Equilibrium in a Pure Exchange Economy

1. Investor endowments, objectives, and preferences

2. Arrow-Debreu equilibrium

(a) Characterization of the sdf process

(b) Existence and uniqueness of equilibrium

(c) Representative agent

3. Security market equilibrium

4. Consumption CAPM

Selected Readings and References

Karatzas and Shreve, 1998, chapter 4.

Duffie, chapter 10.

Merton, R., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867-888.

Breeden, D., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *Journal of Financial Economics* 7, 265-296.

Karatzas, I., J.P. Lehoczky, and S. Shreve, 1990, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model, *Mathematics of Operations Research* 15, 80-128.

Summary of the Continuous-Time Financial Market

- Security prices satisfy $\frac{dS_{0,t}}{S_{0,t}} = r_t\,dt$ and $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t})\,dt + \sigma_{k,t} dB_t$.
- Given tight tr. strat. π_t and consumption c_t, portfolio value X_t satisfies the
  - WEE: $dX_t = r_t X_t\,dt + \pi_t(\mu_t - r_t 1)\,dt + \pi_t \sigma_t\,dB_t - c_t\,dt$.
- No arbitrage $\Rightarrow$ if $\pi_t \sigma_t = 0$ then $\pi_t(\mu_t - r_t 1) = 0 \Rightarrow \exists \theta_t$ s.t. $\sigma_t \theta_t = \mu_t - r_t 1$ $\Rightarrow dX_t = r_t X_t\,dt + \pi_t \sigma_t(\theta_t\,dt + dB_t) - c_t\,dt$.
- Under emm $\mathcal{P}^*$ given by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ where $Z_t = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds}$,
  - $B_t^* = B_t + \int_0^t \theta_s\,ds$ is Brownian motion.

  Let $\beta_t = e^{-\int_0^t r_s\,ds}$ and sdf process $M_t = \beta_t Z_t$. Then the WEE can also be written:
  - WEE*: $d\beta_t X_t + \beta_t c_t\,dt = \beta_t \pi_t \sigma_t\,dB_t^*$
  - WEE-M: $dM_t X_t + M_t c_t\,dt = M_t[\pi_t \sigma_t - \theta_t X_t]\,dB_t$
- So $X_t = \mathrm{E}_t^*\{\int_t^T \frac{\beta_u}{\beta_t} c_u\,du + \frac{\beta_T}{\beta_t} X_T\} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t} c_u\,du + \frac{M_T}{M_t} X_T\}$ if π is mtgale-gen.
- If σ is nonsingular, every c.plan (c, X_T) can be generated by a mtgale-gen. tr.strat.

Investors and Endowments

- Suppose there are m investors in the economy.
- Each investor j is endowed with an exogenous flow $e_{j,t}$ of the single, nonstorable, consumption good. Each $e_{j,t}$ is nonnegative and adapted on $[0, T]$.
- The *aggregate endowment*

$$e_t \equiv \sum_{j=1}^{m} e_{j,t} \tag{1}$$

is an Itô process described by

$$\frac{de_t}{e_t} = \mu_{e,t}\,dt + \sigma_{e,t}\,dB_t\ , \tag{2}$$

where μ_e and σ_e are adapted and bounded.

Investor Objectives, Preferences, and Optimal Consumption Plans

Each investor wants to maximize $\mathrm{E}\int_0^T U_j(c_{j,t},t)\,dt$ where each consumption plan $c_{j,t}$ is a nonnegative, adapted process satisfying $\mathrm{E}\int_0^T c_{j,t}\,dt < \infty$ and U_j is a utility function satisfying

$$U_j(c,t) = e^{-\rho t} u_j(c) \ , \tag{3}$$

where $u_j : (0,\infty) \to \mathcal{R}$ is C^3 and satisfies

- $u_j'(c) > 0$,
- $u_j''(c) < 0$, $\lim_{c\downarrow 0} u_j'(c) = \infty$,
- $\lim_{c\to\infty} u_j'(c) = 0$, and
- $\lim_{c\downarrow 0} \frac{u_j'''(c)}{(u_j''(c))^2}$ exists and is finite.

Thus, the nonnegativity constraint on consumption is nonbinding and there exists an imuf $I_j : (0,\infty)\times[0,T] \to (0,\infty)$ s.t. $I_j(U_j'(c,t),t) = c$ for every $c \in (0,\infty)$, where $U_j'(c,t) \equiv \frac{\partial U_j(c,t)}{\partial c}$. The imuf $I_j(y,t)$ is C^2 and strictly decreasing in y.

In a complete market in which investors could trade consumption plans at prices described by a sdf process M_t, each investor would trade so as to

$$\max_{c_j} \mathrm{E}\int_0^T U_j(c_{j,t},t)\,dt \text{ s.t. } \mathrm{E}\int_0^T M_t c_{j,t}\,dt \le \mathrm{E}\int_0^T M_t e_{j,t}\,dt \ . \tag{4}$$

From our results on optimal consumption, the solution would be

$$c_{j,t}^* = I_j(\lambda_j M_t, t) \ , \tag{5}$$

where the Lagrange multiplier λ_j solves

$$\mathrm{E}\int_0^T M_t I_j(\lambda_j M_t,t)\,dt = \mathrm{E}\int_0^T M_t e_{j,t}\,dt \ . \tag{6}$$

Arrow-Debreu Equilibrium

Definition 1 An *Arrow-Debreu equilibrium* (ADE) is a sdf M_t and allocations $c^*_{j,t}$, $j = 1, \ldots, m$, such that each $c^*_{j,t}$ solves investor j's optimization problem (4), and markets clear, i.e.,

$$\sum_{j=1}^{m} c^*_{j,t} = e_t \ , 0 \le t \le T \ . \tag{7}$$

Proposition 1 In an ADE, there exists $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{R}^m_{++}$ satisfying

$$\mathrm{E} \int_0^T M_t I_j(\lambda_j M_t, t)\, dt = \mathrm{E} \int_0^T M_t e_{j,t}\, dt \quad , \ \forall j = 1, \ldots, m \text{ and} \tag{8}$$

$$\sum_{j=1}^{m} I_j(\lambda_j M_t, t) = e_t \ , 0 \le t \le T \ . \tag{9}$$

Conversely, if there exist M_t and $\Lambda \in \mathcal{R}^m_{++}$ satisfying equations (8) and (9) then M is an equilibrium sdf. In either case, $c^*_{j,t} = I_j(\lambda_j M_t, t)$ are the optimal consumption plans for investors $j = 1, \ldots, m$.

Sketch of Proof The c^*_j are optimal and markets clear iff equations (8)-(9) hold.

- So the search for an Arrow-Debreu equilibrium reduces to a search for M and Λ satisfying equations (8) and (9).
- First use equation (9) to define M as a function of e and Λ as follows. Let $\Lambda \in \mathcal{R}^m_{++}$ be given. Then for each (ω, t) there exists a unique $M(\omega, t) \in (0, \infty)$ solving equation (9) (because the left-hand side is continuous and strictly decreasing in M, goes to ∞ as M goes to zero, and goes to 0 as M goes to ∞).
- Thus, define $\mathcal{M}(e_t, t; \Lambda)$ as the M_t that clears markets in equation (9), i.e.,

$$\sum_{j=1}^{m} I_j(\lambda_j \mathcal{M}(e_t, t; \Lambda), t) = e_t \ . \tag{10}$$

- Note that $\mathcal{M}(e_t, t; \Lambda/a) = a\mathcal{M}(e_t, t; \Lambda) \ \forall a > 0$.

Theorem 1 (Existence and Uniqueness of Equilibrium)
There exists $\Lambda \in \mathcal{R}^m_{++}$ that solves equations (8) and (9) with $M_t = \mathcal{M}(e_t, t; \Lambda)$. Moreover, if $-\frac{cU''_j(c,t)}{U'_j(c,t)} \le 1 \ \forall j = 1, \ldots, m$, then the equilibrium is unique in the sense that if $\Lambda' \in \mathcal{R}^m_{++}$ together with $\mathcal{M}'(e_t, t; \Lambda')$ are another solution, then $\mathcal{M}'(e_t, t; \Lambda') = a\mathcal{M}(e_t, t; \Lambda)$ and $\Lambda' = \Lambda/a$ for some positive constant a, and the allocations $c^*_{j,t} = I_j(\lambda_j \mathcal{M}(e_t, t; \Lambda), t) = I_j(\lambda'_j \mathcal{M}'(e_t, t; \Lambda'), t)$ are unique.

Construction of the "Representative Agent"

Define

$$U(c,t;\Lambda) \equiv \max_{(c_1,\ldots,c_m)\in\mathcal{R}^m} \sum_{j=1}^{m} U_j(c_j,t)/\lambda_j \text{ s.t. } \sum_{j=1}^{m} c_j \le c \ . \tag{11}$$

Let $(\widehat{c}_1(c),\ldots,\widehat{c}_m(c))$ denote the solution to the maximization problem above, where the dependence of $\widehat{c}$ on t and Λ is suppressed for brevity.

Proposition 2 The function $U(\cdot,t;\Lambda) : (0,\infty) \to \mathcal{R}$ has the following properties.

$$U(\cdot,t;\Lambda) \in C^2 \ , \tag{12}$$

$$U'(c,t;\Lambda) = U_j'(\widehat{c}_j(c),t)/\lambda_j \ \forall j = 1,\ldots,m \ , \tag{13}$$

$$\lim_{c\downarrow 0} U'(c,t;\Lambda) = \infty \ , \tag{14}$$

$$\lim_{c\to\infty} U'(c,t;\Lambda) = 0 \ , \tag{15}$$

$$U''(c,t;\Lambda) < 0 \ . \tag{16}$$

Proof Homework

Theorem 2

$$U'(c,t;\Lambda) = \mathcal{M}(c,t;\Lambda) \quad \text{and} \tag{17}$$

$$\widehat{c}_j(c) = I_j(\lambda_j \mathcal{M}(c,t;\Lambda),t) \ . \tag{18}$$

Proof Let $c_j^* = I_j(\lambda_j\mathcal{M}(c,t;\Lambda),t)$. Note that $\sum_{j=1}^m c_j^* = c$, by construction of $\mathcal{M}(c,t;\Lambda)$. Now suppose $(c_1,\ldots,c_m)$ is a different feasible choice for the maximization problem in equation (11). Then $\sum_{j=1}^m c_j \le c$ and

$$\sum_{j=1}^{m} U_j(c_j,t)/\lambda_j < \sum_{j=1}^{m} [U_j(c_j^*,t) + (c_j - c_j^*)U_j'(c_j^*,t)]/\lambda_j \tag{19}$$

$$= \sum_{j=1}^{m} [U_j(c_j^*,t) + (c_j - c_j^*)U_j'(I_j(\lambda_j\mathcal{M}(c,t;\Lambda),t),t)]/\lambda_j \tag{20}$$

$$= \sum_{j=1}^{m} U_j(c_j^*,t)/\lambda_j + \mathcal{M}(c,t;\Lambda)\sum_{j=1}^{m}(c_j - c_j^*) \tag{21}$$

$$\le \sum_{j=1}^{m} U_j(c_j^*,t)/\lambda_j \ . \tag{22}$$

Thus $\hat{c}_j(c) = c_j^* = I_j(\lambda_j \mathcal{M}(c,t;\Lambda),t)$, $j = 1,\ldots,m$, so

$$U'(c,t;\Lambda) = U_j'(I_j(\lambda_j \mathcal{M}(c,t;\Lambda),t),t)/\lambda_j \quad (23)$$
$$= \mathcal{M}(c,t;\Lambda)\ .\ \square \quad (24)$$

- Now fix Λ at an equilibrium value.
- Then we have $U'(e_t,t) = \mathcal{M}(e_t,t;\Lambda) = M_t$.
- I.e., we can interpret the equilibrium sdf as the marginal utility of a "representative agent" consuming the aggregate endowment.

Security Market Equilibrium

- The ADE can be implemented in a market where investors trade securities dynamically, rather than making one-time trades of consumption bundles.
- To construct a complete, correctly priced securities market, we first deduce r_t and θ_t from $M_t = \mathcal{M}(e_t,t;\Lambda) = U'(e_t,t)$, then specify a nonsingular $d \times d$-matrix-valued volatility process σ_t and set $\mu_t = r_t 1 + \sigma_t\theta_t$.
- Deduce the securities-market clearing interest rate r_t and mpr θ_t from the ADE sdf by recognizing that

$$M_t = e^{-\int_0^t r_s\,ds - \int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds} = U'(e_t,t) \quad (25)$$

and matching their drift and diffusion terms from Itô's lemma:

$$\frac{dM_t}{M_t} = -r_t\,dt - \theta_t'\,dB_t = \frac{dU'(e_t,t)}{U'(e_t,t)} \quad (26)$$
$$= \frac{1}{U'(e_t,t)}[U''(e_t,t)e_t\mu_{e,t} + \frac{\partial U'(e_t,t)}{\partial t} + \frac{1}{2}U'''(e_t,t)(e_t)^2|\sigma_{e,t}|^2]\,dt$$
$$+ \frac{1}{U'(e_t,t)}U''(e_t,t)e_t\sigma_{e,t}\,dB_t\ . \quad (27)$$

- Therefore, letting

$$R_t \equiv -\frac{U''(e_t,t)e_t}{U'(e_t,t)} \qquad (28)$$

denote the relative risk aversion of the representative agent,
- the equilibrium interest rate is

$$r_t = R_t\mu_{e,t} - \frac{1}{U'(e_t,t)}[\frac{\partial U'(e_t,t)}{\partial t} + \frac{1}{2}U'''(e_t,t)(e_t)^2|\sigma_{e,t}|^2] \qquad (29)$$

- and the equilibrium mpr is

$$\theta_t = R_t\sigma'_{e,t} \; . \qquad (30)$$

- To construct security prices, let σ_t be a nonsingular, $d \times d$-matrix-valued volatility process, let $\mu_t = r_t 1 + \sigma_t\theta_t$, and let initial prices $S_{k,0}$, $k = 0, 1, \ldots, d$, be given. Let $S_{0,t} = S_{0,0}e^{\int_0^t r_s\,ds}$ and let $S_{k,t}$ be given by

$$\frac{dS_{k,t}}{S_{k,t}} = \mu_{k,t}\,dt + \sigma_{k,t}\,dB_t \;,\; k = 1, \ldots, d \; . \qquad (31)$$

- These securities are in zero net supply ("side bets").

Trading Strategies, Investor Wealth Evolution, and Securities Market Clearing

- Finally, suppose π is a trading strategy satisfying the usual integrability conditions. The wealth process generated by π is $X_{j,t}$ such that

$$dX_{j,t} = r_t X_{j,t}\,dt + (e_{j,t} - c_{j,t})\,dt + \pi_t\sigma_t(\theta_t\,dt + dB_t) \qquad (32)$$

- The consumption-portfolio plan (c_j, π^j) is feasible for investor j if $X_{j,t}$ satisfies

$$X_{j,t} + \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t}e_{j,u}\,du\} \geq 0 \;,\; 0 \leq t \leq T \; . \qquad (33)$$

- If c_j is a consumption plan that satisfies

$$\mathrm{E}\int_0^T M_t c_{j,t}\,dt = \mathrm{E}\int_0^T M_t e_{j,t}\,dt \; , \qquad (34)$$

then there exists a trading strategy π^j s.t. (c_j, π^j) is feasible for investor j, and the corresponding wealth process is

$$X_{j,t} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t}[c_{j,u} - e_{j,u}]\,du\} \; . \qquad (35)$$

Definition 2 The securities market is in $equilibrium$ if

1. investors' consumption plans and trading strategies are optimal,

2. the commodities market clears, i.e., $\sum_{j=1}^{m} c^*_{j,t} = e_t$,

3. security markets clear, i.e.,

$$\sum_{j=1}^{m} \pi_t^{j*} = 0 \text{ and } \sum_{j=1}^{m} \pi_{0,t}^{j*} \equiv \sum_{j=1}^{m} (X^*_{j,t} - \pi_t^{j*} 1) = 0 \ , \ 0 \le t \le T \ . \quad (36)$$

Theorem 3 (Existence of a Securities Market Equilibrium) Let Λ and M be as in an Arrow-Debreu equilibrium and let σ_t be a given nonsingular, $d \times d$-matrix-valued volatility process. Define r_t, θ_t, μ_t and security prices as above. Then the securities market is in equilibrium.

Proof For homework, prove that security markets clear.

Consumption CAPM

Note that in equilibrium, security k's instantaneous excess return is

$$\mu_{k,t} - r_t = \sigma_{k,t}\theta_t = R_t \sigma_{k,t}\sigma'_{e,t} = R_t \text{cov}_{ke,t} \ . \quad (37)$$

In other words, the excess expected return on each security is equal to the representative agent coefficient of relative risk aversion times the security's instantaneous covariance with aggregate endowment (consumption).

Let $\sigma^*_t = \sigma_{e,t}$ and let

$$\mu^*_t = r_t + \sigma^*_t \theta_t = r_t + R_t \sigma_{e,t}\sigma'_{e,t} \quad (38)$$

be the volatility vector and equilibrium appreciation rate on a security that is perfectly correlated with aggregate consumption. Then we have

$$\mu_{k,t} - r_t = \beta_{k,t}(\mu^*_t - r_t) \ , \quad (39)$$

where $\beta_{k,t} = \frac{\sigma_{k,t}\sigma^{*\prime}_t}{\sigma^*_t \sigma^{*\prime}_t}$ is security k's "consumption beta."

Problem

Consider an economy with identical agents each deriving expected utility from consumption plan $\{c_t\}$ equal to

$$\mathrm{E}\int_0^T e^{-\rho t}\frac{c_t^\gamma}{\gamma}\,dt\ ,\ \gamma \in\ (0,1)\ .$$

The aggregate endowment process, e_t, satisfies

$$\frac{de_t}{e_t} = \mu_{e,t}\,dt + \sigma_{e,t}\,dB_t$$

where B_t is d-dimensional Brownian motion and μ_e and σ_e are bounded, progressively measurable processes taking values in $\mathcal{R}$ and $\mathcal{R}^d$, respectively.

a) Determine the interest rate process r_t in the Arrow-Debreu equilibrium for this economy.

b) Suppose $\widehat{r}_t$ is an arbitrary bounded, nonnegative, progressively measurable process. Construct an equilibrium in which the interest rate is $\widehat{r}_t$.

c) Suppose $\widehat{r}_t$ above is an Itô process with $d\widehat{r}_t = \mu_{r,t}\,dt + \sigma_{r,t}\,dB_t$. What is the drift of $\widehat{r}_t$ under the martingale measure $\mathcal{P}^*$ in the equilibrium you constructed?