The Continuous-Time Financial Market

1. Security prices – Itô processes
- Brownian motion
- Stochastic integrals
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Readings and References

Back, chapter 13.

Karatzas and Shreve, 1998, chapter 1.

Karatzas and Shreve, 1991.

Now let's develop these results more explicitly in a rich but tractable setting with continuous trading and security price processes constructed from Brownian motion.

- There is a finite time horizon $[0, T]$.
- The filtered probability space is $(\Omega, \mathcal{F}, \mathcal{P}, \mathbf{F^B})$ where $\mathbf{F^B}$ is the filtration (information) generated by a d-dim'l Brownian motion $B = (B_1, B_2, \dots, B_d)$..
- The consumption space $\mathcal{C}$ is the set of pairs (c, W) where c is an adapted $consumption\ rate\ process$ with $\int_0^T |c_t|\, dt < \infty$ a.s. and W is a random variable representing terminal (time T) wealth.
- There are $n + 1$ securities traded, with ex-dividend prices $S = (S_0, S_1, \dots, S_n)$.
- Security 0 is a "bond" or $locally\ riskless$ money market account earning the $instantaneous\ riskless\ rate$ r_t. I.e.,

$$\frac{dS_{0,t}}{S_{0,t}} = r_t\, dt \Leftrightarrow S_{0,t} = S_{0,0} e^{\int_0^t r_u\, du} \tag{1}$$

  where r is an adapted process with $\int_0^T |r_t|\, dt < \infty$ a.s.

- The n "risky" asset prices are strictly positive Itô processes, each satisfying

$$\frac{dS_{k,t}}{S_{k,t}} = [\mu_{k,t} - \delta_{k,t}]\, dt + \underbrace{\sigma_{k,t}}_{1\times d}\, dB_t\ . \tag{2}$$

  The n-dimensional $instantaneous\ expected\ return$ process $\mu = (\mu_1, \dots, \mu_n)$ is adapted and satisfies $\int_0^T |\mu_t|\, dt < \infty$ a.s., the n-dimensional $dividend\ payout\ rate$ process $\delta = (\delta_1, \dots, \delta_n)$ is adapted and satisfies $\int_0^T |\delta_t|\, dt < \infty$ a.s., and the $n \times d$-matrix-valued $volatility$ process $\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$ is adapted and satisfies $\int_0^T ||\sigma_{k,t}||^2\, dt < \infty$ a.s. for each k.
- Equation (2) above is shorthand for

$$S_{k,t} = S_{k,0} e^{\int_0^t [\mu_{k,u} - \delta_{k,u} - |\sigma_{k,u}|^2/2]\, du + \int_0^t \sigma_{k,u}\, dB_u}\ . \tag{3}$$

- W.l.o.g. assume $n \leq d$ unless you want to track redundant securities for a reason.
- Let's briefly review Brownian motion, stochastic integrals, and Itô processes.

Brownian Motion

Definition 1 A continuous, adapted process B is a *standard Brownian motion* if $B_0 = 0$ and for any $0 \le t \le s \le T$, the increment $B_s - B_t$ is independent of $\mathcal{F}_t$ and normally distributed with mean zero and variance $s - t$. A process X is a *Brownian motion* if $X_t = X_0 + \mu t + \sigma B_t \ \forall\ [t \in [0, T]$, where μ is constant.

Proposition 1 A process X is continuous with stationary independent increments if and only if X is a Brownian motion.

Proposition 2 The sample paths of a Brownian motion have infinite variation and finite quadratic variation. I.e., for all $t \in [0, T]$,

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} |B_{t*(i+1)/2^n} - B_{t*i/2^n}| = \infty \text{ a.s.,} \tag{4}$$

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} |B_{t*(i+1)/2^n} - B_{t*i/2^n}|^2 = t \text{ a.s.} \tag{5}$$

Definition 2 An *d-dimensional Brownian motion* is a vector-valued process $B = (B_1, \ldots, B_d)$, where each B_j is a Brownian motion, $\forall\ j = 1, \ldots, d$ and B_i is independent of B_j for all $i \neq j$.

Stochastic Integrals

Now let's define the stochastic integral $\int_0^t \theta_s \, dB_s$ of a process θ w.r.t. a Brownian motion B. A path-by-path Riemann-Stieltjes definition won't work when the integrand θ has infinite variation, so we build it up starting with "simple" integrands.

Definition 3 An adapted process θ satisfying $\mathrm{E} \int_0^T |\theta_t|^2 \, dt < \infty$ is *simple* if there exists a finite partition
$0 = t_0 < t_1 < \cdots < t_J = T$ of $[0, T]$ and random variables $\theta_j \in \mathcal{F}_{t_j}$ such that

$$\theta_t = \begin{cases} \theta_0 & \text{if} \quad t \in [t_0, t_1] \\ \theta_j & \text{if} \quad t \in (t_j, t_{j+1}] \end{cases} \tag{6}$$

$\forall\ j = 0, \ldots, J-1$.

The stochastic integral of a simple integrand θ can be defined path by path as

$$I_t = \int_0^t \theta_s \, dB_s = \sum_{j=0}^{N-1} \theta_j [B_{t_{j+1} \wedge t} - B_{t_j \wedge t}] \ . \tag{7}$$

Proposition 3 The stochastic integral I_t of a simple integrand θ is continuous, adapted to $\mathbf{F}$, linear in θ, an $L^2(\mathcal{P})$-martingale, and for simple θ_1 and θ_2 satisfies

$$\mathrm{E}[(\int_0^t \theta_{1,s}\, dB_s)(\int_0^t \theta_{2,s}\, dB_s)] = \mathrm{E}\int_0^t \theta_{1,s}\theta_{2,s}\, ds \ . \tag{8}$$

- Next, it turns out that every process θ satisfying $\mathrm{E}\{\int_0^T |\theta_t|^2\, dt\} < \infty$, which we'll call "strongly square-integrable," has a sequence of simple processes that converge to it, and the limit of the integrals of these processes exists and is unique.
- So we define the stochastic integral of a strongly square-integrable process θ as the limit of the integrals of any sequence of simple processes that converges to θ.
- This is the so-called Itô-integral and it satisfies the same properties as the integrals of the simple processes listed above: the Itô-integral is continuous, adapted, linear in its integrand, an $L^2(\mathcal{P})$-martingale, and the expectation of the product of stochastic integrals is the expectation of time-integral of the product of the integrands.
- Finally, it is also possible to define the Itô integral of adapted processes θ that satisfy $\int_0^T |\theta_t|^2\, dt < \infty$ a.s., which we'll call "weakly square-integrable," as follows.

Definition 4 An $\mathcal{F}$-measurable map $\tau : \Omega \to [0, T] \cup \{\infty\}$ is a *stopping time* if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t\ \forall t \in [0, T]$.

- It is always possible to interpret a stopping time as the first time an event occurs.

Definition 5 A process X is a *local martingale* if $\exists$ a sequence of stopping times $\tau_n \uparrow T$ a.s. s.t. each stopped process X^{τ_n} is a martingale.

- Now, for any weakly square-integrable process θ there exists a sequence of *stopping times* $\tau_n \uparrow T$ a.s. s.t. each of the *stopped processes* $\theta_n = \theta \cdot 1_{t \leq \tau_n}$ is strongly square-integrable. Then we can define the Itô integral of a weakly square-integrable process as the limit of the integrals of the stopped processes that converges to it.
- The Itô integral of a weakly square-integrable process will have all of the properties above, except that it may be only a *local martingale*, not a martingale.
- An example of a local martingale that is not a martingale is a wealth process under a doubling strategy.

Itô Processes

Definition 6 A process X adapted to the filtration $\mathcal{F}^B$ generated by a Brownian motion B is an *Itô process* if $\exists$ an adapted real-valued process μ with $\int_0^T |\mu_t|\, dt < \infty$ a.s., which we'll call "absolutely integrable," and an $\mathcal{R}^n$-valued weakly square-integrable process σ s.t.

$$X_t = X_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma'_s\, dB_s \;\forall t \in [0, T] \text{ a.s.} \tag{9}$$

The process μ is called the *drift* of X and the process σ is called the *diffusion* of X.

- Equation (9) can be written in differential form as $dX_t = \mu_t\, dt + \sigma'_t dB_t$.
- An Itô process is a local martingale iff it has zero drift.

Definition 7 If X_1 and X_2 are Itô processes with $dX_{it} = \mu_{it}\, dt + \sigma_{it}\, dB_t$, the *quadratic variation* of X_i is

$$\langle X_i, X_i\rangle_t \equiv \lim_{n\to\infty} \sum_{j=0}^{2^n-1} (X_{i,(j+1)t/2^n} - X_{i,jt/2^n})^2 = \int_0^t |\sigma_s|^2\, ds\ , \tag{10}$$

and the *covariation* of X_1 and X_2 is

$$\langle X_1, X_2\rangle_t \equiv \lim_{n\to\infty} \sum_{j=0}^{2^n-1} (X_{1,(i+1)t/2^n} - X_{1,it/2^n})(X_{2,(j+1)t/2^n} - X_{2,jt/2^n}) \tag{11}$$

$$= \int_0^t \sigma_{1s}\sigma'_{2s}\, ds\ , \tag{12}$$

where the convergence above is in probability.

- As a mnemonically helpful shorthand, some write $d\langle X_i, X_i\rangle_t = (dX_i)^2$ and $d\langle X_i, X_j\rangle_t = (dX_i)(dX_j)$.
- If X is an Itô process, and f is a smooth real-valued function, then $f(X)$ is also an Itô process, and Itô's lemma gives its drift and diffusion:

Itô's Lemma Let X be an m-dimensional Itô process as in equation (9) and let $f : \mathcal{R}^m \times [0, T] \to \mathcal{R}$ be $C^{2,1}$. Then $f(X_t, t)$ is also an Itô process with
$df = f_t\, dt + f_X dX + \frac{1}{2}\text{tr}[f_{XX}\sigma\sigma']\, dt = (\frac{1}{2}\text{tr}[f_{XX}\sigma\sigma'] + f_X\mu + f_t)\, dt + f_X\sigma\, dB$.

- In the shorthand, the Taylor expansion underlying Itô's lemma is more apparent, and this becomes more memorable as
  $df = f_t\, dt + f_X dX + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial X_i \partial X_j}(dX_i)(dX_j)$.

Example 1 To gain intuition for the 2nd-order term with f_{XX}, let $X_t = B_t$ and $f(X) = X^2$. The usual calculus $df = f_X\,dX$ would yield $f(B_1) = \int_0^1 2B_t\,dB_t$ and $\mathrm{E}\{f(B_1)\} = \mathrm{E}\{B_1^2\} = 0$, which is incorrect. Including the 2nd-order term gives $f(B_1) = \int_0^1 2B_t\,dB_t + \frac{1}{2}\int_0^1 2\,dt$ so $\mathrm{E}\{f(B_1)\} = \mathrm{E}\{B_1^2\} = 1$. The 2nd-order term captures the "Jensen's inequality" adjustment to the drift of f which is increasing in both the convexity of f and the volatility of X.

Example 2 Let $X_t = -\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s^2|\,dt$. Let $f(X) = e^X$. Then $df = f\,dX + \frac{1}{2}f\theta^2\,dt$ or $\frac{df}{f} = -\theta\,dB$, so $f = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s^2|\,dt}$ is a local martingale.

Example 3 Consider again the continuous-time model of security prices:
$\frac{dS_{0,t}}{S_{0,t}} = r_t\,dt$ and $\frac{dS_{k,t}}{S_{k,t}} = [\mu_{k,t} - \delta_{k,t}]\,dt + \underbrace{\sigma_{k,t}}_{1\times d}\,dB_t,\ k = 1, \ldots, n.$

- Use Itô's lemma to show $S_{k,t} = S_{k,0}e^{\int_0^t [\mu_{k,u} - \delta_{k,u} - |\sigma_{k,u}|^2/2]\,du + \int_0^t \sigma_{k,u}\,dB_u}$.
- Note that r, μ, δ and σ can be any suitably integrable, adapted processes.
- **Special Case: Markov Model** It is often convenient to specialize to the case in which the coefficients r, μ, δ and σ are functions of (S, Y, t), where Y is a vector of *state variables* with $Y_t = Y_0 + \int_0^t \mu_Y(Y_u, u)\,du + \int_0^t \sigma_Y(Y_u, u)'dB_u$. Under Lipschitz and growth conditions on the coefficients, (S, Y) is Markov.

Continuous-Time Trading Strategies

- We can specify a trading strategy in the $n + 1$ securities either in terms of the *number of shares* of each security held at time t, $N_t = (N_{0,t}, N_{1,t}, \ldots, N_{n,t})$, or in terms of the *value* invested in each security, $\bar{\pi}_t = (\pi_{0,t}, \pi_{1,t}, \ldots, \pi_{n,t}) \equiv (\pi_{0,t}, \pi_t)$, where each $\pi_k = N_k S_k$.
- The integrability condition on the trading strategy is easier to state in terms of the (row-vector) of values invested in the n risky assets, π.

Definition 8 A *trading strategy* is an $n + 1$-dimensional adapted process $\bar{\pi}_t = (\pi_{0,t}, \pi_{1,t}, \ldots, \pi_{n,t}) \equiv (\pi_{0,t}, \pi_t)$ with $\int_0^T |\pi_t\sigma_t|^2\,dt < \infty$ a.s.

- We'll focus on tight trading strategies, eliminate π_0 and just specify π.

Definition 9 Starting from initial wealth x_0, a *tight* trading strategy π *generates* consumption plan (c, W) and wealth process $X_t^{\pi,c,x_0} = X_t$ if

$$X_t = x_0 + \int_0^t r_u X_u\,du + \int_0^t \pi_u(\mu_u - r_u)\,du + \int_0^t \pi_u\sigma_u\,dB_u - \int_0^t c_u\,du \tag{13}$$

(the continuous-time WEE) and $X_T = W$.

- The economic effect of the dividends is that if a share of security k is held in a portfolio for an instant in time, then it changes portfolio value by $dS_k + \delta_k S_k\,dt = \mu_k S_k\,dt + \sigma_k S_k\,dB$. Thus, holding μ_k constant, the effect on the portfolio is invariant to δ_k. Nevertheless, we keep track of the dividend rate, because it affects the ex-dividend security price, which is the basis for many derivative contracts.
- Equation (13) can also be written $N_t S_t = N_0 S_0 + \int_0^t N_u\,dS_u$ if there are no dividends and intermediate consumption.
- The "tightness" of the trading strategy, i.e., the self-financing condition, is essentially the restriction that

$$d(NS) = N dS \tag{14}$$

and the additional terms from Itô's lemma, $S\,dN + d\langle N, S\rangle_t$ are zero. This is the continuous-time analog to more intuitive simple self-financing condition $N_{t_j} S_{t_j} = N_{t_{j-1}} S_{t_j}$ $\forall j = 1, \ldots, J-1$ that we saw for simple trading strategies before.

Market Prices of Risk and Equivalent Martingale Measures

Definition 10 A *market price of risk* (mpr) is an adapted d-dim'l process θ s.t.

$$\underbrace{\mu_t}_{nx1} - r_t \underbrace{1}_{nx1} = \underbrace{\sigma_t}_{nxd} \underbrace{\theta_t}_{dx1} \text{ a.s. a.e.} \tag{15}$$

Proposition 4 No arbitrage $\Rightarrow$ there exists a market price of risk θ.

- The d-factor risk structure together with the wide range of available trading strategies here means that the cross-section of expected returns must respect this structure, i.e., instantaneous excess expected returns must be linear in factor loadings.

Proof No arbitrage $\Rightarrow$ If $\pi_t \sigma_t = 0$ then $\pi_t[\mu_t - r_t 1] = 0$ a.s., a.e.. Otherwise, from WEE (13), one could construct a trading strategy that generated positive consumption from zero wealth. From linear algebra, the statement "If $\pi_t \sigma_t = 0$ then $\pi_t[\mu_t - r_t 1] = 0$" is equivalent to the CSER Equation (15).

- It turns out that from any well-behaved mpr θ we can construct an equivalent martingale measure $\mathcal{P}^*$.

Definition 11 The *riskless discount factor* is $\beta_t \equiv e^{-\int_0^t r_u\,du}$, risklessly discounted security prices are $S_t^* = \beta_t S_t$, and risklessly discounted dividends are

$$D^*(t) \equiv (\int_0^t S_{1,u}^* \delta_{1,u}\,du, \ldots, \int_0^t S_{n,u}^* \delta_{n,u}\,du)\ . \tag{16}$$

Definition 12 A probability measure $\mathcal{P}^*$ on $(\Omega, \mathcal{F}, \mathcal{P})$ is an *equivalent martingale measure* if $\mathcal{P}^* \sim \mathcal{P}$ and discounted cum-dividend stock prices $G^*(t) \equiv S_t^* + D^*(t)$ are local martingales under $\mathcal{P}^*$.

This is a relaxed version of the previous emm definition that's as far as we can go without further restrictions.

Proposition 5 If there exists a mpr θ s.t. $\int_0^T |\theta_t|^2\,dt \leq \infty$ a.s. and the process

$$Z_t \equiv e^{-\int_0^t \theta_u'\,dB_u - \frac{1}{2}\int_0^t |\theta_u)|^2\,du} \tag{17}$$

is a martingale, then $\mathcal{P}^*$ defined by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ is an emm.

The proof uses the Girsanov theorem. Let's review it now.

Girsanov Theorem Let B be an n-dim'l Brownian motion under $\mathcal{P}$, $\mathbf{F}^B$ its natural filtration, and $\mathcal{P}^*$ a probability measure equivalent to $\mathcal{P}$. Then $\exists$ a process θ with $\int_0^T |\theta_t|^2\,dt \leq \infty$ a.s. s.t.

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} = e^{-\int_0^T \theta_t'\,dB_t - \frac{1}{2}\int_0^T |\theta_t|^2\,dt} \tag{18}$$

and $B_t^* \equiv B_t + \int_0^t \theta_s\,ds$ is an n-dim'l Brownian motion under $\mathcal{P}^*$. Moreover, if $X_t = X_0 + \int_0^t \mu_s\,ds + \int_0^t \sigma_s\,dB_s$ is an Itô process under $\mathcal{P}$, then X is an Itô process under $\mathcal{P}^*$ with representation $X_t = X_0 + \int_0^t (\mu_s - \sigma_s \theta_s)\,ds + \int_0^t \sigma_s\,dB_s^*$.

- Here is some intuition. Suppose $X \sim N(0,1)$ under $\mathcal{P}$ and $X \sim N(-\theta, 1)$ under $\mathcal{P}^*$. The p.d.f. of X is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ under $\mathcal{P}$ and $f^*(x) = \frac{1}{\sqrt{2\pi}} e^{-(x+\theta)^2/2}$ under $\mathcal{P}^*$. So $\frac{f^*(x)}{f(x)} = e^{-\theta x - \theta^2/2}$. Further, for any function $g(X)$, the $\mathcal{P}^*$-mean is

$$\mathrm{E}^*\{g(X)\} = \mathrm{E}\{g(X)\frac{f^*(X)}{f(X)}\} = \mathrm{E}\{g(X) e^{-\theta X - \theta^2/2}\}.$$

  Girsanov says the mean shift can be done to Itô processes Brownian increment by Brownian increment, with an adapted, mean shift process θ.
- Note that equivalent changes of measure on a Brownian space can only change drift, not diffusion.

Proof of Proposition 5 Note $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T > 0$ and $\mathrm{E}\{Z_T\} = 1$, so $\mathcal{P}^* \sim \mathcal{P}$. Next, it follows from the Girsanov Theorem that $B_t^* \equiv B_t + \int_0^t \theta_s \, ds$ is an n-dim'l Brownian motion under $\mathcal{P}^*$, so $G_k^*(t)$ is a local martingale for all $k = 1, \ldots, n$:

$$\begin{aligned} dG_{k,t}^* &= dS_{k,t}^* + dD_{k,t}^* = d(\beta_t S_{k,t}) + \beta_t S_{k,t}\delta_{k,t}\, dt && (19)\\ &= \beta_t\, dS_{k,t} - r_t\beta_t S_{k,t}\, dt + \beta_t S_{k,t}\delta_{k,t}\, dt = \beta_t S_{k,t}[(\mu_{k,t} - r_t)\, dt + \sigma_{k,t}\, dB_t] && (20)\\ &= \beta_t S_{k,t}\sigma_{k,t}[\theta_t\, dt + dB_t] = \beta_t S_{k,t}\sigma_{k,t}\, dB_t^* \ . && (21) \end{aligned}$$

Proposition 6 If θ is a mpr and $\mathcal{P}^*$ is its associated emm, then the discounted cum-consumption value of wealth $\beta_t X_t + \int_0^t \beta_s c_s\, ds$ is also a $\mathcal{P}^*$-local martingale under any tight trading strategy π.

Proof From Itô's lemma, $d(\beta_t X_t) = -r_t\beta_t X_t + \beta_t dX_t$, so discounting with the riskless discount factor β absorbs the interest term in WEE 13, and switching from dB to $dB^* = dB + \theta\, dt$ absorbs excess returns, giving WEE*:

$$\begin{aligned} \beta_t X_t &= x_0 + \int_0^t \beta_u\pi_u(\mu_u - r_u)\, du + \int_0^t \beta_u\pi_u\sigma_u\, dB_u - \int_0^t \beta_u c_u\, du && (22)\\ &= x_0 + \int_0^t \beta_u\pi_u\sigma_u\, dB_u^* - \int_0^t \beta_u c_u\, du \ . && (23) \end{aligned}$$

Tame Trading Strategies and Supermartingales

Definition 13 A trading strategy π starting from wealth x_0 is *tame* if $\beta X^{x_0,\pi} \geq -K$ for some finite constant K.

Proposition 7 A local martingale that is bounded below is a supermartingale.

The proof uses Fatou's lemma.

Fatou's lemma If $\{X_n\}$ is a sequence of nonnegative random variables, then $\liminf_{n\to\infty} \mathrm{E}[X_n|\mathcal{G}] \geq \mathrm{E}[\liminf_{n\to\infty} X_n|\mathcal{G}]$.

Apply this to the sequence $\{X_s^{\tau_n}\}$ with $\mathcal{G} = \mathcal{F}_t$ to get $X_t \geq \mathrm{E}\{X_s|\mathcal{F}_t\}$.

Proposition 8 If $\mathcal{P}^*$ is an emm and π is a tame, tight trading strategy generating a nonnegative consumption plan (c, W), then the discounted cum-consumption wealth $\beta_t X_t^{x_0,\pi,c} + \int_0^t \beta_u c_u\, ds = x_0 + \int_0^t \beta_u\pi_u\sigma_u\, dB_u^*$ is a $\mathcal{P}^*$-local martingale bounded below, and thus a $\mathcal{P}^*$-supermartingale. Therefore,

$$X_t^{x_0,\pi,c} \geq \mathrm{E}^*\{\int_t^T e^{-\int_t^u r_s\, ds} c_u\, du + e^{-\int_t^T r_s\, ds} W|\mathcal{F}_t\} \ \forall t \in [0, T].$$

Corollary 9 If there exists an emm, there are no tame arbitrage opportunities.

Complete Markets and Martingales

Lemma 1 No arbitrage $\Rightarrow$ there exists a unique mpr θ s.t. every mpr $\widehat{\theta}$ can be written as $\widehat{\theta} = \theta + \nu$ where $\sigma\nu = 0$ a.s. a.e. If rank$(\sigma) = n$ then $\theta = \sigma'[\sigma\sigma']^{-1}(\mu - r1)$.

We'll focus on this mpr θ and its associated emm $\mathcal{P}^*$ and Brownian motion B^*.

Definition 14 A trading strategy π is *martingale-generating* if $\int_0^t \beta_u \pi_u \sigma_u \, dB_u^*$ is a $\mathcal{P}^*$-martingale, not just a $\mathcal{P}^*$-local martingale.

Proposition 10 If a tame tr. str. π starting from wealth x_0 generates a consumption plan (c, W) that is bounded below, then π is tight and martingale-generating $\Leftrightarrow$

$$X_t^{x_0,\pi,c} = \mathrm{E}^*\{\int_t^T e^{-\int_t^u r_s \, ds} c_u \, du + e^{-\int_t^T r_s \, ds} W | \mathcal{F}_t\} \ \forall t \in [0, T]. \qquad (24)$$

Definition 15 The market is *complete* if every cons. plan (c, W) with $\mathrm{E}^*\{\int_0^T \beta_u |c_u| \, du + \beta_T |W|\} < \infty$ can be generated by a tight, martingale-generating trading strategy.

Theorem 1 The market is complete $\Leftrightarrow$ $n = d$ and σ is nonsingular.

The proof uses the Martingale Representation Theorem.

Martingale Representation Theorem Let B be an n-dim'l Brownian motion, $\mathbf{F}^B$ its natural filtration, and X a local martingale w.r.t. $\mathbf{F}^B$. Then there exists an n-dim'l adapted process with $\int_0^T |\theta_t|^2 \, dt < \infty$ s.t. $X_t = X_0 + \int_0^T \theta_u' \, dB_u$. Moreover, X is an $L^p(\mathcal{P})$-martingale for some $p \in [1, \infty)$ iff $\mathrm{E}[(\int_0^T |\theta_t|^2 \, dt)^{\frac{p}{2}}] < \infty$.

Lemma 2 (Representation of $\mathcal{P}^*$-martingales) If M^* is a $\mathcal{P}^*$-martingale then there exists an adapted, square-integrable process ψ s.t. $M_t^* = M_0 + \int_0^t \psi_u' \, dB_u^*$.

Proof of Theorem 1 $\Leftarrow$: Given consumption plan (c, W), let

$$M_t^* = \mathrm{E}^*\{\int_0^T \beta_u c_u \, du + \beta_T W | \mathcal{F}_t\} \ .$$

By the M.R.T. $M_t^* = M_0^* + \int_0^t \psi_u' \, dB_u^*$ for some adapted, square-integrable process ψ. Let $\pi = \psi'\sigma^{-1}/\beta$ and let $x_0 = M_0^*$. Then, by the WEE*, π generates (c, W) starting from wealth x_0. In particular, evaluating WEE* at time $t = T$ gives

$$\beta_T X_T + \int_0^T \beta_u c_u \, du = x_0 + \int_0^T \beta_u \pi_u \sigma_u \, dB_u^* = M_T^* = \int_0^T \beta_u c_u \, du + \beta_T W. \quad (25)$$

Corollary 11 In a complete market, there exists a unique mpr $\theta = \sigma^{-1}(\mu - r1)$.

Problem Set 2

1. Suppose the price in yen P of a Japanese stock and the exchange rate X dollars per yen are Itô processes given by

$$dP/P = \mu_P \, dt + \sigma_P' \, dB \ , \tag{26}$$
$$dX/X = \mu_X \, dt + \sigma_X' \, dB \ , \tag{27}$$

where B is standard 2-dimensional Brownian motion. Describe the dynamics of the price Y of the stock in dollars.

2. Suppose rank$(\sigma_t) = n$. Let ν be an d-dimensional process with $\sigma_t \nu_t = 0$ and let

$$\theta_t = \sigma_t'(\sigma_t \sigma_t')^{-1}[\mu_t - r_t 1] \ , \tag{28}$$
$$\widehat{\theta}_t = \theta_t + \nu_t \ , \tag{29}$$
$$\widehat{Z}_t \equiv e^{-\int_0^t \widehat{\theta}_s' \, dB_s - \frac{1}{2}\int_0^t |\widehat{\theta}_s|^2 \, ds} \ , \tag{30}$$
$$\beta_t \equiv e^{-\int_0^t r_s \, ds} \ , \text{and} \tag{31}$$
$$m_t \equiv \beta_t \widehat{Z}_t \ . \tag{32}$$

Finally, let

$$Z_t \equiv e^{-\int_0^t \theta_s' \, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 \, ds} \ , \tag{33}$$
$$m_t^* \equiv \beta_t Z_t \ , \tag{34}$$
$$m^* \equiv m_T^* \ . \tag{35}$$

(a) Show that m^* is the only m of the form above in the payoff space, that is, it is the only such m for which there exists a trading strategy that strictly finances a consumption plan $(c, W) \in \mathcal{C}$ with $W = m$.

(b) Show that m_t^* is the m process with the smallest instantaneous volatility, or in other words, that its log has the smallest quadratic variation.

3. Verify the following: if $\theta, \theta_1, \theta_2 \in \mathcal{H}_0^2$, then

(a) I^θ is an $L(\mathcal{P})^2$ martingale;

(b) $\mathrm{E}[I_t^{\theta_1} I_t^{\theta_2}] = \mathrm{E}[\int_0^t \theta_{1,s}\theta_{2,s} \, ds]$.

In addition, verify that $||I^{\theta_1} - I^{\theta_2}||_{\mathcal{M}^2} = ||\theta_1 - \theta_2||_{\mathcal{H}^2}$.

4.(a) Use the Martingale Representation Theorem and Itô's Lemma to prove the following corollary:

Let $\{B_t\}$ be an n-dimensional Brownian motion, $\{\mathcal{F}_t\}$ its natural filtration, and $\{X_t\}$ a strictly positive local martingale adapted to $\{\mathcal{F}_t\}$. Then there exists an n-dimensional process $\theta \in \mathcal{L}^2$ such that

$$X_t = X_0 e^{\int_0^t \theta_s' \, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 \, ds} . \tag{36}$$

(b) Conclude that if $Z \in L^2(\mathcal{P})$ is a strictly positive random variable measurable with respect to $\mathcal{F}_t$, then Z has the representation (36).

5. Use the results in question 4 above, the lemma stated below, and Levy's Theorem (Prop 14 of Domenico Cuoco's lecture notes) to prove the Girsanov Theorem:

Let B_t be standard n-dimensional Brownian motion on $[0, T]$ under the probability measure $\mathcal{P}$ with $\{\mathcal{F}_t\}$ its natural filtration and let $\mathcal{P}^*$ be a probability measure equivalent to $\mathcal{P}$. Then there exists an n-dimensional process $\theta \in \mathcal{L}^2$ s.t.

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} = \exp\left(-\int_0^T \theta_t' \, dB_t - \frac{1}{2}\int_0^T |\theta_t|^2 \, dt\right) \tag{37}$$

and

$$B_t^* \equiv B_t + \int_0^t \theta_s \, ds \tag{38}$$

is standard n-dimensional Brownian motion on $[0, T]$ under $\mathcal{P}^*$.

Lemma In the setting described above, define $Z_t \equiv E\left\{\frac{d\mathcal{P}^*}{d\mathcal{P}} \,|\mathcal{F}_t\right\}$, a strictly positive martingale w.r.t $\{\mathcal{F}_t\}$. If Y is an Itô process and ZY is a $\mathcal{P}$-local martingale, then Y is $\mathcal{P}^*$-local martingale.

6. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ and let the σ-field $\mathcal{F}$ be the set of all subsets of Ω. Define the probability measure $\mathcal{P}$ by

$$\mathcal{P}\{\omega_1\} = \mathcal{P}\{\omega_2\} = \mathcal{P}\{\omega_3\} = \mathcal{P}\{\omega_4\} = \mathcal{P}\{\omega_5\} = \frac{1}{5}.$$

Finally, suppose X_1, X_2 are random variables on $(\Omega, \mathcal{F}, \mathcal{P})$ that take on the following values.

state	X_1	X_2
ω_1	1	2
ω_2	1	3
ω_3	2	3
ω_4	3	4
ω_5	3	4

(a) What is the σ-field $\mathcal{F}_{X_1}$ generated by X_1? What is the σ-field $\mathcal{F}_{X_2}$ generated by X_2? (Formally, $\mathcal{F}_X$ is defined as

$$\mathcal{F}_X = \left\{X^{-1}(B) \,|B \in \mathcal{B}\right\}, \tag{39}$$

where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}$. That is, $\mathcal{F}_X$ is the smallest σ-field with respect to which X is measurable.)

(b) Specify the values of the random variables $\mathrm{E}\{X_2|\mathcal{F}_{X_1}\}$ and $\mathrm{E}\{X_1|\mathcal{F}_{X_2}\}$ for each ω_i.

(c) Suppose X_1 and X_2 are the time 1 and 2 values of a stochastic process X with $X_0 = 1$. Let $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ be the filtration generated by X. What is $\mathcal{F}_0$? $\mathcal{F}_1$? $\mathcal{F}_2$? (In the filtration generated by a stochastic process X, each $\mathcal{F}_t$ is the σ-field generated by the complete history of X up to and inclusing time t. That is,

$$\mathcal{F}_t = \sigma\left\{\mathcal{F}_{X_s} |s \le t\right\} \ , \tag{40}$$

where the notation $\sigma\{\}$, or "σ-closure," or the "σ-field generated by" is necessary because the union of $\mathcal{F}_{X_1}, ..., \mathcal{F}_{X_t}$ may not by itself represent a valid σ-field.)