Contingent Claims Pricing

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Readings and References

Back, chapters 15 and 16..

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Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637-654.

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Summary of the Continuous-Time Financial Market

- Security prices satisfy $\frac{dS_{0,t}}{S_{0,t}} = r_t\,dt$ and $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t})\,dt + \sigma_{k,t} dB_t$.
- Given tight tr. strat. π_t and consumption c_t, portfolio value X_t satisfies the
  - WEE: $dX_t = r_t X_t\,dt + \pi_t(\mu_t - r_t 1)\,dt + \pi_t \sigma_t\,dB_t - c_t\,dt$.
- No arbitrage $\Rightarrow$ if $\pi_t\sigma_t = 0$ then $\pi_t(\mu_t - r_t 1) = 0 \Rightarrow \exists \theta_t$ s.t. $\sigma_t\theta_t = \mu_t - r_t 1$ $\Rightarrow dX_t = r_t X_t\,dt + \pi_t\sigma_t(\theta_t\,dt + dB_t) - c_t\,dt$.
- Under emm $\mathcal{P}^*$ given by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ where $Z_t = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds}$,
  - $B_t^* = B_t + \int_0^t \theta_s\,ds$ is Brownian motion.

  Let $\beta_t = e^{-\int_0^t r_s\,ds}$ and sdf process $M_t = \beta_t Z_t$. Then the WEE can also be written:
  - WEE*: $d\beta_t X_t + \beta_t c_t\,dt = \beta_t \pi_t \sigma_t\,dB_t^*$
  - WEE-M: $dM_t X_t + M_t c_t\,dt = M_t[\pi_t\sigma_t - \theta_t X_t]\,dB_t$
- So $X_t = \mathrm{E}_t^*\{\int_t^T \frac{\beta_u}{\beta_t} c_u\,du + \frac{\beta_T}{\beta_t} X_T\} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t} c_u\,du + \frac{M_T}{M_t} X_T\}$ if π is mtgale-gen.
- If σ is nonsingular, every c.plan (c, X_T) can be generated by a mtgale-gen. tr.strat.

European Contingent Claims

Definition 1 A *European contingent claim* (ecc) is a payoff $(c, W) \in \mathcal{C}$.

- In an incomplete market, it's hard to nail down a unique replication cost of a ecc because there is so much flexibility in the choice of trading strategies that wealth may be only a local martingale or supermartingale.

Definition 2 The *price* of a ecc $x = (c, W)$ is $S_t^x = \min\{\bar{\pi}_t 1 : \bar{\pi}$ is a tame, tight trading strategy that generates $x\}$, provided the minimum exists.

Proposition 1 If the market is complete then we have the RNPE

$$S_t^x = \mathrm{E}^*\{\int_t^T e^{-\int_t^u r_s\,ds} c_u\,du + e^{-\int_t^T r_s\,ds} W | \mathcal{F}_t\}\ \forall t \in [0, T]. \tag{1}$$

Proof In a complete market, there exists a tight, tame, martingale-generating strategy π that finances x and must satisfy

$$X_t = \mathrm{E}^*\{e^{-\int_t^u r_s\,ds} c_u\,du + e^{-\int_t^T r_s\,ds} W | \mathcal{F}_t\}\ . \tag{2}$$

For any other tame trading strategy that finances x, risklessly discounted cum-consumption portfolio value is a $\mathcal{P}^*$-supermartingale, so

$$X_t \geq \mathrm{E}^*\{e^{-\int_t^u r_s\,ds} c_u\,du + e^{-\int_t^T r_s\,ds} W|\mathcal{F}_t\}. \ \Box \tag{3}$$

Definition 3 A *replicating portfolio* for an ecc x is a tame, martingale-generating trading strategy that tightly finances x.

- In what sense is S^x an equilibrium price for an ecc x?
- If a replicating portfolio of x exists, we would like to argue that the equilibrium price of x must be the price of its replicating portfolio.
- But when doubling strategies are possible, the replication cost may not be unique.
- And, if we are restricted to tame trading strategies, we may not be able turn an apparent mispricing into an arbitrage, because the natural long-short position to implement might get closed out if its value gets marked below an institutionally imposed lower bound before the end of the trading horizon.
- Nevertheless, the minimum replication cost is a pretty economically natural notion of the price of x.

SDF Process and Pricing Equation

- If the market is complete, we can define the unique SDF process M_t by

$$M_t = \beta_t \mathrm{E}\{\frac{d\mathcal{P}^*}{d\mathcal{P}}|\mathcal{F}_t\} = \beta_t Z_t = e^{-\int_0^t r_u\,du - \int_0^t \theta_u'\,dB_u - \frac{1}{2}\int_0^t |\theta_u|^2\,du}, \tag{4}$$

- see that stochastically discounted wealth process MX satisfies WEE-M:

$$M_t X_t = x_0 + \int_0^t M_u[\pi_u \sigma_u - X_u \theta_u']\,dB_u - \int_0^t M_u c_u\,du, \tag{5}$$

  and rewrite the RNPE with true expectation and stochastic discounting (SDPE):

$$S_t^x = \mathrm{E}^*\{\int_t^T \frac{\beta_u}{\beta_t} c_u\,du + \frac{\beta_T}{\beta_t} W|\mathcal{F}_t\} \tag{6}$$

$$= \mathrm{E}\{\frac{d\mathcal{P}^*}{d\mathcal{P}}[\int_t^T \frac{\beta_u}{\beta_t} c_u\,du + \frac{\beta_T}{\beta_t} W]|\mathcal{F}_t\} / \mathrm{E}\{\frac{d\mathcal{P}^*}{d\mathcal{P}}|\mathcal{F}_t\} \tag{7}$$

$$= \mathrm{E}\{\int_t^T \frac{M_u}{M_t} c_u\,du + \frac{M_T}{M_t} W|\mathcal{F}_t\} \ \forall t \in [0, T]. \tag{8}$$

Fundamental PDE for Path-Independent Claims

From the WEE in undiscounted terms, but with B^* instead of B, cum-consumption or cum-dividend portfolio value always appreciates at the riskless rate r under $\mathcal{P}^*$:

$$dX_t + c_t\, dt = r_t X_t\, dt + \pi_t(\mu_t - r_t)\, dt + \pi_t \sigma_t\, dB_t = r_t X_t\, dt + \pi_t \sigma_t\, dB_t^* \; . \qquad (9)$$

Intuitively, once B^* absorbs the Sharpe ratios of all the securities, it also absorbs the Sharpe ratios of all the portfolio processes.

Definition 4 A ecc $x = (c, W) \in \mathcal{C}_b^1$ is *path-independent* if $c_t = \varphi_1(S_{1,t}, \ldots, S_{n,t}, t)$ and $W = \varphi_2(S_{1,T}, \ldots S_{n,T})$ for continuous functions $\varphi_1 : \mathcal{R}_+^n \times [0, T] \to \mathcal{R}$ and $\varphi_2 : \mathcal{R}_+^n \to \mathcal{R}$.

- In the Markovian model, if x is path-independent then

  $$S_t^x = \mathrm{E}^*\{e^{-\int_t^u r_s\, ds} c_u\, du + e^{-\int_t^T r_s\, ds} W | \mathcal{F}_t\} = F(S_{1,t}, \ldots, S_{n,t}, Y_t, t)$$

  for some real-valued function F.

- If F is smooth enough for an application of Itô's lemma, then it must satisfy the following PDE, which says the drift of F from the WEE must equal the drift of F from Itô's lemma:

  $rF - \varphi_1 = F_S' I_S(r1 - \delta) + F_Y'(\mu_Y - \sigma_Y \theta) + F_t + \frac{1}{2}\mathrm{tr}(F_{SS} I_S \sigma \sigma' I_S)$
  $+\frac{1}{2}\mathrm{tr}(F_{YY} \sigma_Y \sigma_Y') + \mathrm{tr}(F_{SY}' I_S \sigma \sigma_Y')$

- where I_S is a diagonal matrix with diagonal elements $S_1, S_2, \ldots, S_n$ and the derivatives F_S and F_{SS} are with respect to the vector $(S_1, S_2, \ldots, S_n)$.

- This is subject to the boundary condition $F(S_{1,T}, \ldots S_{n,T}, Y_T, T) = \varphi_2(S_{1,T}, \ldots S_{n,T})$.

- Furthermore, by matching the diffusion coefficient from the WEE with the diffusion coefficient from Itô's lemma, we have that the diffusion coefficient of S_t^x must satisfy

  $$\pi^x \sigma = F_S' I_S \sigma + F_Y' \sigma_Y \; , \qquad (10)$$

  which gives a way to compute the replicating trading strategy π^x.

Special Case: Complete Market with Constant Coefficients

Suppose r, μ, σ, and δ are constant.

Definition 5 A function $f : \mathcal{R}^d \times [0,T] \longrightarrow \mathcal{R}$ *satisfies a polynomial growth condition* (pgc) if there exist positive constants k_1 and k_2 such that

$$|f(x,t)| \leq k_1(1 + |x|^{k_2}) \quad \forall (x,t) \in \mathcal{R}^d \times [0,T] \; . \tag{11}$$

Theorem 1 (Feynman-Kac) Suppose the functions φ_1, φ_2, and
$F(S_{1,t}, \ldots, S_{n,t}, t) \equiv \mathrm{E}^*\{\int_t^T e^{-r(u-t)}\varphi_1(S_{1,u}, \ldots, S_{n,u}, u)\, du$
$+ e^{-r(T-t)}\varphi_2(S_{1,T}, \ldots S_{n,T})|\mathcal{F}_t\}$
each satisfy a pgc. Then F satisfies the PDE

$$rF - \varphi_1 = F_S' I_s(r1 - \delta) + F_t + \frac{1}{2}\mathrm{tr}(F_{SS} I_S \sigma\sigma' I_S) \tag{12}$$

subject to $F(S_{1,T}, \ldots S_{n,T}, T) = \varphi_2(S_{1,T}, \ldots S_{n,T})$. There is no other solution to this PDE that satisfies a pgc.

In addition, the trading strategy π^x that generates $x = (\varphi_1, \varphi_2)$ is given by $\pi_k^x = S_k \partial F / \partial S_k$, i.e., $N_k = \partial F / \partial S_k$, for $k = 1, 2, \ldots, n$, and $\pi_0^x = F - \pi^x 1$.

Black-Scholes-Merton Call Option Model

- Assume $n = d = 1$, r and σ are constant, and for ease of exposition, begin by assuming no dividends. The price of the risky asset or "stock" follows

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma\, dB_t \; . \tag{13}$$

- Consider a call on the stock with time T payoff $\varphi_2(S_T) = (S_T - K)^+$ for some positive constant K.
- The Black-Scholes argument (assume μ is constant):
  - The time t call price $C_t = c(S_t, t)$ because S is Markov.
  - Consider a portfolio short one call and long c_S shares of the stock. The portfolio value is $X = c_S S - c + \pi_0$ where π_0 adjusts to make the portfolio self-financing.
  - On one hand, from the wealth evolution equation $dX = N\, dS$ and Itô's lemma,
    $dX = c_S\, dS - dc + r\pi_0\, dt$
    $= c_S\, dS - (c_S\, dS + c_t\, dt + \frac{1}{2} c_{SS}\sigma^2 S^2\, dt) + r\pi_0\, dt$
    $= -c_t\, dt - \frac{1}{2} c_{SS}\sigma^2 S^2\, dt + r\pi_0\, dt.$

- On the other hand, by no arbitrage, since the portfolio is locally riskless, i.e., has zero diffusion, it must appreciate at the riskless rate:

$$dX = rX\,dt = r(c_S S - c + \pi_0)\,dt\ . \tag{14}$$

  Therefore, it must be that

$$rSc_S + \frac{1}{2}\sigma^2 S^2 c_{SS} + c_t = rc\ . \tag{15}$$

- They recognized this as the heat equation from physics, with solution

$$c(S_t,t) = \mathrm{E}^*\{e^{-r(T-t)}(S_T - K)^+ | S_t\}\ , \tag{16}$$

  which is the same as our result from equation (1).

► Alternative derivation with change of numeraire/change of measure:

- Now we can easily incorporate a constant dividend rate δ. Write
  $C_t = \mathrm{E}^*\{e^{-r(T-t)}(S_T - K)1_{\{S_T>K\}}|\mathcal{F}_t\}$
  $= S_t e^{-\delta(T-t)}\mathrm{E}^*\{e^{\sigma(B_T^*-B_t^*)-\sigma^2(T-t)/2}1_{\{S_T>K\}}|\mathcal{F}_t\} - e^{-r(T-t)}K\mathcal{P}^*\{S_T > K|\mathcal{F}_t\}$.

- Next, simplify the first term by introducing a new measure $\mathcal{P}^{(s)}$ defined by $d\mathcal{P}^{(s)}/d\mathcal{P}^* \equiv e^{\sigma B_T^* - \sigma^2 T/2}$ (under which prices are martingales with the stock as numeraire). Then

$$C_t = S_t e^{-\delta(T-t)}\mathcal{P}^{(s)}\{S_T > K|\mathcal{F}_t\} - e^{-r(T-t)}K\mathcal{P}^*\{S_T > K|\mathcal{F}_t\}\ . \tag{17}$$

- Under the new measure, $B_t^{(s)} = B_t^* - \sigma t$ is a standard Brownian motion, so

$$C_t = c(S_t,t) = S_t e^{-\delta(T-t)}N(d_1(S_t,t)) - e^{-r(T-t)}KN(d_2(S_t,t)) \tag{18}$$

  where N is the cumulative normal distribution function,

$$d_1(S_t,t) = \frac{\ln(S_t e^{-\delta(T-t)}/(Ke^{-r(T-t)})) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\ , \tag{19}$$

  and $d_2(S(t),t) = d_1(S(t),t) - \sigma\sqrt{T-t}$.

- The option "delta," or number of shares in the replicating portfolio, is $c_S = e^{-\delta(T-t)}N(d_1(S_t,t))$.

Forward Contracts

- Consider a forward contract to buy a risky asset at time T for price F. In this case $x = (0, S_T - F)$ and the time t value of the contract is

$$V_t^F = \mathrm{E}^*\{e^{-\int_t^T r_s\,ds}(S_T - F)|\mathcal{F}_t\} \tag{20}$$

The *forward price* F_t at time t is such that $V_t^{F_t} = 0$, i.e.,

$$F_t = \frac{\mathrm{E}^*\{e^{-\int_t^T r_s\,ds} S_T|\mathcal{F}_t\}}{\mathrm{E}^*\{e^{-\int_t^T r_s\,ds}|\mathcal{F}_t\}} \, . \tag{21}$$

If the risky asset dividend rates δ are deterministic, then

$$F_t = \frac{S_t e^{-\int_t^T \delta_s\,ds}}{P_t^T} = S_t e^{+y_t^T(T-t) - \int_t^T \delta_s\,ds} \tag{22}$$

where P_t^T is the time t price of a zero-coupon bond maturing at time T and y_t^T is the continuously compounded $(T-t)$-year "zero rate" or "zero yield" at time t.

Futures Contracts (Duffie and Stanton, 1992)

- A *futures contract* on an *underlying asset* with associated *futures price* G_t, *delivery date* T, and *continuous resettlement* is a contract which produces a cumulative cash flow of $G_u - G_t$ between any two dates t and u with $0 \leq t \leq u \leq T$. In addition, at time T, the contract obliges the holder to buy one share of the asset at price G_T.
- Given the futures price process G, the time t value of a futures contract is $V_t^G = \mathrm{E}^*\{\int_t^T e^{-\int_t^u r_s\,ds}\,dG_u + e^{-\int_t^T r_s\,ds}(S_T - G_T)|\mathcal{F}_t\}$.
- Buying and selling futures contracts is costless, so the equilibrium futures price process G must be such that $V^G \equiv 0$ and $G_T = S_T$.

Theorem 2 Suppose $\mathrm{E}^*\{S_T^2\} < \infty$ and $e^{-\int_0^t r_s\,ds}$ is bounded above and below away from zero. Then there exists a unique Itô process G with $\mathrm{E}^*\langle G, G\rangle_T < \infty$ satisfying $V^G \equiv 0$ and $G_T = S_T$. It is given by $G_t = \mathrm{E}^*\{S_T|\mathcal{F}_t\}$.

Sketch of Proof The conditions $V^f \equiv 0$ and $G_T = S_T$ imply G is a $\mathcal{P}^*$-martingale with last element S_T, so $G_t = \mathrm{E}^*\{S_T|\mathcal{F}_t\}$.

- To compare futures and forward prices, note that

$$F_t = G_t + \frac{\mathrm{cov}^*\{S_T, e^{-\int_t^T r_s\,ds}|\mathcal{F}_t\}}{\mathrm{E}^*\{e^{-\int_t^T r_s\,ds}|\mathcal{F}_t\}} \ . \tag{23}$$

- In particular, they are equal if and only if S_T and $e^{-\int_t^T r_s\,ds}$ are uncorrelated.
- This will be the case if interest rates are deterministic.
- But if interest rates are stochastic, they may differ. For example, bond futures prices are typically lower than their forward prices.

Forward Measure vs. Futures Measure

- When the interest rate r is stochastic, it can be convenient for contingent claims pricing to work with the so-called *forward measure*, $\mathcal{P}^{(T)}$, under which prices normalized by the price of the zero-coupon bond maturing at time T are martingales.
- The forward measure $\mathcal{P}^{(T)}$ is defined by $\frac{d\mathcal{P}^{(T)}}{d\mathcal{P}^*} = \frac{\beta_T}{P_0^T}$, where $P_0^T = \mathrm{E}^*\{\beta_T\}$ is the time 0 price of the zero maturing at time T.
- Note that the forward price F_0 of a risky asset with price S satisfies

$$0 = \mathrm{E}^*\{\beta_T(S_T - F_0)\} = P_0^T \mathrm{E}^{(T)}\{S_T - F_0\} \ , \tag{24}$$

  which implies $\mathrm{E}^{(T)}\{S_T\} = F_0$, which is why $\mathcal{P}^{(T)}$ is called the forward measure.
- By contrast, $\mathrm{E}^*\{S_T\} = G_0$, so $\mathcal{P}^*$ is sometimes called the *futures measure*.
- Both are "risk-neutral" measures in the sense that the price of time T payoff W can be written as

$$W_0 = \mathrm{E}^*\{\beta_T W\} = P_0^T \mathrm{E}^{(T)}\{W\} \tag{25}$$

  with discounting at a "riskless" rate. They are the same when r is nonstochastic.

State Prices Implicit in Options Prices

- The price of a call on a risky asset with strike K expiring at time T can be written

$$C_0 = P_0^T \mathrm{E}^{(T)}\{(S_T - K)^+\} = P_0^T \int_K^\infty (s-K) f^{S,T}(s)\, ds \ , \tag{26}$$

  assuming S_T has a well-defined probability density function $f^{S,T}(s)$ under $\mathcal{P}^{(T)}$.

- If C_0 is a twice-differentiable function of the strike price K, then

$$\frac{\partial C_0}{\partial K} = P_0^T \int_K^\infty (-1) f^{S,T}(s)\, ds \ \text{ and } \ \frac{\partial^2 C_0}{(\partial K)^2} = P_0^T f^{S,T}(K) \ , \tag{27}$$

  so we can recover the pdf of the future asset price with a continuum of call prices.

- Then, if a claim's payoff is a path-independent function of the future risky asset price, $W = \varphi(S_T)$, its price can be written as

$$W_0 = P_0^T \int_0^\infty \varphi(s) f^{S,T}(s)\, ds = P_0^T \int_0^\infty \varphi(s) \frac{\partial^2 C_0}{(\partial K)^2}|_{K=s}\, ds \ . \tag{28}$$

- Finally, note that the call price above can be written

$$C_0 = P_0^T [F\mathcal{P}^{(S)}\{S_T > K\} - K\mathcal{P}^{(T)}\{S_T > K\}] \ , \tag{29}$$

  where $\frac{d\mathcal{P}^{(S)}}{d\mathcal{P}^{(T)}} = \frac{S_T}{F}$, which is a generalization of the Black-Scholes formula.

Problem Set 3

1. Derive the (Margrabe) valuation formula for a European option to exchange asset 1 for asset 2 under the assumption that each asset's dividend rate and volatility is nonstochastic. Describe the dynamics of the replicating trading strategy.

2. Derive the (Merton) European call option valuation formula in the case that both the underlying stock and the zero-coupon bond maturing on the option expiration date have nonstochastic volatility and the stock has a nonstochastic dividend rate. Determine the replicating trading strategy.

3. Suppose the market coefficients are constant. Derive the (Black-Scholes) European call and put option valuation formulas (with dividends) and the replicating trading strategies.