# American Options

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## Readings and References


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## Summary of the Continuous-Time Financial Market

- Security prices satisfy $\frac{dS_{0,t}}{S_{0,t}} = r_t\,dt$ and $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t})\,dt + \sigma_{k,t}dB_t$.
- Given tight tr. strat. $\pi_t$ and consumption $c_t$, portfolio value $X_t$ satisfies the
  - WEE: $dX_t = r_t X_t\,dt + \pi_t(\mu_t - r_t 1)\,dt + \pi_t\sigma_t\,dB_t - c_t\,dt$.
- No arbitrage $\Rightarrow$ if $\pi_t\sigma_t = 0$ then $\pi_t(\mu_t - r_t 1) = 0 \Rightarrow \exists\theta_t$ s.t. $\sigma_t\theta_t = \mu_t - r_t 1$ $\Rightarrow dX_t = r_t X_t\,dt + \pi_t\sigma_t(\theta_t\,dt + dB_t) - c_t\,dt$.
- Under emm $\mathcal{P}^*$ given by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ where $Z_t = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds}$,
  - $B_t^* = B_t + \int_0^t \theta_s\,ds$ is Brownian motion.

  Let $\beta_t = e^{-\int_0^t r_s\,ds}$ and sdf process $M_t = \beta_t Z_t$. Then the WEE can also be written:
  - WEE*: $d\beta_t X_t + \beta_t c_t\,dt = \beta_t\pi_t\sigma_t\,dB_t^*$
  - WEE-M: $dM_t X_t + M_t c_t\,dt = M_t[\pi_t\sigma_t - \theta_t X_t]\,dB_t$
- So $X_t = \mathrm{E}_t^*\{\int_t^T \frac{\beta_u}{\beta_t}c_u\,du + \frac{\beta_T}{\beta_t}X_T\} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t}c_u\,du + \frac{M_T}{M_t}X_T\}$ if $\pi$ is mtgale-gen.
- If $\sigma$ is nonsingular, every c.plan $(c, X_T)$ can be generated by a mtgale-gen. tr.strat.

## Elements and Examples of American Options

Assume we're in a complete continuous-time financial market.

**Definition 1** An *American contingent claim* (acc) associated with a continuous, adapted *payoff process* $G_t$ and *expiration date* $T$ is a claim to the payoff $G_\tau$ at the stopping time $\tau \le T$ chosen by the holder of the claim. A stopping time $\tau$ defines an *exercise policy* for the option.

### Examples of Payoff Processes

- American call: $G_t = (S_t - K)^+$
- American put: $G_t = (K - S_t)^+$
- Levered equity's default option: $G_t = (P_t - V_t)^+$ where $P_t$ is the time $t$ pv of the nondefaultable remaining debt payments and $V_t$ is the value of the firm assets.
- Option embedded in a nondefaultable callable bond: $G_t = (P_t - K_t)^+$ where $P_t$ is the pv of the remaining noncallable, nondefaultable bond payments and $K_t$ is the time $t$ contractual call price.

- Option embedded in a defaultable callable bond: $G_t = (P_t - V_t \wedge K_t)^+$ where $P_t$ is the time $t$ pv of the remaining noncallable, nondefaultable bond payments, $V_t$ is the time $t$ value of firm assets, and $K_t$ is the time $t$ contractual call price.
- Prepayment option embedded in a fixed rate mortgage: $G_t = (P_t - V_t \wedge PB_t)^+$ where $P_t$ is the time $t$ present value of the remaining mortgage payments, $V_t$ is the time $t$ value of the real estate collateral, and $PB_t$ is the time $t$ remaining principal balance.

**Assumption** The discounted payoff process $\beta_t G_t$ is bounded below uniformly in $t \in [0, T]$ and satisfies $\mathrm{E}^*\{\sup_{0 \le t \le T} \beta_t G_t\} < \infty$.

## Optimal Exercise Policy

- For any given exercise policy, $\tau$, the claim to $G_\tau$ at time $\tau$ can be replicated by a tight, martingale-generating trading strategy, since we're in a complete market.
- For example, given $\tau$, the ecc with time $T$ payoff $x = G_\tau e^{\int_\tau^T r_s\, ds}$ can be replicated.
- The replication cost, or value of the option under exercise policy $\tau$ is

$$V_t^\tau = \mathrm{E}_t^*\{\frac{\beta_\tau}{\beta_t} G_\tau\} \ . \tag{1}$$

- In standard option theory, we assume that the acc holder can buy or sell the option freely, or equivalently, hedge or synthesize the option freely, so that the acc holder's objective is to maximize the value of the option, regardless of his or her risk preferences or beliefs.

**Definition 2** An *optimal exercise policy* $\tau^*$, if it exists, is the solution to the *optimal stopping problem*

$$V_0^* = \sup_{0 \le \tau \le T} V_0^\tau = \sup_{0 \le \tau \le T} \mathrm{E}^*\{\beta_\tau G_\tau\} \tag{2}$$

## Existence of an Optimal Policy and Replication of the ACC

Establishing that $V_0^*$ is the no-arbitrage price of the acc is complicated by the fact that hedging a short position in an acc must deal with uncertainty about the holder's exercise policy.

**Definition 3** A *super-replicating trading strategy* is a tight, martingale-generating trading strategy $\pi$ satisfying $\bar{\pi}_\tau 1 \geq G_\tau$ for all $0 \leq \tau \leq T$.

**Definition 4** The *value of the acc* at time 0 is

$$V_0^{acc} = \inf\{\bar{\pi}_0 1 : \bar{\pi} \text{ is a super replicating trading strategy}\} . \qquad (3)$$

**Definition 5** A *replicating trading strategy* $\pi^*$ for the acc is a super-replicating trading strategy s.t. $\bar{\pi}_0^* 1 = V_0^{acc}$.

**Theorem 1** $V_0^{acc} = V_0^* = \sup_{0 \leq \tau \leq T} \mathrm{E}^*\{\beta_\tau G_\tau\}$. Furthermore, there exists a stopping time $\tau^*$ attaining this supremum and there exists a replicating trading strategy $\pi^*$ s.t. $\beta_{\tau^*} G_{\tau^*} = V_0^{acc} + \int_0^{\tau^*} \beta_t \pi_t^* \sigma_t \, dB_t^*$.

Therefore, $V_0^{acc} = V_0^*$ is the no-arbitrage price of the acc.

**Lemma (Snell Envelope)** Let $X$ be a continuous, adapted process bounded below.

- Then there exists a RCLL supermartingale $\zeta$ called the *Snell envelope* of $X$ s.t. $\zeta_t \geq X_t \; \forall t \in [0, T]$ a.s., and $\zeta_t = \sup_{t \leq \tau \leq T} \mathrm{E}_t\{X_\tau\}$.
- Moreover, for the stopping time $\tau^* = \inf\{t \in [0, T] : \zeta_t = X_t\}$, $\zeta_0 = \mathrm{E}\{X_{\tau^*}\}$ and $\zeta$ is a martingale on $[0, \tau^*]$.
- $\zeta$ has Doob-Meyer decomposition $\zeta = M - \Lambda$, where $M$ is a martingale and $\Lambda$ is an increasing process with $\Lambda_0 = \Lambda_{\tau^*} = 0$.

**Proof of Theorem 1** Let $\zeta = M - \Lambda$ be the $\mathcal{P}^*$-Snell envelope of $\beta_t G_t$. Since the market is complete, $\exists$ a martingale-generating tr. strat. $\bar{\pi}^*$ s.t. $M_t = \int_0^t \beta_u \pi_u^* \sigma_u \, dB_u^*$. Thus $\beta_t G_t \leq \zeta_t = \zeta_0 + \int_0^t \beta_u \pi_u^* \sigma_u \, dB_u^* - \Lambda_t \leq \zeta_0 + \int_0^t \beta_u \pi_u^* \sigma_u \, dB_u^*$, which implies $\bar{\pi}^*$ is a super-replicating trading strategy starting from wealth $\zeta_0$, so $V_0^{acc} \leq \zeta_0$.

O.T.O.H., suppose $\bar{\pi}$ is a super-replicating trading strat. Then for every exercise policy $\tau$, $\beta_\tau \bar{\pi}_\tau 1 = \bar{\pi}_0 1 + \int_0^\tau \beta_u \pi_u \sigma_u \, dB_u^* \Rightarrow \mathrm{E}^*\{\beta_\tau G_\tau\} \leq \mathrm{E}^*\{\beta_\tau \bar{\pi}_\tau 1\} \leq \bar{\pi}_0 1$, which implies, by taking $\sup$ on the left-hand side and $\inf$ on the right, that $\zeta_0 \leq V_0^{acc}$.

Therefore, $V_0^{acc} = \zeta_0$, $\tau^* = \inf\{t \in [0, T] : \zeta_t = \beta_t G_t\}$ is an optimal policy, and $\pi^*$ is a replicating trading strategy for the acc.

**Proposition 1** If $\beta_t G_t$ is a $\mathcal{P}^*$-submartingale, then $\tau^* = T$ is an optimal exercise policy. (Prove for homework.)

**Corollary 1 (No early exercise of calls on non-dividend-paying stocks)** If $r \geq 0$, $\delta = 0$, and $\sigma$ is bounded, or satisfies the Novikov condition $\mathrm{E}^*\{e^{\int_0^T |\sigma_t|^2\, dt/2}\} < \infty$, then exercising the American call with payoff $G_t = (S_t - K)^+$ at expiration date $T$ is an optimal policy. In this case, the American option value equals the European option value.

It's hard to say much more about the acc analytically, or even numerically, when there are path-dependencies in security prices or the payoff function. To go further, we specialize to a path-independent setting.

## Markov Model with Path-Independent Payoffs

- In the Markov model, if $G_t = g(S_t, t)$ for a well-behaved function $g$, then

$$V_t^{acc} = \sup_{0 \leq \tau \leq T} \mathrm{E}_t^*\{\frac{\beta_\tau}{\beta_t} G_\tau\} = h(S_t, Y_t, t) \tag{4}$$

  for some function $h : \mathcal{R}_+^n \times \mathcal{R}^m \times [0, T] \to \mathcal{R}$ with $h(S_t, Y_t, t) \geq g(S_t, t)$.
- An optimal exercise policy is $\tau^* = \inf\{t \in [0, T] : h(S_t, Y_t, t) = g(S_t, t)\}$.
- So $h(S_t, Y_t, t) > g(S_t, t)\ \forall t < \tau^*$.

**Definition 6** The *exercise region* for this optimal stopping problem is

$$\mathcal{E} = \{(s, y, t) \in \mathcal{R}_+^n \times \mathcal{R}^m \times [0, T] : h(s, y, t) = g(s, t)\}$$

and its complement, the *continuation region* is

$$\mathcal{U} = \{(s, y, t) \in \mathcal{R}_+^n \times \mathcal{R}^m \times [0, T] : h(s, y, t) > g(s, t)\}\ .$$

The *exercise boundary* $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$.

## Value-Matching and Smooth-Pasting Conditions

The boundary condition $h = g$ on $\partial\mathcal{U}$ is called the *value-matching condition*. In fact, it must hold under any exercise policy that calls for stopping at a boundary.

Under the optimal exercise policy, an additional boundary condition, called the *smooth-pasting condition*, must hold.

For simplicity, let's restrict attention to the case of a one-dim'l state variable $S$.

**Definition 7** The *smooth-pasting condition* is the condition $h_s = g_s$ on $\partial\mathcal{U}$.

**Proposition 2** Smooth-pasting is necessary for optimality of the exercise boundary.

**Sketch of Proof** Given the Markov setting and the openness of $\mathcal{U}$, the search for an optimal stopping time is a search for an optimal exercise boundary $\partial\mathcal{U}$. Fix $t < T$. The $t$-section of $\mathcal{U}$, $\mathcal{U}_t \equiv \{s : (s,t) \in \mathcal{U}\}$ is an open set in $\mathcal{R}^+$, at worst a countable collection of open intervals, so the choice of optimal policy is a choice of endpoints. Let $a^*$ be an optimal endpoint. Let $J(s,t;a)$ be the value of a plan to exercise according to $\mathcal{U}$ everywhere except that at time $t$, the endpoint at $a^*$ is moved to $a$. $J(s,t;a^*) = h(s,t)$, so $J_s(s,t;a^*) = h_s(s,t)$. Since $a^*$ is optimal, $J_a(s,t;a^*) = 0\ \forall s$. Also, $J(a,t;a) = g(a,t)\ \forall a$, so $J_s(a,t;a) + J_a(a,t;a) = g_s(a,t)$, which implies $J_s(a^*,t;a^*) = g_s(a^*,t)$. This implies $h_s(a^*,t) = g_s(a^*,t)$.

Conversely, smooth-pasting is sufficient for optimality in the sense that if a function $\widehat{h}$ satisfies the smooth-pasting condition, together with the necessary martingale, value-matching, and payoff-dominating properties, then $\widehat{h}$ is the value of the acc, $h$.

**Proposition 3** Suppose $\widehat{h} : \mathcal{R}_+ \times [0,T] \to \mathcal{R}$ is $C^{2,1}$ on an open set $\widehat{\mathcal{U}}$, $g$ is $C^{2,1}$, $\widehat{h}$ satisfies $\widehat{h} > g$ on $\widehat{\mathcal{U}}$, $\widehat{h} = g$ on $\widehat{\mathcal{U}}^c$, $\widehat{h}_s = g_s$ on $\partial\widehat{\mathcal{U}}$, $\widehat{h}_s$ is bounded, $\mathcal{D}(\beta\widehat{h}) \le 0$, where $\mathcal{D}(\beta\widehat{h})$ denotes the drift of $\beta\widehat{h}$ given by Itô's lemma, and $\mathcal{D}(\beta\widehat{h}) = 0$ on $\widehat{\mathcal{U}}$. Then $\widehat{h} = h$ and $\widehat{\mathcal{U}} = \mathcal{U}$.

**Sketch of Proof** Because $\widehat{h}_s$ is bounded and $C^1$, Itô's lemma gives

$$\beta_u \widehat{h}(s_u,u) = \beta_t \widehat{h}(s_t,t) + \int_t^u \mathcal{D}(\beta\widehat{h})_v \, dv + \int_t^u \beta_v \widehat{h}_s(S_v,v)\sigma_v \, dB_v^* \,, \tag{5}$$

where the stochastic integral is $\mathcal{P}^*$-martingale. Let $\tau = \inf\{t : (S_t,t) \in \widehat{\mathcal{U}}^c\}$. Then $\mathrm{E}_t^*\{\beta_\tau g(S_\tau,\tau)\} = \mathrm{E}_t^*\{\beta_\tau \widehat{h}(S_\tau,\tau)\} = \beta_t \widehat{h}(S_t,t)$ from (5), which implies $\widehat{h} \le h$.

On the other hand, suppose $\widehat{\tau}$ is any other exercise policy. Then

$$\mathrm{E}_t^*\{\beta_{\widehat{\tau}} g(S_{\widehat{\tau}},\widehat{\tau})\} \le \mathrm{E}_t^*\{\beta_{\widehat{\tau}} \widehat{h}(S_{\widehat{\tau}},\widehat{\tau})\} = \beta_t \widehat{h}(S_t,t) + \mathrm{E}_t^* \int_t^{\widehat{\tau}} \mathcal{D}(\beta\widehat{h})(S_v,v) \, dv \le \beta_t \widehat{h}(S_t,t) \,,$$

which implies $\widehat{h} \ge h$. Thus, $\widehat{h} = h$ and $\widehat{\mathcal{U}} = \mathcal{U}$ and $\widehat{\tau} = \tau$.

## Optimal Stopping and the American Put (Jacka, 1991)

Suppose $n = d = 1$, $r > 0$ and $\sigma$ are constant, $\delta = 0$, and $G_t = (K - S_t)^+ = g(S_t)$.

**Theorem 2** $V_t^{ap} = \sup_{t \le \tau \le T} \mathrm{E}_t^*\{e^{-r(\tau-t)}(K - S_\tau)^+\} = h(S_t, t)$ for some continuous function $h(S_t, t) \ge (K - S_t)^+$. The optimal stopping time is $\tau^* = \inf\{t : h(S_t, t) = (K - S_t)^+\}$.

It follows that $\tau^* = \inf\{t : (S_t, t) \in \mathcal{U}^c\}$ where $\mathcal{U} = \{(s, t) \in \mathcal{R}^+ \times [0, T] : h(S_t, t) > (K - S_t)^+\}$ is the continuation region for this optimal stopping problem.

**Lemma 1** $h(s, t) > 0 \ \forall s \ge 0, t < T$.

**Proposition 4** For each $t < T$, the $t$-section of $\mathcal{U}$ is an open ray:
$\mathcal{U}_t \equiv \{s : (s, t) \in \mathcal{U}\} = (b_t, \infty)$ for some number $b_t \in (0, K)$.

**Proof** $0 \notin \mathcal{U}_t$ since $h(0, t) \le K = g(0, t)$, but $K \in \mathcal{U}_t$ since $h(K, t) > 0 = g(K, t)$. So it remains to show $x \in \mathcal{U}_t \Rightarrow y \in \mathcal{U}_t \ \forall y > x$. Suppose $x \in \mathcal{U}_t$ and $y > x$. Let $\tau$ be the optimal stopping time given $S_t = x$. I.e., $\tau = \inf\{u \ge 0 : (S_u^x, u) \in \mathcal{U}^c\}$, where $S_u^x = x e^{\sigma(B_u^* - B_t^*) + (r - \sigma^2/2)(u-t)}$.

Then, since $\tau$ is feasible starting from $S_t = y$,

$$
\begin{aligned}
h(y, t) - h(x, t) &= h(y, t) - \mathrm{E}^*\{e^{-r\tau}(K - S_\tau^x)^+\} && (6)\\
&\ge \mathrm{E}^*\{e^{-r\tau}[(K - S_\tau^y)^+ - (K - S_\tau^x)^+]\} && (7)\\
&\ge \mathrm{E}^*\{e^{-r\tau}[S_\tau^x - S_\tau^y]\} && (8)\\
&= (x - y)\mathrm{E}^*\{e^{\sigma(B_\tau^* - B_t^*) - \sigma^2\tau/2}\} && (9)\\
&= x - y\ . && (10)
\end{aligned}
$$

Thus, $h(y, t) \ge h(x, t) + x - y > (K - x)^+ + x - y \ge K - y$.
Also $h(y, t) > 0 \Rightarrow h(y, t) > (K - y)^+ \Rightarrow y \in \mathcal{U}_t$.

**Lemma 2** $h(\cdot, t)$ is decreasing for each $t$. $h(s, \cdot)$ is decreasing for each $s$.

**Proposition 5** The boundary $b_t$ is increasing in $t$.

**Proof** For any $t > 0, S \ge 0, \varepsilon > 0$,

$$
\begin{aligned}
h(b_t + \varepsilon, t - s) &\ge h(b_t + \varepsilon, t) \ \text{(since } h \text{ is decreasing in } t\text{)} && (11)\\
&> g(b_t + \varepsilon) \ \ \text{(since } b_t + \varepsilon \in \mathcal{U}_t\text{)}\ . && (12)
\end{aligned}
$$

This implies $b_t + \varepsilon \in \mathcal{U}_{t-s} \ \forall \varepsilon > 0$, which implies $b_t \ge b_{t-s}$.

## American Calls (Kim, 1990)

Suppose $n = d = 1$, $r > 0, \delta > 0$, and $\sigma > 0$ are constant, and the American call payoff is $G_t = (S_t - K)^+$. Then $\exists$ an optimal exercise boundary of critical stock prices $\bar{s}_t$ below which it is optimal for the option-holder to continue, and above which, it is optimal to exercise.

**Proposition 6** The critical stock price boundary $\bar{s}_t$ is nonincreasing in $t$.

- Put-call parity, that a European call equals a forward contract plus a European put, gives the intuition that the choice between exercising and waiting is a trade-off between the benefit of capturing the dividends on the underlying stock, and the costs of accelerating payment of the strike price and throwing away the put.
- Thus, the critical stock price $\bar{s}_t$ is lower the higher the dividend rate, higher the higher the riskless rate, and higher the higher the stock return volatility.
- Note that the interest cost alone exceeds the dividend benefit of exercising when $rK > \delta S \Leftrightarrow S < \frac{rK}{\delta}$.

**Proposition 7** If $r > \delta, \lim_{t \to T} \bar{s}_t = \frac{rK}{\delta} > K$. If $r \leq \delta, \lim_{t \to T} \bar{s}_t = K$.

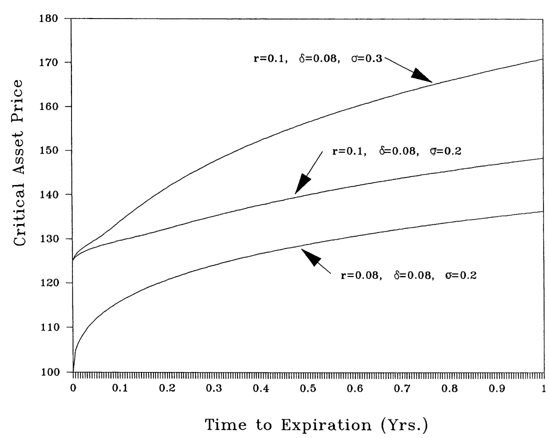


**Figure 1**
**Critical asset price as a function of time to expiration**
The critical asset price for American calls with an exercise price of \$100 as a function of time to expiration obtained from the implicit finite-difference method. $r$, interest rate; $\delta$, dividend yield; $\sigma$, volatility.

## Employee Stock Options (Carpenter, Stanton, Wallace, 2010)

- A major component of corporate compensation. What is the subjective value to employees? What are the incentive effects? What is the cost to firms? Accounting valuations matter because they affect allocations.
- Standard option pricing theory does not directly apply, because the options are nontransferable, so executives don't follow value-maximizing exercise policies as the standard theory for tradeable options assumes. These are 10-year American options. We see exercises of options on non-dividend paying stock well before expiration. This significantly reduces the option's present value.
- The cost of the option to shareholders who can trade freely can be represented as the risk-neutral expectation of its risklessly discounted value, $\mathrm{E}^*\{e^{-r\tau}(S_\tau - K)^+\}$ .
- Determining the cost of these options to shareholders must correctly account for the exercise policies, $\tau$ of the option holders. A structural theory model provides intuition for what drives employee exercise and how to specify a reduced-form empirical model that can be taken to data.
- In addition, when the option is nontransferable, its value to the executive is different from its present value or cost to shareholders. The private value to the executive is lower. The difference is the cost of providing performance incentives.

## General Formulation of the Executive's Problem

- Executive has $n$ options with strike $K$ and expiration date $T$ and additional wealth that can be invested subject to prohibition of short sales of the underlying stock.
- The investment opportunity set includes riskless bonds, a market portfolio, and the underlying stock which is priced by CAPM. Constant market coefficients.
- The optimal holding of stock is zero in the absence of the option.
- The executive chooses an exercise policy $\tau$ and a trading strategy $\pi$ to maximize the expected utility of time $T$ wealth:

$$\sup_{\{\tau \le T, \pi^m, \pi^s \ge 0\}} \mathrm{E}\{V(W_\tau + n(S_\tau - K)^+, \tau)\}$$

subject to

$$dW_t = rW_t\,dt + \pi_t^m(\mu\,dt + \sigma_m\,dB_t) + \pi_t^s(\lambda\,dt + \sigma\,dB_t) \ ,$$

where

$$V(W_t, t) \equiv \sup_{\pi^m} \mathrm{E}_t\{U(W_T)\} \text{ s.t. } dW_u = rW_u\,du + \pi_u^m(\mu\,du + \sigma_m\,dB_u) \ ,$$

and the utility function $U$ is $C^2$, strictly increasing, and strictly concave.

**Special Case with Outside Wealth in the Riskless Asset**

- Suppose the market contains only the stock and riskless bonds, the stock has zero excess expected return, so the optimal portfolio for the executive includes no long position in the stock. The executive's problem becomes

$$f(S_t, t) \equiv \max_{\{t \le \tau \le T\}} \mathrm{E}_t\{g(S_\tau, \tau)\} \ ,$$

  where $g(s,t) \equiv U(n(s-K)^+ e^{r(T-t)} + W)$, the constant $W$ is outside wealth at time T with $W > nKe^{rT}$.
- Assuming $U$ satisfies a polynomial growth condition, $f : (0, \infty) \times [0, T] \to \mathcal{R}$ is continuous and satisfies $f(S_t, t) \ge U(n(S_t - K)^+ e^{r(T-t)} + W)$ and $f(S_T, T) = U(n(S_T - K)^+ + W)$.
- An optimal exercise time is

$$\tau^* \equiv \inf\{t \in [0, T] : f(S_t, t) = U(n(S_t - K)^+ e^{r(T-t)} + W)\} \ .$$

- The continuation region for the problem is the open set

$$\mathcal{U} \equiv \{(s,t) \in (0, \infty) \times [0, T] : f(s,t) > U(n(s-K)^+ e^{r(T-t)} + W)\} \ .$$

**Proposition 8 (Existence of Critical Stock Price Boundary)** Suppose the drift of $g$ is nonincreasing in the stock price $s$. Then for each time $t \in [0, T)$, if there is any stock price at which exercise is optimal, then there exists a critical stock price $\bar{s}(t)$ such that it is optimal to exercise the option if and only if $S_t \ge \bar{s}(t)$.

The hypothesis is satisfied for CRRA utility functions with RA coefficient less than or equal to one, but the result appears to hold for all CRRA utility functions. On the other hand, for $U(W) = \frac{W^{1-A}}{1-A} + cW$ the continuation region is split.

**Proposition 9 (Monotonicity in Risk Aversion)** An executive with less absolute risk aversion has a larger continuation region.

**Corollary (Monotonicity in Wealth)** If the executive has decreasing absolute risk aversion, then the continuation region is larger with greater wealth.

**Proposition 10 (Monotonicity in the Dividend Rate)** If a given state $(s,t)$ is in the continuation when the dividend rate is $\delta_1$, then it is also in the continuation region when the dividend rate is $\delta_2$ for any $\delta_2 < \delta_1$.

**Non-Monotonicity in Volatility** Higher stock return volatility can shrink the continuation region.

## Problem Set 4

**1.** Prove that if an American option's discounted payoff process $\beta G$ is a $\mathcal{P}^*$-submartingale, then it is optimal not to exercise early.

**2.** Consider a complete, continuous-time financial market where $B_t^*$ is a standard 2-dimensional Brownian motion under the martingale measure $\mathcal{P}^*$. The interest rate $r_t$ is a nonnegative one-factor diffusion described by the equation

$$dr_t = \mu(r_t, t)\, dt + \sigma_r(r_t, t)\, dB_{1,t}^* \; , \tag{1}$$

where $\mu$ and $\sigma_r$ are continuous and satisfy Lipschitz and linear growth conditions. The price of a stock, $S_t$, satisfies

$$\frac{dS_t}{S_t} = (r_t - \delta)\, dt + \sigma\, dB_{2,t}^* \; , \tag{2}$$

where $\delta > 0$ and $\sigma \neq 0$ are constants. Let $c(S_t, r_t, t)$ be the time $t$ price of an American call on the stock with strike price $k$ and expiration date $T$.

(a) Prove that for each $(r, t) \in \mathcal{R}^+ \times [0, T)$ and for all $s_2 > s_1 > 0$,

$$0 \le \frac{c(s_2, r, t) - c(s_1, r, t)}{s_2 - s_1} \le 1 \; . \tag{3}$$

(b) Prove that for each $(r, t) \in \mathcal{R}^+ \times [0, T)$, if it is optimal to continue at $(s_2, r, t)$, then it is optimal to continue at $(s_1, r, t)$ for all $s_2 > s_1 > 0$.