided by the difference between the two possible values of the short interest rate at time 0.5:

Time 0.5

$$\begin{matrix} \text{Assets} \\ \text{Future short rate} \end{matrix} \begin{cases} B_u \quad {}_{0.5}r_1^{\,u} \\ B_d \quad {}_{0.5}r_1^{\,d} \end{cases}$$

$$ir\Delta = -\frac{B_u - B_d}{{}_{0.5}r_1^{\,u} - {}_{0.5}r_1^{\,d}}$$

Remember: $\$dur \approx -\frac{\Delta\text{price}}{\Delta\text{rate}}$

So an asset's interest rate delta is like its dollar duration at time 0.5

Examples

An instrument's i. r. delta today is like its dollar duration one period from now.

Time 0		Time 0.5
	0.5	1 0.970857 0.941787 0.913180
0.973047 0.947649 0.922242 0.897166		
	0.5	1 0.976941 0.953790 0.930855

0.5- year zero:

$$ir\Delta = -\frac{1-1}{0.06004-0.04721} = 0$$

1- year zero:

$$ir\Delta = -\frac{0.970857-0.976941}{0.06004-0.04721} = 0.4742$$

1.5- year zero:

$$ir\Delta = -\frac{0.941787-0.953790}{0.06004-0.04721} = 0.9356$$

2- year zero:

$$ir\Delta = -\frac{0.913180-0.930855}{0.06004-0.04721} = 1.3776$$

Connection between Hedge Ratios and Interest Rate Deltas

The hedge ratio of asset 1 with respect to asset 2 is just the ratio of the interest rate delta of asset 1 to the interest rates delta of asset 2:

$$N_2 = \frac{B_{1,u} - B_{1,d}}{B_{2,u} - B_{2,d}} = \left(\frac{\dfrac{B_{1,u} - B_{1,d}}{\cancel{{}_{0.5}r_1^u - {}_{0.5}r_1^d}}}{\dfrac{B_{2,u} - B_{2,d}}{\cancel{{}_{0.5}r_1^u - {}_{0.5}r_1^d}}} \right) = \frac{ir\Delta_1}{ir\Delta_2}$$

Example: The hedge ratio of the 1-year zero with respect to the 2-year zero is the ratio of their interest rate deltas:

$$\frac{0.4742}{1.3776} = 0.34422$$