NYU Stern
Financial Theory IV
Continuous-Time Finance

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Course Outline

1. The continuous-time financial market, stochastic discount factors, martingales

2. European contingent claims pricing, options, futures

3. Term structure models

4. American options and dynamic corporate finance

5. Optimal consumption and portfolio choice

6. Equilibrium in a pure exchange economy, consumption CAPM

7. Exam - in class - closed-note, closed-book

Recommended Books and References

Back, K., *Asset Pricing and Portfolio Choice Theory*, Oxford University Press, 2010.

Duffie, D., *Dynamic Asset Pricing Theory*, Princeton University Press, 2001.

Karatzas, I. and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 1991.

Karatzas, I. and S. E. Shreve, *Methods of Mathematical Finance*, Springer, 1998.

Merton, R., *Continuous-Time Finance*, Blackwell, 1990.

Shreve, S. E., *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, 2004.

Arbitrage, martingales, and stochastic discount factors

1. Consumption space – random variables in $L^p(\mathcal{P})$

2. Preferences – strictly monotone, convex, lower semi-continuous

3. One-period market for payoffs

(a) Marketed payoffs

(b) Prices – positive linear functionals

(c) Arbitrage opportunities

(d) Viability of the price system

(e) Stochastic discount factors

4. Securities market with multiple trading dates

(a) Security prices – right-continuous stochastic processes

(b) Trading strategies – simple, self-financing, tight

(c) Equivalent martingale measures

(d) Dynamic market completeness

Readings and References

Duffie, chapter 6.

Harrison, J., and D. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory*, 20, 381-408.

Harrison, J., and S. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Their Applications*, 11, 215-260.

Dybvig, P, and C. Huang, 1989, Nonnegative wealth, absence of arbitrage, and feasible consumption plans, *Review of Financial Studies* 1, 377-401.

Back, K. and S. Pliska, 1991, On the fundamental theorem of asset pricing with an infinite state space, *Journal of Mathematical Economics* 20, 1-18.

Cox, J., and S. Ross, 1976, The valuation of options for alternative stochastic processes, *Journal of Financial Economics*, 3, 145-166.

Vasicek, O., 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics*, 5, 177-188.

Overview – When Can Prices be Represented with a SDF/EMM?

- A stochastic discount factor (sdf) is a random variable m such that for every marketed payoff P_T with price P_0, the price P_0 can be represented as

$$P_0 = \mathrm{E}\{mP_T\} \ . \tag{1}$$

- A risk-neutral pricing measure, or equivalent martingale measure (emm), associated with a "riskless" numeraire asset with price S is a probability measure $\mathcal{P}^*$ such that for every marketed payoff P_T with price P_0,

$$P_0 = \mathrm{E}^*\{\frac{S_0}{S_T}P_T\} \ . \tag{2}$$

- In a given asset market, there is typically a one-to-one correspondence between sdf's and emm's given by

$$m = \frac{S_0}{S_T}\frac{d\mathcal{P}^*}{d\mathcal{P}} \ , \tag{3}$$

where $\frac{d\mathcal{P}^*}{d\mathcal{P}}$ is the so-called Radon-Nikodym derivative of the emm $\mathcal{P}^*$ w.r.t. the true probability measure $\mathcal{P}$.

- In a securities market with multiple trading dates, these relations expand to

$$P_t = \mathrm{E}_t\{\frac{M_{t+1}}{M_t}P_{t+1}\} = \mathrm{E}_t^*\{\frac{S_t}{S_{t+1}}P_{t+1}\} \ , \tag{4}$$

(ignoring dividends) where it turns out this sdf process M_t can be written as

$$M_t = \mathrm{E}_t\{\frac{d\mathcal{P}^*}{d\mathcal{P}}\}\frac{S_0}{S_t} = \mathrm{E}_t\{S_T m\}/S_t \ . \tag{5}$$

- In other words, $M_t P_t$ is a $\mathcal{P}$-martingale and P_t/S_t is a $\mathcal{P}^*$-martingale.
- The sdf and emm representations of prices provide important insight into the structure of asset prices and powerful computational machinery for modeling.
- This lecture illustrates the generality of this representation, sets up the basic financial market model with discrete trading dates, and develops the key results.

Simple Construction of a SDF from FOC of Portfolio Optimization

- Suppose the space of marketed claims is spanned by a finite number of securities with time 0 prices $p = (p_1, \ldots, p_n)$ and time T payoffs $x = (x_1, \ldots, x_n)$, and an investor with preferences described by a strictly increasing, strictly concave, differentiable utility function u is able to find an optimal portfolio, i.e., a row vector of security holdings $N^* = (N_1^*, \ldots, N_n^*)$.
- In particular, the investor chooses security holdings $N = (N_1, \ldots, N_n)$ to

$$\max_N \mathrm{E}u(c_T) = \mathrm{E}u(e + Nx) \text{ s.t. } Np \leq 0. \tag{6}$$

  The first-order condition for the investor's optimal holding N_k is $\mathrm{E}\{u'(c_T)x_k\} = \lambda p_k$, which implies

$$p_k = \mathrm{E}\{\frac{u'(c_T)}{\lambda}x_k\} \tag{7}$$

- Then $m = \frac{u'(e_T + N^*x)}{\lambda}$ is a sdf for this market.

- If the market is incomplete, there could be different investors with different utility functions, which would produce different sdfs, but the different sdfs would generate the same prices for the marketed securities.
- Introducing a previously unspanned payoff to the market would in general lead to re-optimization and change the prices of all assets.

Formal Building Blocks of the Economy for More General Construction

- Probabilistic Setting
  - Finite time horizon $[0, T]$.
  - Probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t=0}^T$.
  - Each $\omega \in \Omega$ is a complete description of what happens from time 0 to T.
  - $\mathcal{P}$ is the subjective probability measure believed by people in the economy.
  - Each $\mathcal{F}_t$ represents information set at time t. Formally, it is a special collection of events, or subsets A of Ω, called a sigma-field.
  - Each $A \in \mathcal{F}_t$ represents an event that is measurable, or distinguishable, at time t, i.e., you know whether or not $\omega \in A$ at time t.
  - $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}_T = \mathcal{F}$, i.e., people don't forget, and all there is to know is revealed by time T.
  - A random variable X is a real-valued function on Ω that is measurable w.r.t $\mathcal{F}$ (its value is known by time T).
  - A reason to be precise about "measurability" w.r.t., say, $\mathcal{F}_t$, is, e.g., to be precise about statements like "the trading strategy doesn't look ahead."
  - Notational short-hand: $\mathrm{E}_t\{X\}$ means $\mathrm{E}\{X|\mathcal{F}_t\}$

- Consumption
  - There is a single consumption good consumed only at time T.
  - A *consumption plan* $x(\omega)$ is an $\mathcal{F}$-measurable random variable.
  - The *consumption space* $\mathcal{C}$ is $L^p(\mathcal{P})$ for some $p \in [1, \infty)$, the complete, normed vector space (Banach space) of random variables with finite norm $||X||_p \equiv (E(||X||^p)^{1/p}$

- Preferences
  - People are represented by their preferences $\succeq$ on $\mathcal{C}$. Without loss of generality, $\succeq$ represents preferences for time T net trades x (net of endowment $\bar{x}$).
  - Assume these preferences satisfy *lower semi-continuity*, *strict monotonicity*, and *convexity*, which means the sets $\{x \in \mathcal{C} : x \succeq \hat{x}\}$ are convex $\forall \hat{x} \in \mathcal{C}$.
  - For example, a preference relation defined by $x \succeq y \iff \mathrm{E}U(x) \geq \mathrm{E}U(y)$ where U is concave, strictly increasing, and grows at no more than quadratic rate satisfies these conditions.

The One-Period Market

- Let $\mathcal{M} \subset \mathcal{C}$ be the subspace of marketed consumption plans.
- Let p_m be the price functional on $\mathcal{M}$.
- Assume $\mathcal{M}$ is a linear subspace and p_m is a linear functional. I.e., the price of a portfolio is the sum of the prices of its pieces, there are no transaction costs, short sale constraints, etc.
- Suppose there exists $\widehat{x} > 0$ a.s. in $\mathcal{M}$.

Definition 1 An *arbitrage opportunity* is an $x \in \mathcal{M}$ s.t. $x \geq 0, \mathcal{P}\{x > 0\} > 0$, and $p_m(x) \leq 0$.

- No arbitrage implies p_m is a strictly positive linear functional.

Definition 2 The price system $(\mathcal{M}, p_m)$ is *viable* if $\exists$ preferences $\succeq$ and net trade $x^* \in \mathcal{M}$ such that $p_m(x^*) \leq 0$ and $x^* \succeq x$ for every $x \in \mathcal{M}$ such that $p_m(x) \leq 0$.

Definition 3 A *stochastic discount factor* is a random variable $m > 0$ such that

$$p_m(x) = \mathrm{E}\{mx\} \;\forall\; x \in \mathcal{M} \;. \tag{8}$$

Definition 4 The market is *complete* if $\mathcal{M} = \mathcal{C}$.

Lemma 1 A positive linear functional on $L^p(\mu)$ is continuous.

Riesz Representation Theorem Let ϕ be a continuous linear functional on $L^p(\mu)$, where $p \in [1, \infty)$. Then there exists a unique $Y \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, s.t.

$$\phi(X) = \int_{\Omega} X(\omega)Y(\omega)d\mu(\omega) \;\; \forall X \in L^p(\mu) \;. \tag{9}$$

If μ is a probability measure $\mathcal{P}$, this can be written $\phi(X) = \mathrm{E}\{XY\} \;\forall X \in L^p(\mathcal{P})$.

Proposition 1 In a complete market with no arbitrage, there exists a unique sdf.

Proof of Proposition 1: No arbitrage $\Rightarrow$ p_m is a strictly positive linear functional. Complete markets $\Rightarrow$ p_m is defined on all of $\mathcal{C} = L^p(\mathcal{P})$. So by the R.R.T., $\exists!$ sdf.

Theorem 1 (Harrison and Kreps) The price system $(\mathcal{M}, p_m)$ is viable $\Leftrightarrow$ there exists a strictly positive continuous linear extension p of p_m to all of $\mathcal{C}$.

Corollary The price system $(\mathcal{M}, p_m)$ is viable $\Leftrightarrow$ there exists a sdf.

Sketch of Proof of Harrison and Kreps Theorem 1

Separating Hyperplane Theorem Let A and B be two convex subsets of a topological vector space X and assume that A has an interior point. If $\text{Int}(A) \cap B = \emptyset$ then there exists a nontrivial continuous linear functional ϕ and real number α s.t.

$$\phi(x) \geq \alpha \geq \phi(y) \;\; \forall x \in A, y \in B \; .$$

Theorem 1 (Harrison and Kreps) The price system $(\mathcal{M}, p_m)$ is viable $\Leftrightarrow$ there exists a strictly positive continuous linear extension p of p_m to all of $\mathcal{C}$.

$\Leftarrow$: Suppose such a p exists. Define $\succeq$ on $\mathcal{C}$ by $x \succeq y \Leftrightarrow p(x) \geq p(y)$. Then these preferences together with $x^* = 0$ satisfy the viability condition (vc) in Definition 2.

$\Rightarrow$: Suppose $(\mathcal{M}, p_m)$ is viable. Let $\succeq$ and x^* satisfy the vc, and w.l.o.g. set $x^* = 0$.

- Let $A \equiv \{x \in \mathcal{C} : x \succ 0\}$, the set of consumption plans strictly preferred to x^*.
- Let $B \equiv \{y \in \mathcal{M} : p_m(y) \leq 0\}$, the set of affordable consumption plans.
- By the continuity and convexity of preferences and linearity of p_m, A is open and convex, B is convex, A is nonempty since it contains $\widehat{x}$, and A and B are disjoint by the viability condition.
- Therefore, by the S.H.T., $\exists$ a nontrivial continuous linear functional ϕ on all of $\mathcal{C}$ s.t. $\phi(x) \geq 0$ on A and $\phi(x) \leq 0$ on B.

- It can be shown that ϕ is strictly positive, using the $\widehat{x} > 0$ and the continuity and strict monotonicity of preferences.

- Finally, it can be shown that ϕ can be scaled to equal p_m on $\mathcal{M}$. In particular, let $x \in \mathcal{M}$ and let $b = -\widehat{x} p_m(x)/p_m(\widehat{x}) + x$. Both b and $-b$ are in B since $p_m(b) = p_m(-b) = 0$. So $\phi(b) \leq 0$ and $\phi(-b) \leq 0 \implies \phi(b) = 0 \implies \phi(x) = \phi(\widehat{x}) p_m(x)/p_m(\widehat{x})$. So let

$$p \equiv \frac{p_m(\widehat{x})}{\phi(\widehat{x})} \phi.$$

  Then $p|\mathcal{M} = p_m$ and p is a strictly positive continuous linear functional on $\mathcal{C}$. $\square$

- Note that when the market is incomplete, i.e., packages of Arrow-Debreu state contingent claims cannot be unpacked, then distinct sdfs can exist that would price the individual claims differently if they were separately tradable, but price the package the same.

Securities Market with Multiple Trading Dates

- Now let's operationalize these one-period results in a more practical setting where securities are traded on multiple dates and consumption plans are generated as payoffs of dynamic trading strategies.

Definition 5 A *stochastic process* on a time interval $[0, T]$ in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathbf{F})$ is a mapping $X(\omega, t)$ from $\Omega \times [0, T]$ to $\mathcal{R}$. Each $X(\omega, \cdot)$ is a *sample path* and each $X(\cdot, t)$ is a random variable. The process is *adapted* to $\mathbf{F}$ if each X_t is measurable w.r.t. $\mathcal{F}_t$. It is *right-continuous* if it has right-continuous sample paths.

- Suppose there are $n+1$ long-lived securities traded with adapted, right-continuous price processes $S = (S_0, S_1, \ldots, S_n)$, where each $S_{k,t} \in L^p(\mathcal{P})$ and S_0 is strictly positive.

Definition 6 A *trading strategy* $N = (N_0, N_1, \ldots, N_n)$ is an adapted $n+1$-dimensional row-vector-valued process where $N_{k,t}$ denotes the number of shares of security k held at time t.

- We'll restrict attention to trading strategies with a finite number of trading dates:

Definition 7 A trading strategy is *simple* if there exists a finite partition $0 = t_0 < t_1 < \cdots < t_J = T$ of $[0, T]$ and random variables $N_{kj} \in \mathcal{F}_{t_j}$ such that

$$N_{k,t} = \begin{cases} N_{k0} & \text{if} \quad t \in [t_0, t_1] \\ N_{kj} & \text{if} \quad t \in (t_j, t_{j+1}] \end{cases} \tag{10}$$

$\forall$ security $k = 0, 1, \ldots, n$ and trading date $j = 0, \ldots, J-1$.

Definition 8 A simple trading strategy is *self-financing* if

$$\underbrace{N_{t_j} S_{t_j}}_{\text{cost of new portfolio at } t_j} \leq \underbrace{N_{t_{j-1}} S_{t_j}}_{\text{proceeds from sale of old portfolio at } t_j} \tag{11}$$

for each $j = 1, \ldots, J-1$. The trading strategy is *tight* (tightly self-financing) if the above holds with equality, i.e.,

$$N_{t_j} S_{t_j} = N_{t_{j-1}} S_{t_j} \; \forall j = 1, \ldots, J-1 \; . \tag{12}$$

Proposition 1 A simple trading strategy is tight $\Leftrightarrow$ the following wealth evolution equation (WEE) holds:

$$\underbrace{N_t S_t}_{\text{ending ptf. value}} = \underbrace{N_0 S_0}_{\text{beginning value}} + \underbrace{\sum_{i=0}^{j} N_{t_i}[S_{t_{i+1}\wedge t} - S_{t_i}]}_{\text{trading gains}} \tag{13}$$

for every $t \in (t_j, t_{j+1}]$ and $j = 0, \ldots, J-1$.

Proof: Homework.

Remark 1 If S has continuous sample paths, then the summation above can be written$\int_0^t N_u dS_u$, a traditional Riemann-Stieltjes integral path by path.

Definition 9 A consumption plan $x \in \mathcal{C}$ is *marketed* if there exists a tight trading strategy that *generates* x, i.e., N s.t. $N_T S_T = x$. Let $\mathcal{M}$ be the linear subspace of consumption plans that are marketed.

Definition 10 $\mathcal{M}$ is *dynamically complete* if $\mathcal{M} = \mathcal{C}$.

Remark 2 The market will not generally be complete unless $\mathcal{C}$ is finite-dimensional, i.e., there are only a finite number of states ω.

Definition 11 An *arbitrage opportunity* is a self-financing trading strategy N s.t. $N_T S_T \geq 0, \mathcal{P}\{N_T S_T > 0\} > 0$, and $N_0 S_0 \leq 0$.

- Assume there are no arbitrage opportunities.

Proposition 2 (The Law of One Price/The Law of One Process) If two simple, tight trading strategies N and $\hat{N}$ generate the same consumption plan x, then the portfolio processes $N_t S_t$ and $\hat{N}_t S_t$ are indistinguishable.

Definition 12 The *price process* S^x of a marketed consumption plan x is $S_t^x = N_t S_t$ where N is any trading strategy that generates x. The *price functional* p_m on $\mathcal{M}$ is $p_m(x) \equiv S_0^x = N_0 S_0$, the start-up cost of any trading strategy that generates x.

Proposition 3 p_m is linear and strictly positive.

Proposition 4 If the market is dynamically complete then p_m is a continuous linear functional on all of $L^p(\mathcal{P})$ so there exists a unique sdf m s.t. $p_m(x) = \mathrm{E}\{mx\}$.

Theorem 2 (Harrison and Kreps) The price system S is viable, in that there exists an investor with continuous, strictly increasing, convex preferences who can find an optimal trading strategy $\Leftrightarrow$ p_m can be represented with a sdf, i.e., there exists an sdf m s.t. $p_m(x) = \mathrm{E}\{mx\}$.

The Martingale Property

Definition 13 An adapted process X with $E|X_t| < \infty$ $\forall t \in [0,T]$ is a *martingale* if

$$\mathrm{E}_t\{X_s\} = X_t \ \text{ a.s. } \forall 0 \le t \le s \le T \ . \tag{14}$$

X is a *submartingale* if $\mathrm{E}_t\{X_s\} \ge X_t$ a.s. $\forall 0 \le t \le s \le T$.
X is a *supermartingale* if $\mathrm{E}_t\{X_s\} \le X_t$ a.s. $\forall 0 \le t \le s \le T$.

- Let $S^* \equiv \frac{S}{S_0} = (1, \frac{S_1}{S_0}, \frac{S_2}{S_0}, \ldots, \frac{S_n}{S_0})$ be the "discounted" security price processes.

Proposition 5 A simple trading strategy is tight $\Leftrightarrow$ the wealth evolution equation holds in discounted terms (WEE*):

$$N_t S_t^* = N_0 S_0^* + \sum_{i=0}^{j} N_{t_i}[S^*_{t_{i+1}\wedge t} - S^*_{t_i}] \tag{15}$$

for every $t \in (t_j, t_{j+1}]$ and $j = 0, \ldots, J-1$. Proof: Homework.

Definition 14 A probability measure $\mathcal{P}^*$ on $(\Omega, \mathcal{F})$ is an *equivalent martingale measure* (emm) if $\mathcal{P}^*$ is equivalent to $\mathcal{P}$, and S^* is a vector martingale under $\mathcal{P}^*$.

Definition 15 A probability measure $\mathcal{P}^*$ on $(\Omega, \mathcal{F})$ is *equivalent* to $\mathcal{P}$, written $\mathcal{P}^* \sim \mathcal{P}$, if they agree on which events $A \in \mathcal{F}$ have zero probability.

- In particular, equivalent measures agree about which trading strategies are arbitrage opportunities.

Radon-Nikodym Theorem $\mathcal{P}^* \sim \mathcal{P} \Leftrightarrow$ there exists a strictly positive random variable $\frac{d\mathcal{P}^*}{d\mathcal{P}}$ s.t. for every event $A \in \mathcal{F}$, $\mathcal{P}^*(A) = \mathrm{E}\{1_A \frac{d\mathcal{P}^*}{d\mathcal{P}}\}$, where $1_A(\omega) = 1$ if $\omega \in A$ and zero otherwise. In addition,

1. $\frac{d\mathcal{P}}{d\mathcal{P}^*} = 1/\frac{d\mathcal{P}^*}{d\mathcal{P}}$.
2. $\mathrm{E}^*\{X\} = \mathrm{E}\{X \frac{d\mathcal{P}^*}{d\mathcal{P}}\}$.
3. Conditional expectation under $\mathcal{P}^*$: If $\mathcal{G} \subset \mathcal{F}$ is a coarser information set than $\mathcal{F}$ then
$$\mathrm{E}^*\{X|\mathcal{G}\} = \frac{\mathrm{E}\{X \frac{d\mathcal{P}^*}{d\mathcal{P}}|\mathcal{G}\}}{\mathrm{E}\{\frac{d\mathcal{P}^*}{d\mathcal{P}}|\mathcal{G}\}} \ .$$

Theorem 3 (Harrison and Kreps) There is a one-to-one correspondence between sdf's m and emm's $\mathcal{P}^*$ given by

$$m = \frac{S_{0,0}}{S_{0,T}} \frac{d\mathcal{P}^*}{d\mathcal{P}} \ . \tag{16}$$

- The proof of $\Leftarrow$ involves applying the law of iterated expections to WEE* to verify that $\frac{S_{0,0}}{S_{0,T}} \frac{d\mathcal{P}^*}{d\mathcal{P}}$ is a sdf.
- The proof of $\Rightarrow$ involves working with the formal definition of conditional expection to show that the sdf property implies the martingale condition.

Proposition 6 Let $x \in \mathcal{M}$ and let S^x be its price process. If there exists an emm $\mathcal{P}^*$ then $\frac{S^x}{S_0}$ is a $\mathcal{P}^*$-martingale.

Proof: Homework. Intuition: Once discounted securities prices are martingales, then so are all portfolios of them, because linear combinations of martingales are martingales, and so are portfolio value processes under tight trading strategies, because these are essentially non-forward-looking dynamic linear combinations of martingales.

- In particular, if there is an emm, we get the risk-neutral pricing equation (RNPE):

$$S_t^x = \mathrm{E}_t^* \{ \frac{S_{0,t}}{S_{0,T}} x \} \text{ a.s. } \forall\, t \in [0,T] \ . \tag{17}$$

- Using the R.N.T, we can rewrite the RNPE above using expectation under $\mathcal{P}$:
- Let $Z_t \equiv \mathrm{E}\{ \frac{d\mathcal{P}^*}{d\mathcal{P}} | \mathcal{F}_t \}$.
- Let the sdf process $M_t \equiv Z_t \frac{S_{0,0}}{S_{0,t}} = \mathrm{E}\{ \frac{S_{0,T}}{S_{0,t}} m | \mathcal{F}_t \}$.
- Then

$$S_t^x = \mathrm{E}_t^* \{ \frac{S_{0,t}}{S_{0,T}} x \} = \frac{\mathrm{E}_t \{ \frac{S_{0,t}}{S_{0,T}} x \frac{S_{0,T}}{S_{0,0}} m \}}{Z_t} = \frac{\mathrm{E}_t \{ mx \}}{M_t} \ . \tag{18}$$

- It follows that $M_t S_t^x$ is a $\mathcal{P}$-martingale.
- So, for example, for given dates t and u, a (cum-dividend) price process P satisfies

$$P_t = \mathrm{E}_t \{ \frac{M_u}{M_t} P_u \} = \mathrm{E}_t^* \{ \frac{S_{0,t}}{S_{0,u}} P_u \} \ . \tag{19}$$

Problem Set 1

Prove the following propositions.

1. Let N be a simple trading strategy with trading dates $0 = t_1 < \cdots < t_J = T$ and $S = (S_0, S_1, \ldots, S_n)$ be (cum-dividend) security prices. Then N is tight iff

$$N_t S_t = N_0 S_0 + \sum_{i=1}^{j} N_{t_i}(S_{t_i \wedge t} - S_{t_{i-1}}), \text{ a.s.} \tag{20}$$

for all $t \in (t_{j-1}, t_j]$ and $j = 1, \ldots, J$.

2. Suppose there are no free lunches. For every payoff x in the space $\mathcal{M}$ of marketed payoffs, let the price of x, $p(x)$, be the initial portfolio value $N_0 S_0$ under a trading strategy N that finances x. Prove that p is a well-defined, strictly positive, linear functional on $\mathcal{M}$.

3. If the price system $(\mathcal{M}, p)$ (generated by the security prices S) is viable, then there are no free lunches.