Term Structure Models

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Readings and References

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Summary of the Continuous-Time Financial Market

- Security prices satisfy $\frac{dS_{0,t}}{S_{0,t}} = r_t\,dt$ and $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t})\,dt + \sigma_{k,t} dB_t$.
- Given tight tr. strat. π_t and consumption c_t, portfolio value X_t satisfies the
  - WEE: $dX_t = r_t X_t\,dt + \pi_t(\mu_t - r_t 1)\,dt + \pi_t \sigma_t\,dB_t - c_t\,dt$.
- No arbitrage $\Rightarrow$ if $\pi_t \sigma_t = 0$ then $\pi_t(\mu_t - r_t 1) = 0 \Rightarrow \exists \theta_t$ s.t. $\sigma_t \theta_t = \mu_t - r_t 1$ $\Rightarrow dX_t = r_t X_t\,dt + \pi_t \sigma_t(\theta_t\,dt + dB_t) - c_t\,dt$.
- Under emm $\mathcal{P}^*$ given by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ where $Z_t = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds}$,
  - $B_t^* = B_t + \int_0^t \theta_s\,ds$ is Brownian motion.

  Let $\beta_t = e^{-\int_0^t r_s\,ds}$ and sdf process $M_t = \beta_t Z_t$. Then the WEE can also be written:
  - WEE*: $d\beta_t X_t + \beta_t c_t\,dt = \beta_t \pi_t \sigma_t\,dB_t^*$
  - WEE-M: $dM_t X_t + M_t c_t\,dt = M_t[\pi_t \sigma_t - \theta_t X_t]\,dB_t$
- So $X_t = \mathrm{E}_t^*\{\int_t^T \frac{\beta_u}{\beta_t} c_u\,du + \frac{\beta_T}{\beta_t} X_T\} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t} c_u\,du + \frac{M_T}{M_t} X_T\}$ if π is mtgale-gen.
- If σ is nonsingular, every c.plan (c, X_T) can be generated by a mtgale-gen. tr.strat.

Zero-Coupon Bond Prices and Yields

- Now let's go through some classic models of the prices of default-free bonds.
- Let P_t^T be the time t price of the zero maturing at time T and y_t^T be its yield.
- Suppose we're in a complete market. We can represent zero prices in various ways:

$$P_t^T \equiv e^{-y_t^T (T-t)} = \mathrm{E}_t^*\{e^{-\int_t^T r_u\,du}\} = \mathrm{E}_t\{\frac{M_T}{M_t}\} \ . \qquad (1)$$

- For the purpose of modeling bond prices and yields, we can work entirely under the risk-neutral measure $\mathcal{P}^*$.
- In order to characterize expected bond returns, we need to work under the true measure $\mathcal{P}$.

The Vasicek Model

- In the Vasicek (1977) model, the instantaneous riskless rate r_t is modeled as a constant-volatility mean-reverting *Ornstein-Uhlenbeck process*,

$$dr_t = \kappa(\bar{r} - r_t)\,dt + \sigma\,dB^* \ , \tag{2}$$

where $\bar{r}$ is the "long-run mean" of r_t and $\kappa > 0$ is its "speed of mean-reversion."
- The Ornstein-Uhlenbeck process is the continuous-time analogue of the discrete-time AR(1) process.
- The solution of (2) for any $u \geq t \geq 0$ is

$$r_u = e^{-\kappa(u-t)} r_t + (1 - e^{-\kappa(u-t)})\bar{r} + \sigma \int_t^u e^{-\kappa(u-s)}\,dB^*_s, \tag{3}$$

so r_u is normally distributed with conditional mean and variance

$$\mathrm{E}^*_t\{r_u\} = e^{-\kappa(u-t)} r_t + (1 - e^{-\kappa(u-t)})\bar{r} \ \text{ and} \tag{4}$$

$$\mathrm{var}^*_t\{r_u\} = \sigma^2 \int_t^u e^{-2\kappa(u-s)}\,ds = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(u-t)}) \ . \tag{5}$$

- Normality of r_u means negative interest rates are possible in this model.

- It follows from $P_t^T \equiv e^{-y_t^T(T-t)} = \mathrm{E}^*_t\{e^{-\int_t^T r_u\,du}\}$ that zero yields are "affine" in r_t, and thus also normal:

$$y_t^T = a(T-t) + b(T-t) r_t, \ \text{ where} \tag{6}$$

$$a(\tau) = \bar{r} - \frac{\sigma^2}{2\kappa^2} + (\frac{\sigma^2 - \bar{r}\kappa^2}{\kappa^3})(\frac{1 - e^{-\kappa\tau}}{\tau}) - \frac{\sigma^2}{4\kappa^3}(\frac{1 - e^{-2\kappa\tau}}{\tau}) \ \text{ and} \tag{7}$$

$$b(\tau) = \frac{1}{\kappa}(\frac{1 - e^{-\kappa\tau}}{\tau}) \ . \tag{8}$$

- The derivation of this result from Back (2010) is as follows:

$$\begin{aligned}\int_t^T r_u\,du &= (T-t)\bar{r} - (\bar{r} - r_t)\int_t^T e^{-\kappa(u-t)}\,du + \sigma \int_t^T \int_t^u e^{-\kappa(u-s)}\,dB^*_s\,du \\ &= (T-t)\bar{r} - \frac{1}{\kappa}(\bar{r} - r_t)(1 - e^{-\kappa(T-t)}) + \sigma \int_t^T \int_t^u e^{-\kappa(u-s)}\,dB^*_s\,du \ .\end{aligned}$$

- Switching the order of integration in the last integral gives

$$\sigma \int_t^T \int_t^u e^{-\kappa(u-s)}\,dB^*_s\,du = \sigma \int_t^T \int_s^T e^{-\kappa(u-s)}\,du\,dB^*_s = \frac{\sigma}{\kappa}\int_t^T (1 - e^{-\kappa(T-s)})\,dB^*_s \ .$$

- The integrand of this stochastic integral is nonrandom, so the integral is normal with mean zero and variance

$$\frac{\sigma^2}{\kappa^2}\int_t^T (1-e^{-\kappa(T-s)})^2\,ds = (T-t)\frac{\sigma^2}{\kappa^2}-\frac{2\sigma^2}{\kappa^3}(1-e^{-\kappa(T-t)})+\frac{\sigma^2}{2\kappa^3}(1-e^{-2\kappa(T-t)})\ .$$

- Thus $-\int_t^T r_u\,du$ is normal with

$$\mathrm{E}_t^*\{-\int_t^T r_u\,du\} = -(T-t)\bar{r}+\frac{1}{\kappa}(\bar{r}-r_t)(1-e^{-\kappa(T-t)}) \quad \text{and} \tag{9}$$

$$\mathrm{var}_t^*\{-\int_t^T r_u\,du\} = (T-t)\frac{\sigma^2}{\kappa^2}-\frac{2\sigma^2}{\kappa^3}(1-e^{-\kappa(T-t)})+\frac{\sigma^2}{2\kappa^3}(1-e^{-2\kappa(T-t)})\ , \tag{10}$$

- and it follows from the usual rule for expectations of exponentials of normals that

$$\log P_t^T = \log \mathrm{E}_t^*\{e^{-\int_t^T r_u\,du}\} = -(T-t)\bar{r}+\frac{1}{\kappa}(\bar{r}-r_t)(1-e^{-\kappa(T-t)}) + \frac{1}{2}[(T-t)\frac{\sigma^2}{\kappa^2}-\frac{2\sigma^2}{\kappa^3}(1-e^{-\kappa(T-t)})+\frac{\sigma^2}{2\kappa^3}(1-e^{-2\kappa(T-t)}))]\ . \tag{11}$$

- Homework: Verify that P_t^T satisfies the fundamental PDE for derivative prices.

The Cox-Ingersoll-Ross Model

- In the Cox-Ingersoll-Ross (1985) model, the instantaneous riskless rate r_t is modeled as a mean-reverting *square-root process*,

$$dr_t = \kappa(\bar{r}-r_t)\,dt+\sigma\sqrt{r_t}\,dB^*\ , \tag{12}$$

where $\bar{r}>0$ is the long-run mean of r_t, $\kappa>0$ is its speed of mean-reversion, and the solution to (12) is nonnegative for all t, so it is possible to take the square root. If $\kappa\bar{r}\geq\sigma^2/2$, then r_t is a.s. strictly positive for all t.

- The exact solution to (12) is more complicated than the solution to the O-U process in Vasicek, so we derive zero prices by solving their fundamental PDE.
- Since the square-root process (12) is Markov, zero prices are functions of r_t:

$$P_t^T = \mathrm{E}_t^*\{e^{-\int_t^T r_u\,du}\} = p^T(r_t,t)\ . \tag{13}$$

- Assuming the price function p^T is smooth enough for an application of Itô's lemma,

$$\kappa(\bar{r}-r)p_r^T+\frac{1}{2}\sigma^2 r p_{rr}^T+\frac{\partial P^T}{\partial t} = rp^T \quad \text{s.t.} \quad p^T(r,T)=1\ . \tag{14}$$

- The solution to (14) has yields affine in r_t:

$$p^T(r_t, t) = e^{-\alpha(T-t)-\beta(T-t)r_t} \text{ and } y_t^T = a(T-t) + b(T-t)r_t \ . \tag{15}$$

  where $a(\tau) = \alpha(\tau)/\tau$ and $b(\tau) = \beta(\tau)/\tau$ and

$$\alpha(\tau) = -\frac{2\kappa\bar{r}}{\sigma^2}[\frac{(\kappa+\gamma)\tau}{2} + \log\frac{2\gamma}{c(\tau)}] \text{ and } \beta(\tau) = \frac{2(e^{\gamma\tau}-1)}{c(\tau)} \ , \tag{16}$$

  where $c(\tau) = 2\gamma + (\kappa+\gamma)(e^{\gamma\tau}-1)$ and $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$.
- To verify this solution, note that (15) implies $p_r = -\beta p$, $p_{rr} = \beta^2 p$, and $p_t = (\alpha' + r\beta')p$, so the fundamental PDE holds if and only if

$$-\kappa(\bar{r} - r)\beta + \frac{1}{2}\sigma^2\beta^2 r + \alpha' + \beta' r = r \ . \tag{17}$$

  This holds for all values of r if and only if

$$\kappa\beta + \frac{1}{2}\sigma^2\beta^2 + \beta' = 1 \text{ and } \alpha' = \kappa\bar{r}\beta \ . \tag{18}$$

  The ODE for β above is said to be of the *Riccati* type.

- The terminal condition $p^T(r, T) = 1$ is equivalent to $\alpha(0) = \beta(0) = 0$.
- Differentiating β in (16) shows it is a solution to (18).
- Given β, integrating α' in (18) gives α in (16).

The Multifactor CIR Model

We can get a multifactor version of the CIR model by taking r to be the sum of square-root processes: $r_t = X_{1t} + X_{2t}$ where

$$dX_{it} = \kappa_i(\bar{X}_i - X_{it})\,dt + \sigma_i\sqrt{X_{it}}\,dB_i^* \ . \tag{19}$$

- Independence of the X_i implies

$$p_t^T = \mathrm{E}_t^*\{e^{-\int_t^T r_u\,du}\} = \mathrm{E}_t^*\{e^{-\int_t^T X_{1u}\,du}\}\mathrm{E}_t^*\{e^{-\int_t^T X_{2u}\,du}\} \tag{20}$$
$$= e^{-\alpha_1(T-t)-\alpha_2(T-t)-\beta_1(T-t)X_{1t}-\beta_2(T-t)X_{2t}} \tag{21}$$

  where the α_i and β_i are defined as in the single-factor CIR model.

- This implies $y_t^T = y_{1t}^T + y_{2t}^T$ where $y_{it}^T = \frac{\alpha_i(\tau)}{\tau} + \frac{\beta_i(\tau)}{\tau}X_{it}$.

Other Multifactor Models Similarly, we could construct other multifactor interest rate processes as sums of independent processes, and get product pricing formulas if price formulas exist for each independent process.

Affine Term Structure Models

In an n-factor affine term structure model, the short rate r is affine in n Markov state variables X whose drift and instantaneous covariance matrix are also affine in X:

$$r_t = \delta_0 + \delta' X_t \quad \text{and} \quad dX_t = (\phi + KX_t)\,dt + \sigma(X_t)\,dB^*, \quad \text{where} \tag{22}$$

- B^* is a vector of n independent Brownian motions under $\mathcal{P}^*$,
- δ_0 is a constant, δ and ϕ are a constant n-vectors, K is a constant $n \times n$ matrix,
- σ is an $n \times n$ matrix-valued function s.t. each element of the covariance matrix $\sigma(x)\sigma(x)'$ is affine in x. Parametric assumptions are made to ensure the diagonal elements of $\sigma(x)\sigma(x)'$ are nonnegative and uniqueness of the solution for X.
- Both the Vasicek and CIR models are examples of affine models.
- If σ is constant, then the model is Gaussian, in the sense that conditional on X_t, (r_u, X_u) is multivariate normal for all $u \geq t$.
- It can be shown that in any two-factor Gaussian model, the two factors can be taken to be the short rate and its drift.

- In an affine model, zero prices are exponential-affine in X and yields are affine:

$$p^T(x,t) = e^{-\alpha(T-t)-\sum_{i=1}^n \beta_i(T-t)x_i} \text{ and } y_t^T = a(T-t) + \sum_{i=1}^n b_i(T-t)x_i \ , \quad (23)$$

as shown by verifying the fundamental PDE for p with BC $\alpha(0) = \beta_i(0) = 0$.

- The β_i solve a multi-dimensional ODE, for which no closed-form is available in general, and α can be solved by integrating the β_i.
- A useful feature of affine models is that one can take a vector of zero yields to be the factors. For the vector Y of n yields for fixed times to maturity $\tau_1, \ldots, \tau_n$, let $\mathcal{A}$ be the vector of the $a(\tau_i)$ and $\mathcal{B}$ the matrix of the $b_j(\tau_i)$. Then

$$Y_t = \mathcal{A} + \mathcal{B}X_t \ \Rightarrow \ X_t = \mathcal{B}^{-1}(Y_t - \mathcal{A}) \ , \quad (24)$$

provided $\mathcal{B}$ is invertible. Substituting $X_t = \mathcal{B}^{-1}(Y_t - \mathcal{A})$ and $dX = \mathcal{B}^{-1}\,dY_t$ produces an affine model with the n yields as factors.

- Similarly, one could use $n-1$ zero yields and the short rate as affine factors.

- To get the fundamental PDE, write $P_t^T = e^{-Y_t}$ where $Y = \alpha(T-t) + \beta(T-t)^\top X_t$
- From Itô's lemma

$$\begin{aligned} \frac{dP}{P} &= -dY + \frac{1}{2}(dY)^2 = \alpha'(T-t)\,dt + \beta'(T-t)^\top X\,dt - \beta(T-t)^\top\,dX \\ &\quad + \frac{1}{2}\beta(T-t)^\top(dX)(dX)^\top\beta(T-t)\,dt \quad (25) \\ &= \alpha'(T-t)\,dt + \beta'(T-t)^\top X\,dt - \beta(T-t)^\top[(\phi + KX)\,dt + \sigma(X)\,dB^*] \\ &\quad + \frac{1}{2}\beta(T-t)^\top\sigma(X)\sigma(X)^\top\beta(T-t)\,dt \ . \quad (26) \end{aligned}$$

- Setting the appreciation rate of P equal to the short rate $r = \delta_0 + \delta' X$ gives:

$$\begin{aligned} \delta_0 + \delta' X &= \alpha'(T-t) + \beta'(T-t)^\top X - \beta(T-t)^\top(\phi + KX) \quad (27) \\ &\quad + \frac{1}{2}\beta(T-t)^\top\sigma(X)\sigma(X)^\top\beta(T-t) \ . \quad (28) \end{aligned}$$

- Equating the coefficients of the X_i on each side gives a system of n ODEs in the functions β_i, called *Riccati* equations because they are affine in the β_i' and quadratic in the β_i.
- Matching constant terms on each side, given the β_i, determines α', which can be integrated from $\alpha(0) = 0$ to give α.

Completely Affine Models

In a completely affine model, factor dynamics are affine under both the risk-neutral measure $\mathcal{P}^*$ and the true measure $\mathcal{P}$. This is set up as follows.

- Assume $\sigma(X)$ in the affine model (23) is of the form $\sigma S(X)$ where σ is a constant $n \times n$ matrix and S is a diagonal $n \times n$ matrix-valued function with each squared diagonal element being an affine function of X. Then each element of the covariance matrix $\sigma(x)\sigma(x)' = \sigma S(x)^2\sigma'$, is an affine function of x.
- Assume the market price of risk θ in the Radon-Nikodym derivative $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z(T) = e^{-\int_0^T \theta_u\, dB_u - \frac{1}{2}\int_0^t |\theta_u|^2\, du}$ is of the form $\theta_t = S(X_t)\theta$ for a constant vector θ.
- Then $dB_t^* = dB_t + S(X)\theta\, dt$ and the model is also affine under $\mathcal{P}$:

$$dX_t = (\phi + KX_t)\, dt + \sigma S(X_t)\, dB^* \tag{29}$$
$$= [\phi + KX_t + \sigma S(X_t)^2\theta]\, dt + \sigma S(X_t)\, dB_t \tag{30}$$
$$= (\widehat{\phi} + \widehat{K}X)\, dt + \sigma S(X_t)\, dB_t \tag{31}$$

  for some constant vector $\widehat{\phi}$ and matrix $\widehat{K}$, because $S(X)^2$ is diagonal with affine functions of X on the diagonal.

Bond Risk Premia

In a completely affine model, bond risk premia are also affine in X.

- To see this, note from (26) that the volatility vector of the zero price P^T is $-\beta(T-t)'\sigma S(X_t)$.
- Thus, the risk premium on the $(T-t)$-year zero is

$$\mu_t^T - r_t = -\beta(T-t)'\sigma S(X_t)^2\theta\ , \tag{32}$$

  which is affine in X_t.
- In a Gaussian completely affine model, $S(X)$ is a constant matrix, so the risk premium of a zero depends only on its time to maturity.

Inflation and Nominal Asset Prices

- Until now, we have been working with real prices $S_{k,t}$, in units of the consumption good, and the sdf M_t for these real prices. When we go to the data, we can in some contexts ignore inflation, but it's an issue when pricing nominal bonds, i.e., claims to a dollar rather than a unit of consumption.
- Let q_t be the price level, e.g., dollars per unit of consumption, sometimes measured by CPI.
- Then nominal asset prices are $q_t S_{k,t}$.
- Therefore, the sdf for nominal asset prices is $M_t^q \equiv \frac{M_t}{q_t}$, since $M_t S_t = \mathrm{E}_t\{M_u S_u\}$ implies $M_t^q q_t S_t = \mathrm{E}_t\{M_u^q q_u S_u\}$.
- Suppose $\frac{dq_t}{q_t} = \iota_t + \sigma_{q,t}\, dB_t$, or $q_t = e^{\int_0^t \iota_s\, ds + \int_0^t \sigma_{q,s}\, dB_s - \frac{1}{2}\int_0^t |\sigma_{q,s}|^2\, ds}$.
- Interpret ι_t as expected inflation over the period and $\frac{dq_t}{q_t}$ as realized inflation.

Locally Riskless Inflation

- If $\sigma_{q,t} = 0$, i.e., inflation is locally riskless or "known at the beginning of the period," then nominal asset prices $q_t S_{k,t}$ follow

$$\frac{dq_t S_{k,t}}{q_t S_{k,t}} = (\mu_{k,t} + \iota_t - \delta_{k,t})\, dt + \sigma_{k,t}\, dB_t \ , \tag{33}$$

- the sdf for nominal prices is $M_t^q = \frac{M_t}{q_t} = e^{-\int_0^t (r_s + \iota_s)\, ds - \int_0^t \theta_s'\, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds}$, and
- the nominal riskless rate, i.e., the nominal return on a locally riskless nominal money market account is $r_t + \iota_t$.
- In this case, nominal excess expected returns $(\mu_{k,t} + \iota_t) - (r_t + \iota_t)$ are equal to real excess returns $\mu_{k,t} - r_t$, so theory developed for real excess returns may be able to be applied directly to empirical work with nominal excess returns.
- The price of a "nominal bond" paying a dollar at time T is

$$P_t^{q,T} = \mathrm{E}_t\{\frac{M_T^q}{M_t^q} 1\} = \mathrm{E}_t^*\{e^{-\int_t^T (r_s + \iota_s\}\, ds}\} \ . \tag{34}$$

If both r_t and ι_t are affine in a set of affine state variables, then bond prices will again be exponentially affine, $p^{q,T}(x,t) = e^{-\alpha(T-t) - \sum_{i=1}^n \beta_i(T-t) x_i}$.

Inflation with Shocks

- If $\sigma_{q,t} \neq 0$, i.e., if realized inflation over the period contains both its expectation $\iota_t\, dt$ and a shock $\sigma_{q,t}\, dB_t$, then nominal asset prices $q_t S_{k,t}$ follow

$$\frac{dq_t S_{k,t}}{q_t S_{k,t}} = \frac{dS_{k,t}}{S_{k,t}} + \frac{dq_t}{q_t} + (\frac{dq}{q})(\frac{dS_k}{S_k}) \tag{35}$$

$$= (\mu_{k,t} + \iota_t + \sigma_{k,t}\sigma'_{q,t} - \delta_{k,t})\, dt + (\sigma_{k,t} + \sigma_{q,t})\, dB_t \ , \tag{36}$$

- the sdf for nominal prices is

$$M_t^q = \frac{M_t}{q_t} = e^{-\int_0^t (r_s + \iota_s)\, ds - \int_0^t (\theta'_s + \sigma_{q,s})\, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds + \frac{1}{2}\int_0^t |\sigma_{q,s}|^2\, ds} \ , \tag{37}$$

$$= e^{-\int_0^t (r_s - \sigma_{q,s}\theta_s + \iota_s - |\sigma_{q,s}|^2)\, ds - \int_0^t (\theta'_s + \sigma_{q,s})\, dB_s - \frac{1}{2}\int_0^t |\theta_s + \sigma'_{q,s}|^2\, ds} \ , \tag{38}$$

- and it follows from the form of M_t^q that the nominal riskless rate, the rate on a locally riskless nominal money market account, is $r_t - \sigma_{q,t}\theta_t + \iota_t - |\sigma_{q,t}|^2$.

The Nominal Riskless Rate and the Inflation Risk Premium

- To get to this more intuitively, note that the nominal money market account, which delivers a locally riskless nominal return and zero shocks, must have shocks to its real returns that are exactly opposite the shocks to the price level. In other words, if the nominal money market is security k^*, with real price S_{k^*}, its real return shocks must by equation (36) have loadings σ_{k^*} identically equal to $-\sigma_q$.
- Thus the nominal money market's real risk premium $\mu_{k^*,t} - r_t$ must be $\sigma_{k^*}\theta = -\sigma_q\theta$, which it must pay because its real return in units of consumption is risky.
- As we will see shortly, this $\sigma_q\theta$ gives rise to the so-called "inflation risk premium" in the term structure of nominal bond yields.
- As we will see later, in equilibrium $\sigma_q\theta$ is the instantaneous covariance between shocks to inflation and shocks to aggregate consumption, times the relative risk aversion of the representative agent. Estimates suggest that this has been positive historically, though perhaps not since the crisis.

- Now to get the nominal riskless rate, note that by equation (36) with $\sigma_{k^*} = -\sigma_q$, the nominal money market account's nominal price follows

$$\frac{dq_t S_{k^*,t}}{q_t S_{k^*,t}} = (\mu_{k^*,t} + \iota_t + \sigma_{k^*,t}\sigma'_{q,t})\,dt + (\sigma_{k^*,t} + \sigma_{q,t})\,dB_t \tag{39}$$

$$= (r_t - \sigma_{q,t}\theta_t + \iota_t - |\sigma_{q,t}|^2)\,dt\ . \tag{40}$$

- Therefore, the nominal riskless rate is $r_t - \sigma_{q,t}\theta_t + \iota_t - |\sigma_{q,t}|^2$.
- To gain economic intuition for all this, let's redefine inflation as the rate of decline in the consumption price of dollars $\frac{1}{q}$, rather than the rate of increase in the dollar price of consumption q or CPI, that is more typically quoted in practice. Viewing currency (e.g., dollars) as an asset priced in units of consumption will clarify the economics of its real return and the real returns of nominal assets more generally.
- By Itô's lemma,

$$d\frac{1}{q_t} = -\frac{1}{q_t^2}\,dq_t + \frac{1}{2}\frac{2}{q_t^3}\,(dq) = -\frac{1}{q_t}(\iota_t\,dt + \sigma_{q,t}\,dB_t) + \frac{1}{q_t}|\sigma_{q,t}|^2\,dt \tag{41}$$

- so $d\frac{1/q_t}{1/q_t} = (-\iota_t + |\sigma_{q,t}|^2)\,dt - \sigma_{q,t}\,dB_t = -i_t\,dt + \sigma_{1/q,t}\,dB_t$, where $i_t = \iota_t - |\sigma_{q,t}|^2$ is the rate of decline in the real price of dollars, our alternative measure of inflation, and $\sigma_{1/q,t} = -\sigma_{q,t}$ is the volatility of the real price of dollars.
- Then we can see the nominal riskless rate as $r_t + i_t + \sigma_{1/q,t}\theta_t$, the real riskless rate on rate r_t plus compensation for the decline in the real price of dollars i_t plus compensation for the risk of the real price of dollars $\sigma_{1/q,t}\theta_t$.
- If shocks to inflation and aggregate consumption are positively correlated, i.e., $\sigma_q\theta > 0$, then the nominal money market is a hedge against negative consumpion shocks, because it pays dividends in dollars, which are worth more in real terms, when inflation and consumption are down. In that case, it makes sense that the real risk premium on the nominal money market is negative, $\sigma_{1/q}\theta = -\sigma_q\theta < 0$.

Nominal Bond Prices

- The nominal price of the nominal zero paying a dollar at time T is

$$P_t^{q,T} = \mathrm{E}_t\{\frac{M_T^q}{M_t^q}1\} = \mathrm{E}_t\{e^{-\int_t^T (r_s - \sigma_{q,s}\theta_s + \iota_s - |\sigma_{q,s}|^2)\, ds - \int_t^T (\theta_s' + \sigma_{q,s})\, dB_s - \frac{1}{2}\int_t^T |\theta_s + \sigma_{q,s}'|^2\, ds}\} \tag{42}$$

$$= \mathrm{E}_t^q\{e^{-\int_t^T (r_s - \sigma_{q,s}\theta_s + \iota_s - |\sigma_{q,s}|^2)\, ds}\} \tag{43}$$

under the risk-neutral measure for nominal prices $\mathcal{P}^q$ given by

$$\frac{d\mathcal{P}^q}{d\mathcal{P}} = e^{-\int_0^t (\theta_s' + \sigma_{q,s})\, dB_s - \frac{1}{2}\int_0^t |\theta_s + \sigma_{q,s}'|^2\, ds}\,, \tag{44}$$

under which $B_t^q \equiv B_t + \int_0^t (\theta_s' + \sigma_{q,s})\, ds$ is a standard Brownian motion.

- Therefore, if r_t and ι_t are affine in a set state variables X_t that are affine under $\mathcal{P}^q$, and if $\sigma_{q,t} = \sigma_q S(X_t)$ and $\theta_t = S(X_t)\theta$, where σ_q and θ constant vectors and $S(X)$ a diagonal matrix as in the completely affine model, then the nominal riskless rate $r_t - \sigma_{q,t}\theta_t + \iota_t - |\sigma_{q,t}|^2$ will be affine in the state variables and nominal zero prices will again be exponentially affine, $p^{q,T}(x,t) = e^{-\alpha(T-t) - \sum_{i=1}^n \beta_i(T-t)x_i}$.

Nominal Risk Premia

- Note that under $\mathcal{P}^q$, nominal expected returns on all assets are equal to the nominal riskless rate, as can be seen by substituting $dB_t^q - (\theta_t + \sigma_{q,t})\, dt$ for dB_t in equation (36):

$$\frac{dq_t S_{k,t}}{q_t S_{k,t}} = (\mu_{k,t} + \iota_t + \sigma_{k,t}\sigma_{q,t}' - \delta_{k,t})\, dt + (\sigma_{k,t} + \sigma_{q,t})\, dB_t \tag{45}$$

$$= (r_t - \sigma_{q,t}\theta_t + \iota_t - |\sigma_{q,t}|^2 - \delta_{k,t})\, dt + (\sigma_{k,t} + \sigma_{q,t})\, dB_t^q\,. \tag{46}$$

This justifies calling $\mathcal{P}^q$ a risk-neutral measure for nominal returns.

- Nominal risk premia, expected nominal returns minus the nominal riskless rate, are

$$(\mu_{k,t} + \iota_t + \sigma_{k,t}\sigma_{q,t}') - (r_t - \sigma_{q,t}\theta_t + \iota_t - |\sigma_{q,t}|^2) \tag{47}$$

$$= \mu_{k,t} - r_t + \sigma_{q,t}\theta_t + |\sigma_{q,t}|^2 + \sigma_{k,t}\sigma_{q,t}' \tag{48}$$

$$= \sigma_{k,t}^q(\theta_t + \sigma_{q,t}') \tag{49}$$

where $\sigma_{k,t}^q \equiv \sigma_{k,t} + \sigma_{q,t}$ is the vector of security k's nominal return risk loadings, and $\theta + \sigma_q'$ is the mpr paid by nominal returns.

- Now the nominal risk premia differ from the real risk premia $\mu_{k,t} - r_t$.