Introduction to Production Models

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Selected Readings and References

These notes are taken directly from Back, chapter 22.

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Capital Stock, Investment, and Operating Cash Flow in Discrete Time

- Let K denote a firm's capital stock and I denote its capital expenditures, or real investment, which could be negative, reflecting asset sales.
- Assume the firm's capital stock evolves according to

$$K_{t+1} = f(K_t, I_t) \tag{1}$$

for some strictly increasing function f.

- For example, a simple model with depreciation and costless capital adjustment is

$$f(K_t, I_t) = \delta K + I \text{ , where } 0 < \delta \leq 1 \text{ .} \tag{2}$$

- A model with quadratic adjustment costs is

$$f(K_t, I_t) = \delta K + \frac{\sqrt{1 + 4\gamma I} - 1}{2\gamma}, \text{ so } I_t = K_{t+1} - \delta K_t + \gamma(K_{t+1} - \delta K_t)^2 \text{ .} \tag{3}$$

- Suppose the firm's operating cash flow π, i.e., revenue minus production costs, is a function of its capital K_t and a vector of state variables X_t.
- For example, if the firm is a price-taker and has constant returns to scale in production,

$$\pi(X_t, K_t) = X_t K_t \tag{4}$$

for a univariate X_t.

- Then net firm cash flow at date t is $\pi(X_t, K_t) - I_t$.
- Suppose the market is complete so there is a unique sdf process M_t.
- Then the firm's shareholders should agree to choose investment each date to maximize firm market value:

$$V_t(K_t) = \max_{I_t} \mathrm{E}\{\sum_{u=t}^{\infty} \frac{M_u}{M_t}[\pi(X_u, K_u) - I_u]\} \text{ .} \tag{5}$$

Marginal q

- Assume f and $\pi(X, \cdot)$ are continuously differentiable.
- The Bellman equation for the shareholder's problem is

$$V_t(K) = \max_I[\pi(X_u, K_u) - I_u] + \mathrm{E}_t\{\frac{M_{t+1}}{M_t}V_{t+1}(f(K, I))\}] \ , \tag{6}$$

- and the first-order condition is

$$-1 + \mathrm{E}_t\{\frac{M_{t+1}}{M_t}V'_{t+1}f_I\} = 0 \ . \tag{7}$$

- So the optimal investment policy $(I_0, I_1, \dots)$ sets

$$\mathrm{E}_t\{\frac{M_{t+1}}{M_t}q_{t+1}\} = 1 \ \forall t \ , \text{ where} \tag{8}$$

$$q_t \equiv V'_t(K_t)f_I(K_{t-1}, I_{t-1}) \tag{9}$$

- The variable q_t is the marginal return on investment at date $t-1$. It is called $marginal\ q$, as opposed to $average\ q$ defined by Tobin (1969) as the firm's maket value divided by the replacement cost of its capital.

Proposition 1 When investments are optimally chosen, marginal q depends on the operating cash flow and capital accumulation functions π and f as follows:

$$q_t = f_I(K_{t-1}, I_{t-1})[\pi_K(X_t, K_t) + \frac{f_K(K_t, I_t)}{f_I(K_t, I_t)}] \ . \tag{10}$$

Proof Suppose the optimal investment for the Bellman equation (6) is $I_t = \mathcal{I}_t(K)$, where $\mathcal{I}_t$ is differentiable. Then

$$V_t(K) = \pi(X_t, K) - \mathcal{I}_t(K) + \mathrm{E}_t\{\frac{M_{t+1}}{M_t}V_{t+1}(f(K, \mathcal{I}_t(K)))\} \tag{11}$$

$$\Rightarrow V'_t(K) = \pi_K(X_t, K) - \mathcal{I}'_t(K) + \mathrm{E}_t\{\frac{M_{t+1}}{M_t}V'_{t+1}(f(K, \mathcal{I}_t(K)))\} * \\ [f_K(K, \mathcal{I}_t(K)) + f_I(K, \mathcal{I}_t(K))\mathcal{I}'_t(K)] \ . \tag{12}$$

Using equation (8), this simplifies to

$$V'_t(K_t) = \pi_K(X_t, K_t) + \frac{f_K(K_t, I_t)}{f_I(K_t, I_t)} \ , \tag{13}$$

which together with equation (9) gives the result.

Envelope Theorem for the Optimal Dynamic Investment Policy

To interpret Proposition 1, consider an infinitesimal increase in I_{t-1} offset by a reduction in I_t that leaves the optimized future capital stock $(K_{t+1}, K_{t+2}, \dots)$ unchanged. Then $f(f(K_{t-1}, I_{t-1}), I_t)$ is constant, so totally differentiating gives

$$0 = f_K(K_t, I_t) f_I(K_{t-1}, I_{t-1})\, dI_{t-1} + f_I(K_t, I_t)\, dI_t \tag{14}$$

$$\Rightarrow dI_t = -f_I(K_{t-1}, I_{t-1}) \frac{f_K(K_t, I_t)}{f_I(K_t, I_t)}\, dI_{t-1} \tag{15}$$

Then the derivative of net cash flow at date t given the change in I_{t-1} is

$$\frac{d(\pi(X_t, K_t) - I_t)}{dI_{t-1}} = \pi_K(X_t, K_t) f_I(K_{t-1}, I_{t-1}) - \frac{dI_t}{dI_{t-1}} \tag{16}$$

$$= f_I(K_{t-1}, I_{t-1})[\pi_K(X_t, K_t) + \frac{f_K(K_t, I_t)}{f_I(K_t, I_t)}] \tag{17}$$

$$= q_t\ . \tag{18}$$

Thus, under the optimal policy, the marginal return on investment at date $t-1$ is equal to the marginal date t net operating cash flow, holding all else equal.

Special Case with Costless Capital Adjustment and Constant Returns to Scale

- Suppose $f(K_t, I_t) = \delta K_t + I_t$ and $\pi(X_t, K_t) = X_t K_t$.
- Then $q_t = X_t + \delta$, and the FOC for optimal investment implies $\mathrm{E}\{\frac{M_{t+1}}{M_t}(X_t + \delta)\}$.
- This is independent of the firm's investment decision, so it is a restriction on the equilibrium sdf, under which any investment policy is optimal.
- One version of the equilibrium is one in which individuals hold equity in the firm.
- Another is that individuals invest directly in the CRS production technology, which is just another asset, with return q_t.
- The supply of this asset is *perfectly elastic* – it can be chosen without altering q_t.
- Cox ingersoll Ross (1985a) model an economy with firms that have CRS and no capital adjustment costs. Such a firm is now sometimes called a Cox-Ingersoll-Ross technology.

Costly Reversibility

- The previous section assumed f is differentiable in I at $I = 0$, but this may not be realistic.
- Consider the following example:

$$f(K, I) = \delta K + I \text{ if } I \geq 0 \quad (19)$$
$$\delta K + I/\alpha \text{ if } I < 0 \; , \quad (20)$$

where the capital resale price is $\alpha \leq 1$. I.e., capital expenditures/revenues are

$$I_t = K_{t+1} - \delta K_t \text{ if } K_{t+1} - K_t \geq 0 \quad (21)$$
$$\alpha(K_{t+1} - \delta K_t) \text{ if } K_{t+1} - K_t < 0 \; . \quad (22)$$

- In the case $\alpha < 1$, f is not differentiable at $I = 0$.
- The case of irreversible investment, in which f is undefined for $I < 0$, corresponds to the limit as $\alpha \to 0$.
- A generalized FOC for optimal investment still holds if f is concave and V_t is C^1.

- For all K and I, define

$$\partial_I f(K, I) = \{a : \forall I' \; a \geq \frac{f(K, I') - f(K, I)}{I' - I}\} \; . \quad (23)$$

- The previous FOCs (8) and (9) implied

$$1/\mathrm{E}_t\{\frac{M_{t+1}}{M_t} V'_{t+1}(K_{t+1})\} = f_I(K_t, I_t) \; . \quad (24)$$

- In the current setting, the necessary condition for optimal investment becomes

$$1/\mathrm{E}_t\{\frac{M_{t+1}}{M_t} V'_{t+1}(K_{t+1})\} \in \partial_I f(K_t, I_t) \; , \quad (25)$$

so that one of the lines tangent to the objective function in the Bellman equation (6) at the chosen investment level has a slope of zero, as is necessary at a maximum.
- An additional complication arises if there are fixed costs of adjustment, as well as variable costs, say ξ^+ for positive investment and ξ^- for asset sales. Then f may be neither continuous nor concave in I. In this case the optimum can be computed by optimizing separately over $I < 0$, $I = 0$, and $I > 0$, and then choosing the best of these.

Irreversibility of Investment and Optimal Exercise of Growth Options

- When it is costly to reverse investment, investing is equivalent to exercising an option. This is a so-called real option, as opposed to a financial option.
- The following continuous-time model illustrates the connection between the optimal investment problem and the solutions to a family of American option problems.
- Let I_t now denote cumulative investment from 0 to t, assumed to be increasing.
- Assume there is no depreciation of capital, so $dK_t = dI_t$, and π is concave in K.
- The firm chooses an investment policy to maximize

$$\mathrm{E} \int_0^\infty M_t[\pi(X_t, K_t)\, dt - dI_t] \ . \tag{26}$$

Proposition 2 Let $\tau_k = \inf\{t : K_t > k\}$. Then the firm's objective (26) is

$$\mathrm{E} \int_0^\infty M_t \pi(X_t, K_0)\, dt + \int_{K_0}^\infty \mathrm{E}\{M_{\tau_k}(\int_{\tau_k}^\infty \frac{M_t}{M_{\tau_k}} \pi_K(X_t, k)\, dt - 1)\}\, dk \ . \tag{27}$$

- The first term is the value of the firm's assets in place.
- The second term is the value of the firm's growth options.

Proof Since

$$\pi(X_t, K_t) = \pi(X_t, K_0) + \int_{K_0}^{K_t} \pi_K(X_t, k)\, dk \ , \tag{28}$$

the firm's objective (26) can be written as

$$\mathrm{E} \int_0^\infty M_t \pi(X_t, K_0)\, dt + \mathrm{E} \int_0^\infty M_t \int_{K_0}^{K_t} \pi_K(X_t, k)\, dk\, dt - \mathrm{E} \int_0^\infty M_t dI_t \ . \tag{29}$$

Changing the order of integration, the second term becomes

$$\mathrm{E} \int_{K_0}^\infty \int_{\tau_k}^\infty M_t \pi_K(X_t, k)\, dt\, dk \ . \tag{30}$$

Finally, the third term is

$$-\mathrm{E} \int_0^\infty M_t dI_t = -\mathrm{E} \int_0^\infty M_t dK_t = -\mathrm{E} \int_{K_0}^\infty M_{\tau_k} dk \tag{31}$$

using a change-of-variable formula for Lebesgue-Stieltjes integrals.

- The firm's optimal investment policy maximizes the value of its growth options.
- For each $k > K_0$, consider the problem

$$\max_{\tau} \mathrm{E}\{M_\tau(\int_\tau^\infty \frac{M_t}{M_\tau}\pi_K(X_t, k)\, dt - 1)\} \tag{32}$$

  where the maximization is over stopping times τ. The objective function above is the integrand of the firm's integral of growth options in equation (27).
- Define

$$S_t(k) = \mathrm{E}_t \int_t^\infty \frac{M_u}{M_t}\pi_K(X_u, k)\, du \ , \tag{33}$$

  which is the value of an asset paying the marginal operating cash flow $\pi_K(X_u, k)$ at each date $u \geq t$. Then the optimization (32) is

$$\max_{\tau} \mathrm{E}\{M_\tau(S_\tau(k) - 1)\} \ , \tag{34}$$

  which is the value of a perpetual American call on this asset with strike equal to 1.

- Define the continuation value of this option if not yet exercised at time t as

$$J_t(k) = \max_{\tau \geq t} \mathrm{E}_t\{\frac{M_\tau}{M_t}(S_\tau(k) - 1)\} \ . \tag{35}$$

- From American option theory, an optimal exercise time for this option is

$$\phi_k = \inf\{t : J_t(k) = S_t(k) - 1\} \ . \tag{36}$$

- If K_t is a capital process such that $\tau_k \equiv \inf\{t : K_t > k\} = \phi_k$ for every $k \geq K_0$, then K_t is an optimal capital process.
- Marginal q is $q_t(k) = S_t(k) - J_t(k)$, because given $K_t = k$, making a small investment at time t generates incremental cash flow worth $S_t(k)$ but entails giving up the option to invest, which is worth $J_t(k)$.
- Optimal exercise of the investment option implies $q_t(k) = S_t(k) - J_t(k) = 1$ when investment is made, i.e., the marginal value of investment equals its marginal cost.

Irreversibility of Investment and Perfect Competition

- Suppose firms have CRS in operations, i.e., $\pi(X_t, K_t) = X_t K_t$. Then

$$S_t(k) \equiv \mathrm{E}_t \int_t^\infty \frac{M_u}{M_t} \pi_K(X_u, k)\, du = \mathrm{E}_t \int_t^\infty \frac{M_u}{M_t} X_u \, du \equiv S_t \;\forall k \geq K_0 \,. \tag{37}$$

- Suppose there are infinitely many potential firms with this production technology and no barriers to entry in the market.
- Let $\widehat{Q}_t$ denote industry output and assume $X_t = h(Y_t, \widehat{Q}_t)$ for some stochastic process Y and function h that is decreasing in $\widehat{Q}$.
- It is a Nash equilibrium for firms to exercise the investment option whenever it reaches the money, and given that potential entrants play this strategy, it is fruitless to defer exercise until it becomes more valuable. Therefore, growth options never get strictly in the money in equilibrium, and thus $J_t = 0$.
- It follows that marginal q equals S_t and the value of each firm is the just value of its assets in place, $K_t S_t = K_t q_t$, and so marginal q equals average q in this model.

Irreversibility of Investment and Firm Risk

- Irreversibility of investment increases firm risk. Consider the CRS example above.
- Let $V_t = K_t q_t$ be the market value of a given firm. Total firm return is

$$\frac{\pi(X_t, K_t)dt - dI_t}{V_t} + \frac{dV_t}{V_t} \,. \tag{38}$$

- Investment occurs only when $q_t = 1$, in which case $K_t = V_t$, and thus

$$\frac{dI_t}{V_t} = \frac{dK_t}{V_t} = \frac{dK_t}{K_t} \,. \tag{39}$$

- At the same time, since $V_t = K_t q_t$ and K_t is increasing so $(dK)(dq) = 0$,

$$\frac{dV_t}{V_t} = \frac{dK_t}{K_t} + \frac{dq_t}{q_t} \,. \tag{40}$$

- Combining these three equations shows that total firm return is

$$\frac{\pi(X_t, K_t)dt - dI_t}{V_t} + \frac{dV_t}{V_t} = \frac{\pi(X_t, K_t)}{V_t} dt + \frac{dq_t}{q_t} \,. \tag{41}$$

- If investment were costlessly reversible, then industry capital would continuously adjust so that $q_t \equiv 1$, firm market value would equal book value, and returns on firms would be locally riskless, equal to the riskless rate in equilibrium.
- Here, however, irreversibility of investment introduces the additional risk of a fluctuating market-to-book ratio q_t.
- Berk, Green, and Naik (1999) present a model of optimal investment in the presence of growth options such that in equilibrium, conditional expected returns vary with firm book-to-market and size.