Optimal Consumption and Portfolio Choice

1. Solutions in the general continuous-time financial market setting

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Selected Readings and References

Back, chapter 14.

Karatzas and Shreve, 1998, chapter 3.

Duffie, chapter 9.

Merton, R., 1969, Lifetime portfolio selection under uncertainty: the continuous-time case, *Review of Economics and Statistics* 51, 247-257.

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Summary of the Continuous-Time Financial Market

- Security prices satisfy $\frac{dS_{0,t}}{S_{0,t}} = r_t\,dt$ and $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t})\,dt + \sigma_{k,t} dB_t$.
- Given tight tr. strat. π_t and consumption c_t, portfolio value X_t satisfies the
  - WEE: $dX_t = r_t X_t\,dt + \pi_t(\mu_t - r_t 1)\,dt + \pi_t\sigma_t\,dB_t - c_t\,dt$.
- No arbitrage $\Rightarrow$ if $\pi_t\sigma_t \ = 0$ then $\pi_t(\mu_t - r_t 1) = 0 \Rightarrow \exists\theta_t$ s.t. $\sigma_t\theta_t = \mu_t - r_t 1$ $\Rightarrow dX_t = r_t X_t\,dt + \pi_t\sigma_t(\theta_t\,dt + dB_t) - c_t\,dt$.
- Under emm $\mathcal{P}^*$ given by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ where $Z_t = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds}$,
  - $B_t^* = B_t + \int_0^t \theta_s\,ds$ is Brownian motion.

  Let $\beta_t = e^{-\int_0^t r_s\,ds}$ and sdf process $M_t = \beta_t Z_t$. Then the WEE can also be written:
  - WEE*: $d\beta_t X_t + \beta_t c_t\,dt = \beta_t\pi_t\sigma_t\,dB_t^*$
  - WEE-M: $dM_t X_t + M_t c_t\,dt = M_t[\pi_t\sigma_t - \theta_t X_t]\,dB_t$
- So $X_t = \mathrm{E}_t^*\{\int_t^T \frac{\beta_u}{\beta_t}c_u\,du + \frac{\beta_T}{\beta_t}X_T\} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t}c_u\,du + \frac{M_T}{M_t}X_T\}$ if π is mtgale-gen.
- If σ is nonsingular, every c.plan (c, X_T) can be generated by a mtgale-gen. tr.strat.

Investor's Optimization Problem

Consider the problem of a risk averse investor with inital wealth x_0 who chooses a consumption plan c_t and trading strategy π_t to maximize expected utility:

$$\max_{c_t,\pi_t} \mathrm{E}\{\int_0^T U(c_t,t)\,dt + U(X_T,T)\} \tag{1}$$

$$\text{s.t. } X_T = x_0 + \int_0^T [r_t X_t + \pi_u(\mu_t - r_t 1) - c_t]\,du + \int_0^T \pi_t\sigma_t\,dB_t\ . \tag{2}$$

If c_t and X_t are constrained to be bounded below, e.g., nonnegative, then the WEE (2) above implies the static budget constraint (BC)

$$\mathrm{E}\{\int_0^T M_t c_t\,dt + M_T X_T\} \leq x_0 \tag{3}$$

Suppose the market is complete, i.e., $n = d$ and σ is nonsingular. Then the set of nonnegative consumption plans that satisfy the BC (3) with equality is the same as the set of those that satisfy the WEE (2) for some martingale-generating trading strategy π.

So if the market is complete and consumption is constrained to be bounded below, the investor's problem can be re-stated as a choice of optimal consumption and terminal payoff (c, X_T):

$$\max_{c_t \geq 0, X_T \geq 0} \mathrm{E}\{\int_0^T U(c_t, t)\, dt + U(X_T, T)\} \text{ s.t. } \mathrm{E}\{\int_0^T M_t c_t\, dt + M_T X_T\} \leq x_0 \ . \quad (4)$$

To get a quick feel for the solution, ignore for the moment the nonnegativity constraint, which will not bind for some utility functions, and consider maximizing the Lagrangian

$$\max_{c_t, X_T} \mathcal{L} = \mathrm{E}\{\int_0^T U(c_t, t)\, dt + U(X_T, T) - \lambda[\int_0^T M_t c_t\, dt + M_T X_T]\} \ . \quad (5)$$

The FOCs are $U'(c_t, t) = \lambda M_t$ and $U'(X_T, T) = \lambda M_T$, and the solution to the investor's problem is $c_t^* = U'^{-1}(\lambda M_t, t)$ and $X_T^* = U'^{-1}(\lambda M_T, T)$, assuming the derivatives of utility w.r.t consumption U' have well-defined inverse functions, and where λ solves

$$\mathrm{E}\{\int_0^T M_t U'^{-1}(\lambda M_t, t)\, dt + M_T U'^{-1}(\lambda M_T, T)\} = x_0 \ . \quad (6)$$

Utility Functions and Inverse Marginal Utility Functions

Definition 1 A *utility function* (uf) is a real-valued, strictly increasing, strictly concave C^2 function on a domain $[\bar{x}, \infty)$ or $(\bar{x}, \infty)$ with $\lim_{x \to \infty} U'(x) = 0$.

- Call $\bar{x}$ the *subsistence level* and for ease of exposition set $\bar{x} = 0$.
- Let $U'(0+) = \lim_{x \to 0} U'(x)$.

Example 1 (CRRA Utility) $U(x) = \frac{x^{1-A}}{1-A}$, $U'(x) = x^{-A}$, $U'(0+) = \infty$.

Example 2 (CARA Utility) $U(x) = -e^{-ax}$, $U'(x) = ae^{-ax}$, $U'(0+) = a < \infty$.

- If $U'(0+) < \infty$, define U on $[0, \infty)$. In this case the nonnegativity constraint on consumption will bind.

Definition 2 The *inverse marginal utility function* (imuf) associated with a uf U is the function $I : \mathcal{R}_+ \to [0, \infty)$ given by

$$I(y) = [U'^{-1}(y)]^+ = U'^{-1}(y) \text{ if } 0 < y < U'(0+) \text{ and } 0 \text{ otherwise} \ . \quad (7)$$

Example 1 (CRRA Utility) $U'(x) = x^{-A} = y \Rightarrow I(y) = y^{-1/A}$.

Example 2 (CARA Utility) $U'(x) = ae^{-ax} = y \Rightarrow I(y) = -\frac{1}{a} \ln(\frac{y}{a})$ for $0 < y < a$ and $I(y) = 0$ for $y \geq a$.

Optimal Policies

Consider also two simpler versions of the problem, the pure terminal wealth problem and the pure consumption problem:

$$\max_{X_T \geq 0} \mathrm{E}\{U(X_T)\} \text{ s.t. } \mathrm{E}\{M_T X_T\} \leq x_0 \text{ and} \tag{8}$$

$$\max_{c_t \geq 0} \mathrm{E}\{\int_0^T U(c_t, t)\, dt\} \text{ s.t. } \mathrm{E}\{\int_0^T M_t c_t\, dt\} \leq x_0. \tag{9}$$

We'll prove the result for the terminal wealth problem (8). Maximize the Lagrangian

$$\mathcal{L} = \mathrm{E}\{U(X_T) - \lambda M_T X_T + \gamma X_T\}\ , \tag{10}$$

where the positive constant λ is the Lagrange multiplier for the BC and the nonnegative random variable γ is the state-dependent Lagrange multiplier for the nonnegativity constraint $X_T \geq 0$. The FOCs are

$$U'(X_T) = \lambda M_T - \gamma,\ X_T \geq 0,\ \gamma \geq 0,\ \gamma X_T = 0,\ \lambda > 0, \text{ and } \mathrm{E}\{M_T X_T\} = x_0\ .$$

Theorem Assume $\mathrm{E}M_T < \infty$ and $\mathrm{E}\{M_T I(\lambda M_T)\} = \mathrm{E}\{M_T [U'^{-1}(\lambda M_T)]^+\} < \infty$ for every $\lambda > 0$. Then the solution to the terminal wealth problem (8) is

$$X_{1T} = I(\lambda_1 M_T, T) \tag{11}$$

where λ_1 solves $\mathrm{E}\{M_T I(\lambda_1 M_T)\} = x_0$.

Lemma 1 $\exists\ \lambda_1$ s.t. $\mathrm{E}\{M_T I(\lambda_1 M_T)\} = x_0$.

Proof of Lemma $\mathrm{E}\{M_T I(\lambda_1 M_T)\} = \mathrm{E}\{M_T [U'^{-1}(\lambda M_T)]^+\}$ is continuous and strictly decreasing in λ, goes to infinity as $\lambda \to 0$, and goes to 0 as $\lambda \to \infty$.

Proof of Theorem First note that $X_{1T} \geq 0$ is feasible for the terminal wealth problem, and satisfies the FOCs above with $\gamma = \gamma_1 \equiv \lambda_1 M_T - U'(X_{1T})$.

I.e., $U'(X_{1T}) = \lambda_1 M_T$ if $\lambda_1 M_T < U'(0+)$ and $U'(X_{1T}) = U'(0+)$ if $\lambda_1 M_T \geq U'(0+)$, so $\gamma_1 \geq 0$ and $\gamma_1 X_{1T} = 0$.

Next, suppose W is another feasible policy with $\mathcal{P}\{W = X_{1T}\} < 1$. We verify that W gives strictly less expected utility than X_{1T} with a traditional concavity argument as follows:

$$\mathrm{E}\{U(W)-U(X_{1T})\} < \mathrm{E}\{U'(X_{1T})(W-X_{1T})\} \tag{12}$$
$$= \mathrm{E}\{(\lambda_1 M_T - \gamma_1)(W - X_{1T})\} \tag{13}$$
$$= \lambda_1 \mathrm{E}\{M_T(W-X_{1T})\} - \mathrm{E}\{\gamma_1 W\} + \mathrm{E}\{\gamma_1 X_{1T}\} \tag{14}$$
$$\leq 0 \tag{15}$$

Similarly, assume $\mathrm{E}\{\int_0^T M_t\,dt\} < \infty$ and $\mathrm{E}\{\int_0^T M_t I(\lambda M_t,t)\,dt\} < \infty\ \forall\ \lambda > 0$. Then the solution to the pure consumption problem (9) is

$$c_{2t} = I_2(\lambda_2 M_t, t) \tag{16}$$

where λ_2 solves $\mathrm{E}\{\int_0^T M_t I(\lambda_2 M_t,t)\,dt\} = x_0$, and the solution to the consumption-terminal wealth problem (4) is

$$(c_{3t}, X_{3T}) = (I(\lambda_3 M_t, t), I(\lambda_3 M_T, T)) \tag{17}$$

where λ_3 solves $\mathrm{E}\{\int_0^T M_t I(\lambda_3 M_t,t)\,dt\} + M_T I(\lambda_3 M_T, T) = x_0$.

Optimal Trading Strategy with Deterministic Market Coefficients

- In the general continuous-time financial market, we can solve for the optimal consumption plan (c_t, X_T) using martingale methods, and we know from the Martingale Representation Theorem that an optimal trading strategy π_t exists, but we may not be able to solve for π_t explicitly.
- However, in the special case of deterministic market coefficients, we can solve for π_t explicitly.
- Assume r, σ, and θ are nonrandom continuous functions on $[0,T]$.
- Then $Z_t = e^{-\int_0^t \theta_s'\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds} = e^{-\int_0^t \theta_s'\,dB_s^* + \frac{1}{2}\int_0^t |\theta_s|^2\,ds}$ is Markov and under the optimal policy for any of the problems above, investor wealth will be a deterministic function of Z_t and time. For example, in the pure consumption problem (9),

$$X_{2t} = \mathrm{E}_t^*\{\int_t^T e^{-\int_t^u r_s\,ds} c_{2u}\,du\} \tag{18}$$
$$= \mathrm{E}_t^*\{\int_t^T e^{-\int_t^u r_s\,ds} I(\lambda_2 M_u, u)\,du\} \tag{19}$$
$$= \mathrm{E}_t^*\{\int_t^T e^{-\int_t^u r_s\,ds} I(\lambda_2 e^{-\int_0^u r_s\,ds} Z_t e^{-\int_t^u \theta_s' dB_s^* + \frac{1}{2}\int_t^u |\theta_s|^2\,ds}, u)\,du\} \tag{20}$$
$$= f(Z_t, t)\ . \tag{21}$$

- Suppose f is smooth enough for an application of Itô's lemma. (E.g., either verify through explicit computation for the given utility function, or assume r, θ, and I are Hölder continuous and $I(\cdot,t)$ and $U(I(\cdot,t),t)$ each satisfy a pgc, which imply f is $C^{2,1}$–see Assumption 3.8.2 of Karatzas and Shreve, 1998, p. 120.)
- Then we can solve for the optimal trading strategy π_2 by equating the diffusion of f from the WEE to the diffusion of f from Itô's lemma:

$$\pi_{2t}\sigma_t = -f_z(Z_t,t)Z_t\theta_t' \tag{22}$$
$$\Rightarrow \pi_{2t}' = -f_z(Z_t,t)Z_t\sigma_t'^{-1}\theta_t \tag{23}$$
$$= -f_z(Z_t,t)Z_t[\sigma_t\sigma_t']^{-1}[\mu_t - r_t 1]\ . \tag{24}$$

- Note that the only term above that is specific to the investor (utility function) is the scalar $f_z(Z_t,t)$, which implies the following result:

Merton's Two-Fund Separation Theorem In the deterministic coefficient case here, all investors divide wealth between the riskless asset and the risky portfolio with weights proportional to $[\sigma_t\sigma_t']^{-1}[\mu_t - r_t 1]$.

Optimal Controls in Feedback Form

- In the dynamic programming approach to solving the investor's problem, the optimal controls π_{2t} and c_{2t} are typically expressed as deterministic functions of optimal investor wealth X_{2t}. They can also be expressed that way here:
- From the construction of $X_{2t} = f(Z_t,t)$ in equations (18)-(21), it is clear that for each $t \in [0,T]$, $f(\cdot,t)$ is strictly decreasing and has a strictly decreasing inverse $f^{-1}(\cdot,t)$. Thus,

$$\pi_{2t} = -\frac{f^{-1}(X_{2t},t)}{f_X^{-1}(X_{2t},t)}[\sigma_t\sigma_t']^{-1}[\mu_t - r_t 1] \tag{25}$$
$$= g(X_{2t},t)\ . \tag{26}$$

- Similarly, $c_{2t} = I(\lambda M_t,t) = h(X_{2t},t)$.
- Analogous expressions in feedback form hold for the optimal consumption and trading strategies for the pure terminal wealth problem (8) and the combined problem (4).

Special Case: Log Utility

Suppose $U(c) = \log c$. Then, in fact, we can fully solve the investor's problem even with general market coefficients r_t, σ_t, θ_t. For ease of exposition, consider the terminal wealth problem:

$$\max_{X_T} \mathrm{E}\{\log X_T\} \text{ s.t. } \mathrm{E}\{M_T X_T\} \le x_0 \tag{27}$$

$$\Rightarrow U'(X_T) = \frac{1}{X_T} = \lambda M_T \Rightarrow X_T = \frac{1}{\lambda M_T} . \tag{28}$$

$$\mathrm{E}\{M_T X_T\} = \mathrm{E}\{M_T \frac{1}{\lambda M_T}\} = x_0 \Rightarrow \frac{1}{\lambda} = x_0 \tag{29}$$

$$\Rightarrow X_T = \frac{x_0}{M_T} \tag{30}$$

$$\Rightarrow X_t = \mathrm{E}_t\{\frac{M_T}{M_t} X_T\} = \frac{x_0}{M_t} = x_0 e^{\int_0^t r_u\, du + \int_0^t \theta_u'\, dB_u^* - \frac{1}{2}\int_0^t |\theta_u|^2\, du} \tag{31}$$

$$\Rightarrow dX_t = r_t X_t\, dt + X_t \theta_t'\, dB_t^* = (r_t + |\theta_t|^2) X_t\, dt + X_t \theta_t'\, dB_t \tag{32}$$

$$\Rightarrow \pi_t \sigma_t = X_t \theta_t' \Rightarrow \frac{\pi_t'}{X_t} = \sigma_t'^{-1} \theta_t = [\sigma_t \sigma_t']^{-1} [\mu_t - r_t 1] . \tag{33}$$

Dynamic Programming Approach

Merton (1969, 1971) originally solved the consumption/investment problems using dynamic programming. Let p and c represent portfolio holdings and consumption flow as proportions of wealth and consider the problem

$$\sup_{p,c} \mathrm{E}\{\int_0^T U(c_t X_t, t) dt + U(X_T, T)\} \tag{34}$$

$$\text{s.t. } \frac{dX_t}{X_t} = [r_t - c_t + p_t(\mu_t - r_t 1)]\, dt + p_t \sigma_t\, dB_t, \tag{35}$$

$$X_0 = x_0$$

where p and c are adapted controls taking values in a compact set K, and r, μ, and σ are continuously differentiable functions of time.

The value function for the problem is

$$V_t = \sup_{p,c} \mathrm{E}_t\{\int_t^T U(c_u X_u, u)\, du + U(X_T, T)\} \tag{36}$$

$$\text{s.t. } \frac{dX_t}{X_t} = [r_t - c_t + p_t(\mu_t - r_t 1)]\, dt + p_t \sigma_t\, dB_t .$$

The value of any given feasible strategy $\widehat{p}, \widehat{c}$ is

$$J_t^{\widehat{p},\widehat{c}} = \mathrm{E}_t\{\int_t^T U(\widehat{c}_u X_u^{\widehat{p},\widehat{c}}, u)\, du + U(X_T^{\widehat{p},\widehat{c}}, T)\} \tag{37}$$

$$\text{where } \frac{dX_t^{\widehat{p},\widehat{c}}}{X_t^{\widehat{p},\widehat{c}}} = [r_t - \widehat{c}_t + \widehat{p}_t(\mu_t - r_t 1)]\, dt + \widehat{p}_t \sigma_t \, dB_t \tag{38}$$

where the hat and superscript notation above are just to emphasize the use of a particular, though not necessarily optimal, set of controls, and will be omitted below when the meaning is clear.

Because of the Markov properties of Brownian motion, we may restrict attention to controls that are "Markov" or "feedback," i.e., of the form

$$p_t = g(X_t, t) \text{ and } c_t = h(X_t, t) \ , \tag{39}$$

and assume that the value function is of the form

$$V_t = v(X_t, t) \ . \tag{40}$$

The Hamilton-Jacobi-Bellman Equation

By the Bellman principle, for $t < u < T$,

$$v(X_t, t) = \sup_{p,c} \mathrm{E}\{\int_t^u U(c_s X_s, s)\, ds + v(X_u^{p,c}, u) | X_t\} \tag{41}$$

where $X^{p,c}$ indicates the wealth process under controls p, c. Now suppose v is $C^{2,1}$. Then by Itô's lemma,

$$\begin{aligned} v(X_u^{p,c}, u) = v(X_t, t) + \int_t^u \mathcal{D}^{p,c} v(X_s^{p,c}, s)\, ds \\ + \int_t^u v_X(X_s^{p,c}, s) p(s) \sigma_s X_s^{p,c} \, dB_s \end{aligned} \tag{42}$$

where

$$\mathcal{D}^{p,c} v \equiv v_X[r - c + p(\mu - r1)]X^{p,c} + v_t + \frac{1}{2} v_{XX} p\sigma\sigma' p' (X^{p,c})^2 \tag{43}$$

is the drift of v under the controls p, c.

Thus, by the Bellman equation,

$$
\begin{aligned}
0 &= \sup_{p,c} \mathrm{E}_t\{\int_t^u U(c_s X_s, s)\, ds + v(X_u^{p,c}, u) - v(X_t, t)\} && (44)\\
&= \sup_{p,c} \mathrm{E}_t\{\int_t^u U(c_s X_s, s)\, ds + \int_t^u \mathcal{D}^{p,c} v(X_s^{p,c}, s)\, ds + \int_t^u v_X(X_s^{p,c}, s) p(s) \sigma_s X_s^{p,c}\, dB_s\} \\
&= \sup_{p,c} \mathrm{E}_t\{\int_t^u U(c_s X_s, s)\, ds + \int_t^u \mathcal{D}^{p,c} v(X_s^{p,c}, s)\, ds\} \ , && (45)
\end{aligned}
$$

assuming the integrand of the stochastic integral above is well enough behaved that the stochastic integral has mean zero. Dividing by $u - t$ and letting $u \downarrow t$ gives the Hamilton-Jacobi-Bellman (HJB) equation

$$
\begin{aligned}
0 = &\ \sup_{p,c} U(cX, t) + \mathcal{D}^{p,c} v(X, t) && (46)\\
= &\ \sup_{p,c} U(cX, t) + v_X(X, t)[r - c + p(\mu - r1)]X + v_t(X, t) \\
&+ \frac{1}{2} v_{XX}(X, t) p \sigma \sigma' p' X^2 \ . && (47)
\end{aligned}
$$

In other words, cumulative of the utility dividend, the value function must be a martingale under the optimal policy, and a supermartingale under a suboptimal policy.

The first-order conditions for optimal p and c imply

$$
\begin{aligned}
U'(cX, t) &= v_X(X, t) \ , && (48)\\
p'X &= -\frac{v_X(X, t)}{v_{XX}(X, t)} (\sigma\sigma')^{-1} (\mu - r1) \ , && (49)
\end{aligned}
$$

which yields the two-fund separation result.

In addition, v must satisfy the terminal condition $v(X, T) = U(X, T)$.

Verification Theorem If $Q \in C^{2,1}$ satisfies the HJB equation and the terminal condition, and if Q and U each satisfy a pgc, then

$$
Q(x, t) \geq J^{p,c}(x, t) \qquad (50)
$$

for any predictable K-valued controls (p, c). If, in addition, there exist predictable K-valued controls (p^*, c^*) s.t.

$$
0 = U(c_t^* X_t, t) + \mathcal{D}^{p*,c*} Q(X_t, t) \text{ a.s. a.e.} \ , \qquad (51)
$$

then $Q(x, t) = J^{p*,c*}(x, t) = v(x, t)$ and (p^*, c^*) are optimal controls.

Proof Suppose Q satisfies the HJB equation and the terminal condition, and let (p, c) be feasible controls. Then

$$0 \geq U(c_t X_t, t) + \mathcal{D}^{p,c} Q(X_t, t) \ . \tag{52}$$

Note that by Itô's lemma,

$$Q(X_T, T) - Q(X_t, t) = \int_t^T \mathcal{D}^{p,c} Q(X_s, s)\, ds + \int_t^T Q_X(X_s, s) X_s p_s \sigma_s \, dB_s \ . \tag{53}$$

The conditions imposed on σ, p, and Q ensure that the stochastic integral above has mean zero, so

$$\begin{aligned}
\mathrm{E}_t\{Q(X_T, T) - Q(X_t, t)\} &= \mathrm{E}_t\{\int_t^T \mathcal{D}^{p,c} Q(X(s), s)\, ds\} && (54)\\
&\leq -\mathrm{E}_t\{\int_t^T U(c_s X_s, s)\, ds\} && (55)\\
\Rightarrow Q(X_t, t) &\geq \mathrm{E}_t\{\int_t^T U(c_s X_s, s)\, ds + U(X_T, T)\} && (56)\\
&= J^{p,c}(X_t, t) && (57)\\
\Rightarrow Q(X_t, t) &\geq v(X_t, t) \ . && (58)
\end{aligned}$$

On the other hand, if there exist controls (p^*, c^*) such that

$$0 = U(c_t^* X_t, t) + \mathcal{D}^{p*,c*} Q(X_t, t) \ , \tag{59}$$

then, by the same reasoning, it follows that

$$\begin{aligned}
Q(X_t, t) &= J^{p*,c*}(X_t, t) && (60)\\
\Rightarrow Q(X_t, t) &\leq v(X_t, t) && (61)\\
\Rightarrow Q(X_t, t) &= v(X_t, t) \ , && (62)
\end{aligned}$$

and (p^*, c^*) are optimal controls. $\square$

Problems

1. Investor A chooses an investment policy to maximize expected utility from time T wealth $\mathrm{E}U(X_T)$.

(a) Assume investor A has a HARA utility function with decreasing absolute risk aversion:

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{a(x-\bar{x})}{1-\gamma}\right)^{\gamma}\ ,\ \gamma < 1\ , a > 0\ , \tag{63}$$

and assume $\bar{x} \geq 0$ so the nonnegativity constraint on wealth is nonbinding.

i. Assuming a complete, standard financial market, use martingale methods to solve for investor A's optimal payoff.

ii. Assuming the market has constant coefficients, compute A's optimal trading strategy.

iii. Assuming the market has constant coefficients, use dynamic programming methods to solve for A's optimal trading strategy.

(b) Now assume investor A has constant relative risk averse (CRRA) utility ($\bar{x} = 0$):

$$U(x) = \frac{x^{\gamma}}{\gamma}\ . \tag{64}$$

What are the optimal payoff and optimal trading strategy for investor A?

(c) Now assume investor A has log utility ($\bar{x} = \gamma = 0$):

$$U(x) = \log(x)\ . \tag{65}$$

What are the optimal payoff and optimal trading strategy for investor A?

2. Investor B chooses an investment and consumption policy to maximize expected utility from consumption $\mathrm{E}\int_0^T e^{-\delta t}U(c_t)\,dt$. Solve investor B's problem for the case of HARA utility with $\bar{x} \geq 0$ and constant coefficients. Determine both the optimal consumption plan and the optimal trading strategy.

3. Consider investor A's problem assuming constant absolute risk averse (CARA) utility $U(x) = -e^{-ax}$ and suppose the market has only a single risky asset $(n = d = 1)$ and constant coefficients.

(a) Determine investor A's optimal dynamic holdings of the stock, $\{\pi_t^*\}_{t=0}^T$, in the absence of a nonnegativity constraint on wealth.

(b) Determine investor A's optimal dynamic holdings of the stock, $\{\widehat{\pi}_t\}_{t=0}^T$, under the constraint that the investor's wealth must be nonnegative at all times.

(c) Show that

i. $\widehat{\pi}_t < \pi_t^*$,

ii. $\widehat{\pi}_t$ is increasing in $\widehat{X}_t$, the investor's optimal time t wealth under the nonnegativity constraint,

iii. $\widehat{\pi}_t \to 0$ as $\widehat{X}_t \to 0$, and

iv. $\widehat{\pi}_t \to \pi_t^*$ as $\widehat{X}_t \to \infty$.