# Managerial Risk Incentives

## Selected Readings and References


Carpenter, J., 2000, Does Option Compensation Increase Managerial Risk Appetite?, *Journal of Finance* 55, 2311-2331.

Ross, S., 2004, Compensation, Incentives, and the Duality of Risk Aversion and Riskiness, *Journal of Finance* 59, 207-225.

Basak, S., Pavlova, A., and Shapiro, A., 2007. Optimal Asset Allocation and Risk Shifting in Money Management, *Review of Financial Studies*, 20, 1583-1621.

Panageas, S., and M. Westerfield, 2009, High-Water Marks: High Risk Appetites? Convex Compensation, Long Horizons, and Portfolio Choice, *Journal of Finance* 64, 1-36.

Lan, Y., Wang, N. and J. Yang, 2013, The Economics of Hedge Funds, *Journal of Financial Economics* 110, 300-323.

Drechsler, I., 2014, Risk Choice under High-Water Marks, *Review of Financial Studies* 27, 2052-2096.


# Summary of the Continuous-Time Financial Market

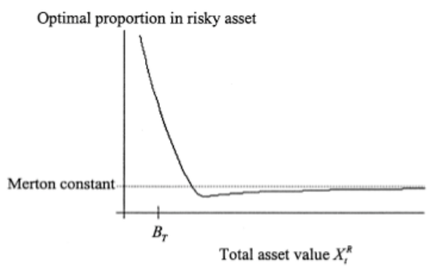
- Security prices satisfy $\frac{dS_{0,t}}{S_{0,t}} = r_t\, dt$ and $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t})\, dt + \sigma_{k,t} dB_t$.
- Given tight tr. strat. $\pi_t$ and consumption $c_t$, portfolio value $X_t$ satisfies the
  - WEE: $dX_t = r_t X_t\, dt + \pi_t(\mu_t - r_t 1)\, dt + \pi_t \sigma_t\, dB_t - c_t\, dt$.
- No arbitrage $\Rightarrow$ if $\pi_t\sigma_t = 0$ then $\pi_t(\mu_t - r_t 1) = 0 \Rightarrow \exists \theta_t$ s.t. $\sigma_t\theta_t = \mu_t - r_t 1$ $\Rightarrow dX_t = r_t X_t\, dt + \pi_t\sigma_t(\theta_t\, dt + dB_t) - c_t\, dt$.
- Under emm $\mathcal{P}^*$ given by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ where $Z_t = e^{-\int_0^t \theta_s' \, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds}$,
  - $B_t^* = B_t + \int_0^t \theta_s\, ds$ is Brownian motion.

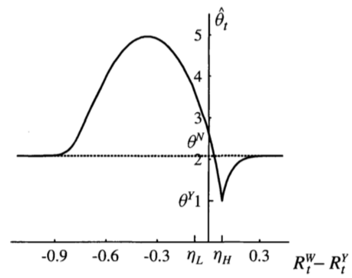
  Let $\beta_t = e^{-\int_0^t r_s\, ds}$ and sdf process $M_t = \beta_t Z_t$. Then the WEE can also be written:
  - WEE*: $d\beta_t X_t + \beta_t c_t\, dt = \beta_t \pi_t \sigma_t\, dB_t^*$
  - WEE-M: $dM_t X_t + M_t c_t\, dt = M_t[\pi_t\sigma_t - \theta_t X_t]\, dB_t$
- So $X_t = \mathrm{E}_t^*\{\int_t^T \frac{\beta_u}{\beta_t} c_u\, du + \frac{\beta_T}{\beta_t} X_T\} = \mathrm{E}_t\{\int_t^T \frac{M_u}{M_t} c_u\, du + \frac{M_T}{M_t} X_T\}$ if $\pi$ is mtgale-gen.
- If $\sigma$ is nonsingular, every c.plan $(c, X_T)$ can be generated by a mtgale-gen. tr.strat.

# Managerial Risk Choice - Theory

Carpenter (2000)
Single option

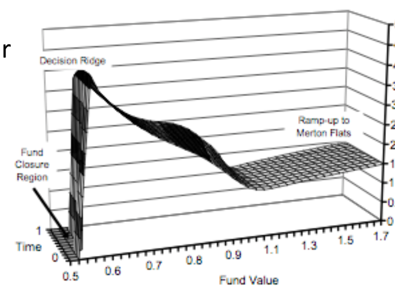
Basak, Pavlova, Shapiro (2007) Fund flows
Buraschi, Kosowski, Sritrakul – Equity, opts




Hodder Jackwerth (2007) Option, equity, shutdown option

Earlier

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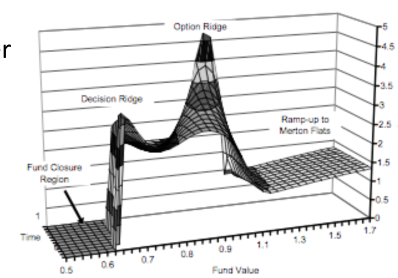
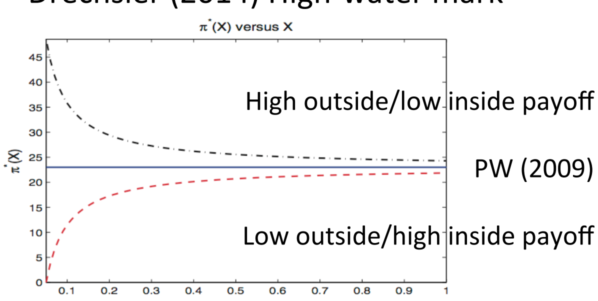
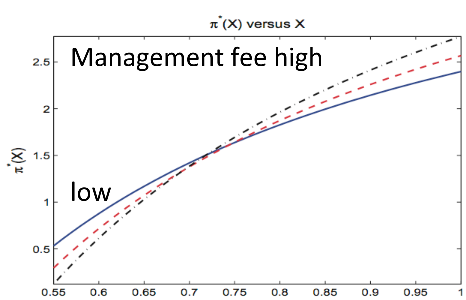


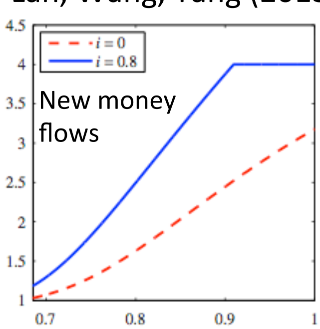

# Managerial Risk Choice - Theory

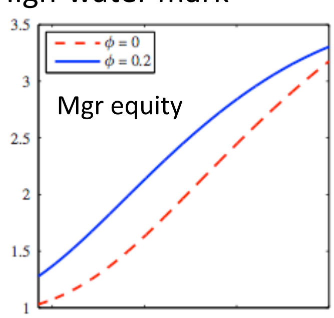
Panageas and Westerfield (2009),
Drechsler (2014) High-water mark

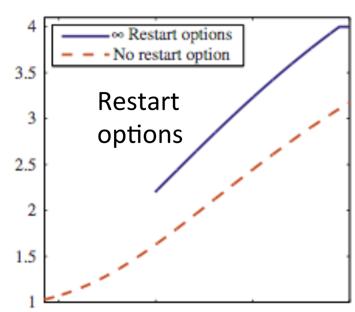



Lan, Wang, Yang (2013) High-water mark



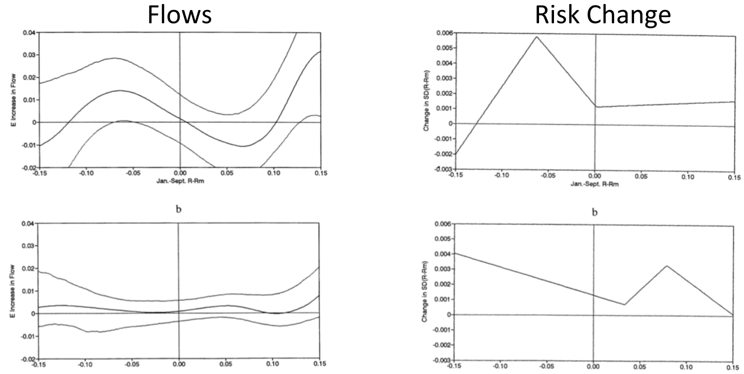



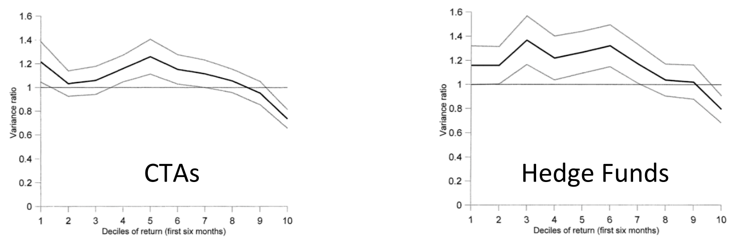
# Managerial Risk Choice - Evidence

## Chevalier, Ellison (1997) Mutual Funds

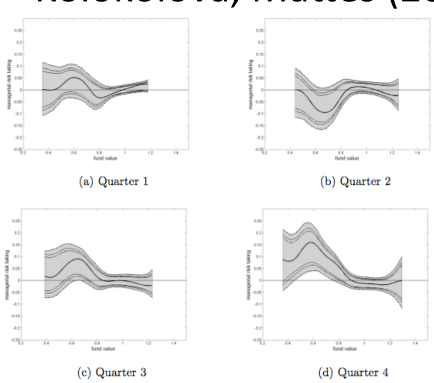
Flows | Risk Change

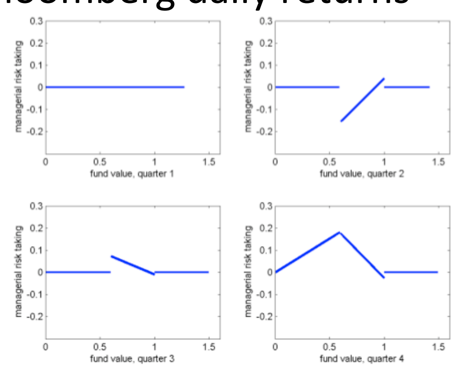


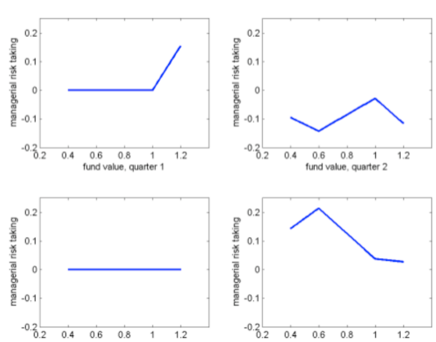

## Brown, Goetzmann, Park (2001) Tass data



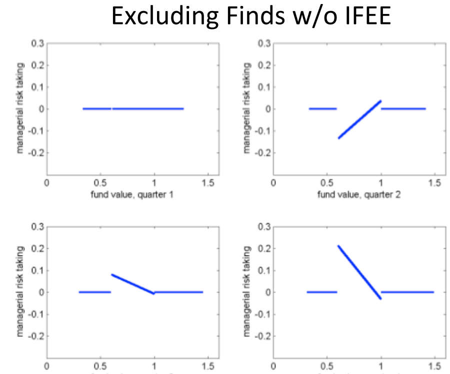

# Kolokolova, Mattes (2014) Bloomberg daily returns




Excluding Finds w/o IFEE



## Does Option Compensation Increase Managerial Risk Appetite?

Consider the dynamic asset risk choice of a manager compensated with an option on the assets $X_t$ that he controls.

**Manager Payoff** The manager's payoff is

$$\alpha(X_T - P_T)^+ + K, \tag{4}$$

where

- $X_T$ is the terminal value of the managed assets,
- $P_T$ is the option strike or benchmark portfolio value,
- $\alpha > 0$ represents the number of options or fraction of positive profits net of the benchmark paid to the manager, and
- $K$ is a constant that includes both cash compensation and any outside wealth, assumined to be invested in riskless bonds.

**Manager Preferences** The manager chooses a trading strategy for the assets to maximize his expected utility of his payoff.

- His uf $U$ is strictly increasing, strictly concave, $C^2$, and defined on a domain including $[K, \infty)$. $U$'' is nondecreasing and $U'(W) \to 0$ as $W \to \infty$.
- Consequently, the imuf $I = U'^{-1}$ is a strictly decreasing, convex, $C^1$ function from $(0, \infty)$ onto a range containing $[K, \infty)$.
- For example, the CARA or DARA uf's $U(W) = \frac{1-\gamma}{\gamma}(\frac{A(W-w)}{1-\gamma})^\gamma$ with $\gamma < 1$, $w < K$, and $A > 0$ satisfy these hypotheses.

**The Manager's Investment Problem** The manager's dynamic problem is to choose a trading strategy $\pi_t$ to

$$\max_\pi \mathrm{E}U(\alpha(X_T - P_T)^+ + K) \tag{5}$$

$$\text{s.t.} \quad dX_t = r_t X_t\,dt + \pi_t(\mu_t - r_t 1)\,dt + \pi_t \sigma_t\,dB_t, \tag{6}$$

$$X_t \geq 0,\ X_0 = x_0\ . \tag{7}$$

The manager's dynamic problem can be re-stated as a choice of a terminal asset value $X_T \geq 0$ that satisfies the budget constraint:

$$\max_{X_T \geq 0} \mathrm{E}U(\alpha(X_T - P_T)^+ + K) \tag{8}$$

$$\text{s.t.} \quad \mathrm{E}\{M_T X_T\} \leq x_0 \ . \tag{9}$$

- The problem modeled here resembles the one-period problem of a hedge fund manager compensated with an incentive fee.
- It is also a stylized description of the leverage problem faced by a corporate CFO.
- It is important that the manager cannot hedge the option in his outside portfolio. Otherwise he would undo the incentives by synthetically selling the option. In that case he would maximize the option value by maximizing asset volatility and the problem would have no solution.
- The technical challenge here is that the manager's objective function is nonconcave in the asset value $X_T$.

**Concavification of the Manager's Objective Function** Define the manager's utility of asset value and benchmark value as

$$u(x,b) = \begin{cases} U(\alpha(x-b)^+ + K) & \text{for } x \geq 0 \ , \\ -\infty & \text{otherwise.} \end{cases} \tag{10}$$

In terms of $u$, the manager's problem is

$$\max_{X_T \geq 0} \mathrm{E}[u(X_T, P_T)] \text{ s.t. } \mathrm{E}[M_T X_T] \leq X_0 \ . \tag{11}$$

- For each $b$, $u(\cdot, b)$ has a concavification $\tilde{u}(\cdot, b)$, illustrated by the dashed line in Figure 1.
- The concavified objective function replaces part of the original function with a chord between $x = 0$ and another point, $x = \widehat{x} > b$, at which the slope of the chord equals the slope of $u$ at $\widehat{x}$, so that the resulting function is concave.
- Lemma 1 establishes that for each value of $b > 0$, such a point $\widehat{x}(b)$ exists.

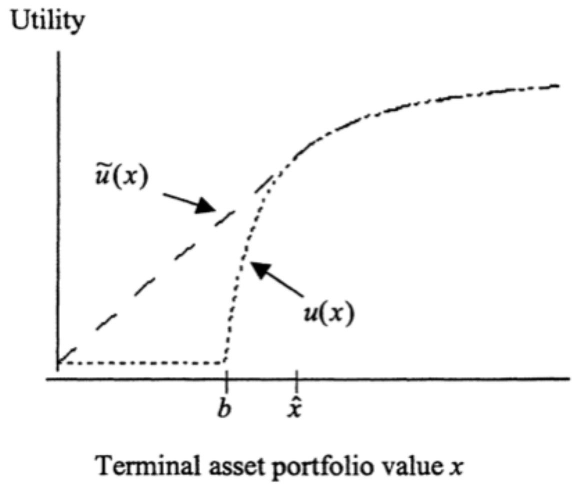



**Figure 1. Manager's original and concavified objective functions.** The dotted line represents the manager's objective function $u(x) = U(\alpha(x-b)^+ + K)$, where $U$ is the manager's concave utility function and $\alpha, B_T$, and $K$ are positive constants. The dashed line represents the concavification of $u(x)$, $\tilde{u}(x)$.

**Lemma 1** Let $u'(x,b) = \frac{\partial u(x,b)}{\partial x}$, for $x > b$. For every $b > 0$, there exists a unique point $\widehat{x}(b) > b$ such that

$$\frac{u(\widehat{x}(b), b) - u(0,b)}{\widehat{x}(b)} = u'(\widehat{x}(b), b) \ . \tag{12}$$

- It follows that $\tilde{u} : \mathcal{R} \times (0, \infty) \to \mathcal{R}$ defined by

$$\begin{aligned} \tilde{u}(x,b) = & -\infty && \text{for } x < 0 \\ & u(0) + u'(\widehat{x}(b), b)x && \text{for } 0 \leq x \leq \widehat{x}(b) \\ & u(x,b) && \text{for } x > \widehat{x}(b) \end{aligned} \tag{13}$$

  is concave in $x$.
- Furthermore, $\tilde{u}(x,b) \geq u(x,b)$ for all $(x,b) \in \mathcal{R} \times (0,\infty)$ and $\tilde{u}(x,b) = u(x,b)$ for $x = 0$ and for all $x \geq \widehat{x}(b)$.

- It turns out the optimal random payoff $X_T$ never takes values between zero and $\widehat{x}(P_T)$ where the true and the concavified ufs differ.
- Since the chord between them lies above the true uf, the average utility of the endpoints zero and $\widehat{x}(P_T)$ exceeds the utility of the average of those endpoints.
- Therefore, maximizing the concavified objective function gives the optimal payoff, and this can be solved with a straightforward extension of the standard method.

**Theorem 1** Let

$$h(y,b) = (I(y/\alpha) - K)/\alpha + b \text{ for all } y > 0, b > 0 \ . \tag{14}$$

Assume that

$$\mathcal{X}(\lambda) \equiv \mathrm{E}[M_T h(\lambda M_T, P_T) 1_{\{h(\lambda M_T, B_T) > \widehat{x}(P_T)\}}] < \infty \text{ for all } \lambda > 0 \ . \tag{15}$$

Then there exists a unique $\lambda^* > 0$ such that $\mathcal{X}(\lambda^*) = X_0$ and the unique optimal payoff for the manager with the option is

$$X_T^* = h(\lambda^* M_T, P_T) 1_{\{h(\lambda^* M_T, P_T) > \widehat{x}(P_T)\}} \ . \tag{16}$$

**Proof** First, although the concavification $\tilde{u}$ is not differentiable at $x = 0$, we can define a set-valued function $\tilde{u}'$ on $[0, \infty) \times (0, \infty)$ by

$$\begin{aligned} \tilde{u}'(x,b) = & [u'(\widehat{x}(b), b), \infty) && \text{for } x = 0 \\ & \{u'(\widehat{x}(b), b)\} && \text{for } 0 < x \le \widehat{x}(b) \\ & \{u'(x,b)\} && \text{for } x > \widehat{x}(b) \ . \end{aligned} \tag{17}$$

The function $\tilde{u}'(x,b)$ is essentially the derivative of $\tilde{u}$ with respect to $x$. In particular, for every $y \in \mathcal{R}$ and $x \ge 0$, and for every $m \in \tilde{u}'(x,b)$,

$$\tilde{u}(y,b) - \tilde{u}(x,b) \le m(y-x) \ . \tag{18}$$

Furthermore, strict inequality holds whenever $x > \widehat{x}(b)$ and $y \ne x$.

Second, we can define an inverse function for $\tilde{u}'(\cdot, b)$, $i : (0,\infty) \times (0,\infty) \to [0,\infty)$, by

$$i(y,b) = [(I(y/\alpha) - K)/\alpha + b] 1_{\{y < u'(\widehat{x}(b), b)\}} \ . \tag{19}$$

The function $i$ is the inverse of $\tilde{u}$ in the sense that

$$y \in \tilde{u}'(i(y, b), b) \text{ for all } b > 0 \ . \tag{20}$$

Third, under assumption (15) in the statement of the theorem, the function $\mathcal{X}(\lambda) = \mathrm{E}[M_T i(\lambda M_T, P_T)]$, for $\lambda > 0$, is continuous and strictly decreasing. Furthermore, $\mathcal{X}(\lambda) \to \infty$ as $\lambda \to 0$, and $\mathcal{X}(\lambda) \to 0$ as $\lambda \to \infty$. Therefore, there exists a unique $\lambda^* > 0$ such that $\mathcal{X}(\lambda^*) = X_0$.

Finally, note that $X_T^* = i(\lambda^* M_T, P_T)$. Let $Y$ be any other feasible payoff that is not almost surely equal to $X_T^*$. Then $Y$ provides lower expected utility than $X_T^*$:

$$\begin{aligned}
\mathrm{E}\{u(Y, B_T) - u(X_T^*, P_T)\} &= \mathrm{E}\{u(Y, P_T) - \tilde{u}(X_T^*, P_T)\} && (21)\\
&\leq \mathrm{E}\{\tilde{u}(Y, P_T) - \tilde{u}(X_T^*, P_T)\} && (22)\\
&< \mathrm{E}\{\lambda^* M_T (Y - X_T^*)\} && (23)\\
&\leq \lambda^*(\mathrm{E}[M_T Y] - X_0) \leq 0 \ . && (24)
\end{aligned}$$

Equation (21) follows from the fact that $X_T^*$ never takes on values where $u$ and $\tilde{u}$ disagree. Equation (23) follows from equations (18) and (20). □

**Optimal Payoff with Riskless Benchmark** If $P_T$ is a constant, the manager's optimal terminal fund value is

$$X_T^R = [(I(\lambda_R M_T/\alpha) - K)/\alpha + B_T] 1_{\{M_T < z_R\}} \ , \tag{25}$$

where $\lambda_R$ solves $\mathrm{E} M_T i(\lambda M_T, P_T) = X_0$ and $z_R = \alpha U'(\alpha(\widehat{x}(P_T) - P_T) + K)/\lambda_R$.

- In particular, the set of states in which the manager's option is in the money is the set of states in which the sdf $M_T$ falls below a certain critical value.
- A plot of the optimal terminal wealth $X_T^R$ as a function of the sdf $M_T$ appears in Figure 2.
- Optimal terminal wealth $X_T^R$ is greater than $\widehat{x}$ and decreasing in $M_T$ until $M_T$ hits $z_R$. Then $X_T^R$ jumps from $\widehat{x}$ down to zero.

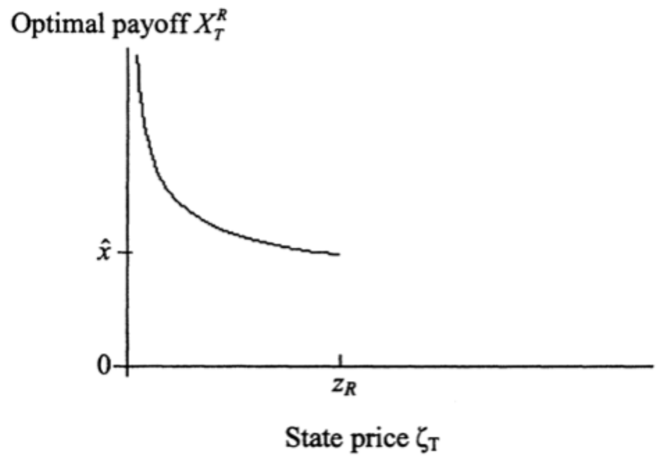



**Figure 2. Optimal random payoff.** The random variable $X_T^R$ maximizes $\mathrm{E}[U(\alpha(X_T - B_T)^+ + K)]$ subject to the constraints that $\mathrm{E}[\zeta_T X_T] \le X_0$, $X_T \ge 0$, and $X_T$ is measurable with respect to the filtration generated by asset prices up to time $T$. $U$ is the manager's concave utility function, $\alpha$, $B_T$, and $K$ are positive constants, and $\zeta_T$ is the state price.

**Optimal Trading Strategy with Riskless Benchmark and Constant Coefficients**
Suppose the market coefficients $r, \mu$, and $\sigma$ are constant, $P_T = P_0 e^{rT}$, and the manager has a DARA uf described earlier. Let $k = K - w$.

Then the manager's optimal asset portfolio value is the process

$$X_t^R = e^{-r(T-t)}[\widehat{x}N(d_{1,t}) + (\widehat{x} - P_T + k/\alpha)(N(d_{2,t})\frac{N'(d_{1,t})}{N'(d_{2,t})} - N(d_{1,t}))] \quad (26)$$

and the manager's optimal trading strategy is

$$\pi_t^R = \{\frac{X_t^R}{1-\gamma} + e^{-r(T-t)}[\frac{\widehat{x}N'(d_{1,t})}{||\theta||\sqrt{T-t}} - \frac{P_T - k/\alpha}{1-\gamma}N(d_{1,t})]\}\Sigma^{-1}\mu \ , \quad (27)$$

where $\widehat{x} = \widehat{x}(P_T)$, $d_{1,t} = \frac{\ln(z_R/M_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$, $d_{2,t} = d_{1,t} + \frac{||\theta||\sqrt{T-t}}{1-\gamma}$, and $N$ is the standard cumulative normal distribution function.

**Proposition 1** With the riskless benchmark and DARA utility,
(i) as $M_t \to 0$,

$$X_t^R \to +\infty \ \text{ and } \ \frac{\pi_t^R}{X_t^R} \to \frac{\Sigma^{-1}\mu}{1-\gamma} \ ,$$

(ii) as $M_t \to +\infty$,

$$X_t^R \to 0 \ , \pi_t^R \to 0 \ , \text{ and } ||\frac{\pi_t^R}{X_t^R}|| \to \infty \ ,$$

(iii) as $t \to \ T$,

$$\frac{\pi_t^R}{X_t^R} \to \frac{X_T^R - B_T + k/\alpha}{X_T^R} \ \frac{\Sigma^{-1}\mu}{1-\gamma} \ , \text{ if } M_T < z_R \ , \tag{28}$$

$$\pi_t^R \to 0 \ \text{ and } ||\frac{\pi_t^R}{X_t^R}|| \to \infty \ , \text{ if } M_T > z_R \ .$$

- Merton (1969, 1971) shows that in the standard investment problem with constant coefficients, the CRRA investor holds risky assets in the constant proportions $\frac{\pi_t^*}{X_t^*} = \frac{\Sigma^{-1}\mu}{1-\gamma}$, which implies that his portfolio has constant volatility.
- Part (i) of Proposition 1 says that when underlying asset value is high, the manager with the option approaches the same constant volatility trading strategy he would follow if he were solving the standard problem, or if he were paid a linear share of profits.

- By contrast, part (ii) of the proposition indicates that when the asset portfolio is performing poorly, the value of the risky asset holdings goes to zero to meet the solvency constraint, but that value goes to zero more slowly than the total portfolio value, so that the proportional risky asset holdings, and thus portfolio volatility, converge to infinity as bankruptcy approaches.
- To illustrate, Figure 3 plots the optimal proportional holdings of risky assets as a function of total asset value for a CRRA manager with a riskless benchmark.
- Part (iii) of Proposition 1 says that as the evaluation date draws near, in states in which the manager finishes in the money, the proportional risky asset holdings converge to his Merton constant net of leverage.
- Equation (28) in part (iii) of Proposition 1 implies that in some states the volatility of the managed assets can actually be less than the Merton constant volatility that a CRRA investor solving the standard investment problem would choose. In particular, this will be the case if $-P_T + k/\alpha < 0$, $t$ is near $T$, and the option is in the money, as is visible in Figure 3. The reason is that the leverage inherent in the option magnifies the manager's exposure to the asset volatility. If the option is a large component of his compensation, he reduces asset volatility to offset the option's leverage effect.

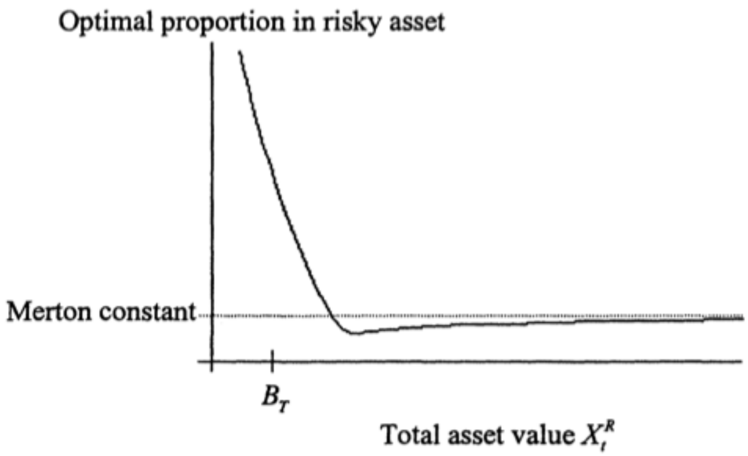



**Figure 3. Optimal trading strategy for a CRRA manager.** Optimal proportion of portfolio value invested in the risky asset as a function of portfolio value, $X_t^R$, one year prior to the evaluation date $T$. Terminal portfolio value $X_T^R$ maximizes $\mathrm{E}[U(\alpha(X_T - B_T)^+ + K)]$ subject to the constraints that $\mathrm{E}[\zeta_T X_T] \le X_0$, $X_T \ge 0$, and $X_T$ is measurable with respect to the filtration generated by asset prices up to time $T$. $\alpha = 0.15$, $B_T = 1$, $K = 0.03$, and the utility function $U$ is CRRA with coefficient $1 - \gamma = 2$. Intermediate fund value is $X_t^R = \mathrm{E}_t\{(\zeta_T/\zeta_t)X_T^R\}$ where the state price density process is $\zeta_t = e^{-rt-\theta W_t - \theta^2 t/2}$ with $r = 0$, $\theta = 0.4$, and $W_t$ Brownian motion at time $t$. The Sharpe ratio $\theta$ on the risky asset is $\mu/\sigma$ where the risky asset's excess expected return is $\mu = 0.08$ and its volatility is $\sigma = 0.2$. The Merton constant is $\mu/\sigma^2(1-\gamma)$, the optimal proportion for a CRRA investor solving the standard terminal wealth problem.

**Proposition 2** With the riskless benchmark, CRRA utility, and outside wealth $K = K_T \equiv K_0 e^{rT}$,

$$\lim_{T\to\infty} \frac{\pi_0^R}{X_0^R} = \frac{\Sigma^{-1}\mu}{1-\gamma}\,. \tag{29}$$

Under the assumptions of the proposition, the solution is essentially invariant to the interest rate. In particular, $e^{-rt}X_t^R$ and $e^{-rt}\pi_t^R$ are invariant to $r$. It follows that increasing the evaluation period $T$ is essentially equivalent to increasing the Sharpe ratio $||\theta||$, which is like putting the manager deeper in the money.

**Proposition 3** With the riskless benchmark and DARA or CARA utility, increasing the number of options, $\alpha$, holding asset value constant, causes the manager to reduce asset volatility.

The intuition for this is that giving the manager more options increases his exposure to the risky assets in the managed portfolio, which he offsets by decreasing its risky holdings.

**Optimal Payoff with the MVE Portfolio as Benchmark** Another benchmark portfolio for which it is possible to give a more explicit solution is $P_T = P_0/M_T$, the instantaneously mean-variance efficient portfolio held by the log utility investor.

**Corollary 1** The optimal asset payoff with benchmark $P_T = P_0/M_T$ is

$$X_T^M = [(I(\lambda_M M_T/\alpha) - K)/\alpha + P_0/M_T]1_{\{M_T < z_M\}} \ , \qquad (30)$$

where $\lambda_M$ solves $\mathrm{E}[M_T h(\lambda M_T, P_0/M_T)1_{\{h(\lambda^* M_T, P_0/M_T) > \widehat{x}(P_0/M_T)\}}] = X_0$ and $z_M$ is the unique zero of

$$g(M_T) = u'(\widehat{x}(M_0/M_T), P_0/M_T) - \lambda_M M_T \ . \qquad (31)$$

- Again, the states in which the manager is in the money are states in which the sdf is below a critical level.
- And as with the riskless benchmark, when the manager is in the money, he first "buys back the benchmark," and then invests managed assets in his optimal portfolio net of his leverage from the option.

**Optimal Trading Strategy with MVE Benchmark and Constant Coefficients** Let $\widehat{b} = P_0/z_M$ and let $\widehat{x} = \widehat{x}(\widehat{b})$. Let $\pi^P = P_t\Sigma^{-1}\mu$, the trading strategy that generates the MVE benchmark portfolio. Then the optimal portfolio process is

$$X_t^M = P_t N(d_{5,t}) + e^{-r(T-t)}[(\widehat{x} - \widehat{b} + k/\alpha)N(d_{4,t})\frac{N'(d_{3,t})}{N'(d_{4,t})} - (k/\alpha)N(d_{3,t})] \qquad (32)$$

and the manager's optimal trading strategy is

$$\begin{aligned}\pi_t^M &= \pi_t^P N(d_{5,t}) \\ &\quad + e^{-r(T-t)}[\frac{\widehat{x}N'(d_{3,t})}{||\theta||\sqrt{T-t}} + \frac{(\widehat{x} - \widehat{b} + k/\alpha)}{1-\gamma}N(d_{4,t})\frac{N'(d_{3,t})}{N'(d_{4,t})}]\Sigma^{-1}\mu \ ,\end{aligned} \qquad (33)$$

where $d_{3,t} = \frac{\ln(z_M/M_t) + (r - ||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$, $d_{4,t} = d_{3,t} + \frac{||\theta||\sqrt{T-t}}{1-\gamma}$, and $d_{5,t} = d_{3,t} + ||\theta||\sqrt{T-t}$.

Note that $d_3$ and $d_4$ are just $d_1$ and $d_2$ with the critical state price $z_R$ replaced by $z_M$.

**Comparison of Trading Strategies** To compare the trading strategies with the riskless and MVE benchmarks, rewrite the riskless benchmark trading strategy in equation (27) as

$$\pi_t^R = e^{-r(T-t)}[\frac{\widehat{x}N'(d_{1,t})}{||\theta||\sqrt{T-t}} + \frac{(\widehat{x} - P_T + k/\alpha)}{1-\gamma}N(d_{2,t})\frac{N'(d_{1,t})}{N'(d_{2,t})}]\Sigma^{-1}\mu \ . \qquad (34)$$

- Comparing the two trading strategies in equations (33) and (34) shows that the trading strategy with the MVE benchmark consists of a component that tracks the benchmark, weighted by the factor $N(d_{5,t})$, and a component that behaves like the riskless benchmark trading strategy.
- As the portfolio looks more and more likely to finish in the money, $N(d_{5,t})$ approaches one, so the manager essentially undoes the effect of the benchmark and then invests in the optimal portfolio for a riskless benchmark, a result similar to that of Admati and Pfleiderer (1997).

## Risk Choice Under High-Water Marks

- A risk-neutral fund manager operates in a complete continuous-time financial market with a single risky asset and constant market coefficients $r, \mu$, and $\sigma$.
- The manager dynamically chooses fraction $\pi_t$ of managed wealth $W_t$ to invest in the risky asset.
- The manager earns a flow management fee $mW_t$ and a fraction $k$ of increments in fund value above its high-water mark $H_t$.
- When the fund value is at the high-water mark, $W_t = H_t$, then when the fund value increases from $W_t = H_t$ to $W_t = H_t + dH_t^\varepsilon$, a performance fee $kdH_t^\varepsilon$ is paid out of the fund to the manager, and the HWM is reset to $H + H_t + dH_t^\varepsilon$.
- Otherwise, the HWM appreciates at rate $r$, and is adjusted downward for fund withdrawals at rate $\phi_t$ and management fees at rate $m$.
- Fund value $W_t$ and the HWM $H_t$ evolve according to

$$\frac{dW_t}{W_t} = r\,dt + \pi_t(\mu - r)\,dt + \pi_t\sigma\,dB_t - (\phi_t + m)\,dt - kdH_t^\varepsilon \ , \qquad (35)$$

$$dH_t = (r - \phi_t + m)H_t\,dt + dH_t^\varepsilon \ . \qquad (36)$$

- Let $X_t = \frac{W_t}{H_t}$. Then $dX_t = \frac{dW_t}{H_t} - \frac{W_t}{H_t^2} dH_t$, or

$$dX_t = X_t \pi_t (\mu - r)\, dt + X_t \pi_t \sigma\, dB_t - (1+k) dH_t^\varepsilon / H_t \ . \tag{37}$$

- The manager is terminated if fund value falls below $CH_t$. The fund is also terminated at exogenous rate $\lambda$.
- Let $V(X_t, H_t)$ denote the manager's value function. When the fund is terminated, the manager receives his "outside option" $\bar{V}_t = gV(1, H_t)$.
- Let $\tau$ be the termination time of the manager or $\infty$ if he is never terminated.
- His objective is to choose a trading strategy to maximize his expected discounted value of his income stream

$$V_t = V(X_t, H_t = \max_{\pi_t} \mathrm{E}\{\int_t^\tau e^{-\rho(s-t)}(mW_s\, ds + kdH_s^\varepsilon + e^{-\rho(\tau - t)}\bar{V}_t\} \ , \tag{38}$$

where $\rho$ is his subjective discount factor.

- The process $e^{-\rho t} V_t + \int_0^t e^{-\rho s}(mW_s + kdH_s^\varepsilon)$ is a martingale under the optimal trading strategy $\pi_t$ so it satisfies the HJB equation

$$0 = -\rho V_t + \lambda(\bar{V}_t - V_t) + mX_t H_t + \sup_{\pi_t}\{V_X X_t \pi_t (\mu - r) + \frac{1}{2} V_{XX} X_t^2 \pi_t^2 \sigma^2\} \tag{39}$$

$$+ V_H H_t (r - \phi_t - m) + kdH_t^\varepsilon) - V_X (1+k) \frac{dH_t^\varepsilon}{H_t} + V_H\, dH_t^\varepsilon \ . \tag{40}$$

- Assuming $V_X \geq 0$ and $V_{XX} < 0$, as will be verified, the optimal strategy is

$$\pi_t^* = \frac{\mu - r}{\sigma^2(-XV_{XX}/V_X)} \ . \tag{41}$$

## Solution

- Conjecture $V(X_t, H_t) = \beta_1 H_t G(X_t)$
  - $\beta_1$ is chosen so $G(1) = 1$ (normalization)

**Proposition**

$$G(X_t) = \left(\frac{X_t - D_0}{D_1}\right)^{\eta} + D_2$$
$$\beta_1 = \frac{k}{G_X(1)(1+k) - 1}$$

where

$$\eta = \frac{\rho + \lambda - r + \phi + m}{\omega + \rho + \lambda - r + \phi + m} \qquad \omega = \frac{1}{2}\frac{(\mu - r_f)^2}{\sigma^2} > 0$$
$$D_2 = \frac{m_H \beta_1^{-1} + \lambda g_0}{\rho + \lambda - r + \phi + m} \qquad = \text{the \textit{“risk-shutdown” payoff}}$$

- Note: $0 < \eta < 1 \Rightarrow V_{XX} < 0$



Introduction Model Solution Risk Choice Optimal Walk-Away Withdrawals Conclusion

## Solution

**Proposition (continued)**

if the outside option is sufficiently large ($g_0 \geq D_2$),

$$D_0 = C - \frac{1 - C}{(1 - D_2)^{1/\eta} - (g_0 - D_2)^{1/\eta}}(g_0 - D_2)^{1/\eta}$$
$$D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta} - (g_0 - D_2)^{1/\eta}}$$

otherwise, if the outside option is small ($g_0 < D_2$),

$$D_0 = C$$
$$D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta}}$$

- If $g_0 \leq D_2$

$\Rightarrow$ $\pi_t^*(C) = 0$ (risk shutdown) and the fund remains at $X_t = C$

$\Rightarrow$ $G(C) = D_2$ and hence $D_2$ represents the risk-shutdown payoff

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## Optimal Risk Choice

$$\pi_t^* = \underbrace{\frac{1}{(1-\eta)\left(\frac{X_t}{X_t - D_0}\right)}}_{\text{effective 'risk-aversion'}} \frac{\mu - r}{\sigma^2}$$

If $D_0 > 0$ (e.g., termination 'strict', risk-shutdown payout large, outside payout small)

1. $X_t \downarrow$ implies effective RA $\uparrow$ and manager *reduces* risk ('**de-risking**')
2. Effective risk aversion is strongly nonlinear in $X_t$: $\frac{d \ln RA}{dX_t} = \frac{-D_0}{X_t(X_t - D_0)}$
3. If $D_0 = C$ then RA $\to \infty$ as $X_t \downarrow C$
4. RA function looks like habits a la Campbell and Cochrane (1999)
   - CC local coefficient of RRA is $\gamma \frac{C_t/H_t}{C_t/H_t - 1}$
   - $X_t$ acts like $C_t/H_t$ for the manager but arises *endogenously*

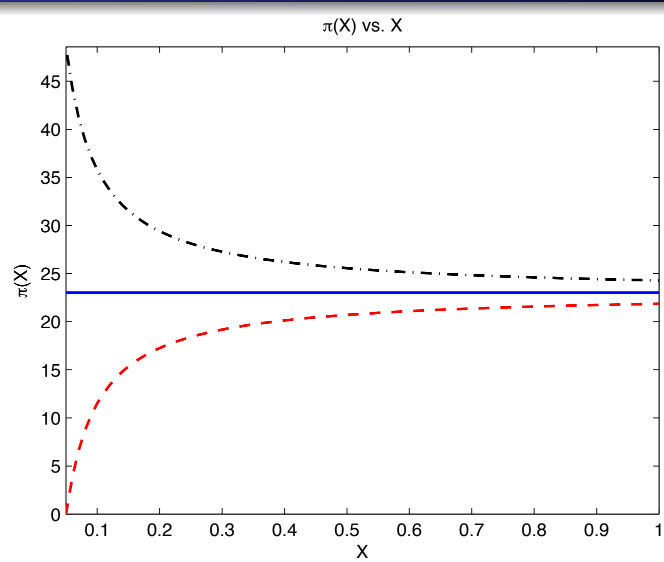
If $D_0 < 0$ (e.g., termination 'loose', risk-shutdown payout small, outside payout large)

1. $X_t \downarrow$ implies effective RA $\downarrow$ and manager *increases* risk ('**gambling**')





## Distance to HWM and Risk Taking: 3 Cases




- dashed line: $D_0 > 0$
- dash-dot line: $D_0 < 0$
- solid line: $D_0 = 0$