

## Incomplete Markets and Portfolio Constraints

1. Portfolio constraint set, support function, effective domain
2. Family of hypothetical financial markets with adjusted appreciation rates
3. Constrained utility maximization problem
  - (a) Unconstrained problem in hypothetical markets with adjusted appreciation rates
  - (b) Constrained solution as unconstrained solution in particular hypothetical market
4. No-arbitrage upper and lower bounds on derivative prices
  - (a) Upper bound price  $\equiv$  infimal superreplication cost
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  - (d) Upper bound price with constant coefficients and path-independent payoff = unconstrained replication cost of supremal adjusted payoff function
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  - (f) Lower bound price with constant coefficients and path-independent payoff = unconstrained replication cost of infimal adjusted payoff function

### Selected Readings and References

- Karatzas and Shreve, 1998, chapters 5 and 6.
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## Summary of the Continuous-Time Financial Market

- ▶ Security prices satisfy  $\frac{dS_{0,t}}{S_{0,t}} = r_t dt$  and  $\frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t}) dt + \sigma_{k,t} dB_t$ .
- ▶ Given tight tr. strat.  $\pi_t$  and consumption  $c_t$ , portfolio value  $X_t$  satisfies the WEE  $dX_t = r_t X_t dt + \pi_t(\mu_t - r_t \mathbf{1}) dt + \pi_t \sigma_t dB_t - c_t dt$ .
- ▶ No arbitrage  $\Rightarrow$  if  $\pi_t \sigma_t = 0$  then  $\pi_t(\mu_t - r_t \mathbf{1}) = 0 \Rightarrow \exists \theta_t$  s.t.  $\sigma_t \theta_t = \mu_t - r_t \mathbf{1} \Rightarrow dX_t = r_t X_t dt + \pi_t \sigma_t \theta_t dt + \pi_t \sigma_t dB_t - c_t dt$ .
- ▶ Under emm  $\mathcal{P}^*$  given by  $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$  where  $Z_t = e^{-\int_0^t \theta'_s dB_s - \int_0^t |\theta_s|^2 ds}$ ,  $B_t^* = B_t + \int_0^t \theta_s ds$  is Brownian motion. Let  $\beta_t = e^{-\int_0^t r_s ds}$  and sdf process  $M_t = \beta_t Z_t$ . Then the WEE can also be written several other potentially convenient ways:

$$dX_t = r_t X_t dt + \pi_t \sigma_t dB_t^* - c_t dt, \quad (1)$$

$$d\beta_t X_t = \beta_t \pi_t \sigma_t dB_t^* - \beta_t c_t dt, \quad (2)$$

$$dM_t X_t = M_t [\pi_t \sigma_t - \theta_t X_t] dB_t - M_t c_t dt. \quad (3)$$

- ▶ So  $X_t = E_t^* \left\{ \int_t^T \frac{\beta_u}{\beta_t} c_u du + \frac{\beta_T}{\beta_t} X_T \right\} = E_t \left\{ \int_t^T \frac{M_u}{M_t} c_u du + \frac{M_T}{M_t} X_T \right\}$  if  $\pi$  is mtgale-gen., and  $X_t \geq E_t^* \left\{ \int_t^T \frac{\beta_u}{\beta_t} c_u du + \frac{\beta_T}{\beta_t} X_T \right\} = E_t \left\{ \int_t^T \frac{M_u}{M_t} c_u du + \frac{M_T}{M_t} X_T \right\}$  if  $\pi$  is tame.
- ▶ If  $\sigma$  is nonsingular, every c.plan  $(c, X_T)$  can be generated by a mtgale-gen. tr.strat.

## Additional Regularity Conditions

For this lecture, suppose we're in a complete standard continuous-time financial market (so the number of securities  $n$  equals the number of Brownian motions  $d$ ) with the following modifications:

1. The volatility process  $\sigma(t)$  is bounded and nonsingular, with  $\sigma^{-1}(t)$  bounded, uniformly in  $(t, \omega) \in [0, T] \times \Omega$ .
2. There exists a positive constant  $s_0$  such that  $S_0(T) \geq s_0$  a.s.

In addition, we'll be working entirely under the true probability measure so the process  $Z(t) = e^{-\int_0^t \theta'(s) dB(s) - \frac{1}{2} \int_0^t |\theta(s)|^2 ds}$  may be a martingale but need not be.

## Portfolio Constraints, Convex Sets, and Support Functions

Suppose the investor is required to keep proportional portfolio holdings  $p(t) \equiv \pi(t)/X^{x,C,\pi}(t)$  in a nonempty, closed, convex set  $K \subseteq \mathcal{R}^n$ . Define  $p(t) \equiv p_*$  for some arbitrary fixed vector  $p_* \in K$  whenever  $X^{x,C,\pi}(t) = 0$ .

**Definition 1** The *support function* of the convex set  $-K$ ,  $\zeta : \mathcal{R}^n \rightarrow \mathcal{R} \cup \{+\infty\}$ , is

$$\zeta(\nu) \equiv \sup_{p \in K} (-p\nu), \nu \in \mathcal{R}^n. \quad (4)$$

Its *effective domain* is the convex cone

$$\tilde{K} \equiv \{\nu \in \mathcal{R}^n : \zeta(\nu) < \infty\}. \quad (5)$$

Note that  $0 \in \tilde{K}$ ,  $\zeta(0) = 0$ ,  $\zeta(\alpha\nu) = \alpha\zeta(\nu)$  for every  $\nu \in \mathcal{R}^n$  and  $\alpha \geq 0$ ,  $\zeta(\nu + \mu) \leq \zeta(\nu) + \zeta(\mu)$  for every  $\nu, \mu \in \mathcal{R}^n$ , and especially

$$p \in K \Leftrightarrow \zeta(\nu) + p\nu \geq 0 \quad \forall \nu \in \tilde{K}. \quad (6)$$

Assume that  $\zeta$  is bounded below on  $\mathcal{R}^n$ . This will be true, for example, if  $0 \in K$ .

### Examples

1. No constraint:  $K = \mathcal{R}^n$ . Then  $\tilde{K} = \{0\}$  and  $\zeta \equiv 0$  on  $\tilde{K}$ .
2. No short sales:  $K = [0, \infty)^n$ . Then  $\tilde{K} = K$  and  $\zeta \equiv 0$  on  $\tilde{K}$ .
3. Incomplete market:  $K = \{p \in \mathcal{R}^n : p_{m+1} = \dots = p_n = 0\}$  for some  $m \in \{1, \dots, n-1\}$ . Then  $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 = \dots = \nu_m = 0\}$  and  $\zeta \equiv 0$  on  $\tilde{K}$ .
4. Incomplete market with no short sales:  $K = \{p \in \mathcal{R}^n : p_1 \geq 0, \dots, p_m \geq 0, p_{m+1} = \dots = p_n = 0\}$  for some  $m \in \{1, \dots, n-1\}$ . Then  $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 \geq 0, \dots, \nu_m \geq 0\}$  and  $\zeta \equiv 0$  on  $\tilde{K}$ .
5. No borrowing:  $K = \{p \in \mathcal{R}^n : p_1 \leq 1\}$ . Then  $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 = \dots = \nu_n \leq 0\}$  and  $\zeta(\nu) = -\nu_1$  on  $\tilde{K}$ .
6. Constraints on short sales:  $K = [-\kappa, \infty)^n$  for some  $\kappa > 0$ . Then  $\tilde{K} = [0, \infty)^n$  and  $\zeta(\nu) = \kappa\nu'_1$  on  $\tilde{K}$ .
7. Constraints on borrowing:  $K = \{p \in \mathcal{R}^n : p_1 \leq \kappa\}$  for some  $\kappa > 1$ . Then  $\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_1 = \dots = \nu_n \leq 0\}$  and  $\zeta(\nu) = -\kappa\nu_1$  on  $\tilde{K}$ .

8. Rectangular constraints:  $K = I_1 \times \cdots \times I_n$  with  $I_j = [\alpha_j, \beta_j]$ ,  $-\infty \leq \alpha_j \leq 0 \leq \beta_j \leq +\infty$ , with the understanding that  $I_j$  is open on the left if  $\alpha_j = -\infty$  and  $I_j$  is open on the right if  $\beta_j = \infty$ . If all the  $\alpha_j$  and  $\beta_j$  are finite, then

$$\tilde{K} = \mathcal{R}^n \text{ and } \zeta(\nu) = - \sum_{j=1}^n (\alpha_j \nu_j^+ - \beta_j \nu_j^-) . \quad (7)$$

More generally,

$$\tilde{K} = \{\nu \in \mathcal{R}^n : \nu_j \geq 0 \forall j \text{ s.t. } \beta_j = \infty \text{ and } \nu_j \leq 0 \forall j \text{ s.t. } \alpha_j = -\infty\} \quad (8)$$

and the above formula for  $\zeta$  remains valid.

### Family of Hypothetical Financial Markets Associated with Constraint Set $K$

1. Let  $\mathcal{D}$  be the set of well-behaved  $\tilde{K}$ -valued adapted processes  $\nu_t$  and  $\mathcal{D}^b \subset \mathcal{D}$  the subset of bounded ones.
2. For each  $\nu_t \in \mathcal{D}$ , construct the  $\nu$ -financial market from the original financial market with the following adjusted expected returns:
  - ▶  $r_{\nu,t} = r_t + \zeta(\nu_t)$  and
  - ▶  $\mu_{\nu,t} = \mu_t + \nu_t + \zeta(\nu_t)\mathbf{1}$ .
3. Then in the  $\nu$ -financial market, the mpr and sdf are
  - ▶  $\theta_{\nu,t} = \sigma_t^{-1}(\mu_{\nu,t} - r_{\nu,t}\mathbf{1}) = \theta_t + \sigma_t^{-1}\nu_t$  and
  - ▶  $M_{\nu,t} = e^{-\int_0^t r_{\nu,s} ds - \int_0^t \theta'_{\nu,s} dB_s - \frac{1}{2} \int_0^t |\theta_{\nu,s}|^2 ds}$ ,
 and portfolio value  $X_\nu^{x,c,p}$  satisfies the  $\nu$ -WEE

$$dX_\nu^{x,c,p} = (r_\nu X_\nu^{x,c,p} - c) dt + X_\nu^{x,c,p} p \sigma (\theta_\nu + dB_t) . \quad (9)$$

## Investor's Constrained Optimization Problem

- ▶ Let  $\tau_0 = \inf\{t \in [0, T] : X_t^{x,c,p} = 0\}$ .
- ▶ Let  $\mathcal{A}(x, K)$  be the set of consumption plans and trading strategies  $(c, p)$  s.t.  $p_t \in K$  a.s. and  $c_t = 0$  for all  $t \in [\tau_0, T]$ , so that  $X_t^{x,c,p} = 0$  for all  $t \in [\tau_0, T]$ .
- ▶ Given  $x > 0$ , the investor's constrained optimization problem is

$$V(x; K) \equiv \sup_{(c,p) \in \mathcal{A}(x,K)} \mathbb{E}\left\{\int_0^T U(c_t, t) dt + U(X_T^{x,c,p})\right\} \quad (10)$$

## Family of Unconstrained Problems in the Associated $\nu$ -Financial Markets:

- ▶ Let  $\tau_\nu = \inf\{t \in [0, T] : X_{\nu,t}^{x,c,p} = 0\}$ .
- ▶ Let  $\mathcal{A}_\nu(x)$  be the set of consumption plans and trading strategies  $(c, p)$  s.t.  $c_t = 0$  for all  $t \in [\tau_\nu, T]$  a.s.
- ▶ Given  $x > 0$  and  $\nu \in \mathcal{D}$ , the investor's unconstrained optimization problem in the  $\nu$ -market is

$$V_\nu(x) \equiv \sup_{(c,p) \in \mathcal{A}_\nu(x)} \mathbb{E}\left\{\int_0^T U(c_t, t) dt + U(X_{\nu,T}^{x,c,p})\right\} . \quad (11)$$

- ▶ Following from our earlier results on optimal consumption and portfolio choice, the solution to the unconstrained problem in the  $\nu$ -financial market is  $(c_\nu, p_\nu)$  s.t.

$$c_{\nu,t} = I(\lambda_\nu M_{\nu,t}, t) , \quad X_{\nu,T}^{x,c_\nu,p_\nu} = I(\lambda_\nu M_{\nu,T}, T) , \quad \text{and} \quad (12)$$

$$X_{\nu,t}^{x,c_\nu,p_\nu} = \mathbb{E}_t\left\{\int_t^T \frac{M_{\nu,u}}{M_{\nu,t}} c_{\nu,u} du + \frac{M_{\nu,T}}{M_{\nu,t}} X_{\nu,T}^{x,c_\nu,p_\nu}\right\} . \quad (13)$$

## Key Results

**Proposition 1** If  $(c, p) \in \mathcal{A}(x, K)$  and  $\nu \in \mathcal{D}$ , then  $X_{\nu,t}^{x,c,p} \geq X_t^{x,c,p}$  a.s., which implies  $\mathcal{A}(x, K) \subset \mathcal{A}_\nu(x)$  and  $V(x, K) \leq V_\nu(x)$  for all  $x > 0, \nu \in \mathcal{D}$ .

Moreover, if  $\zeta(\nu_t) + p_t \nu_t = 0$  for a.e.  $t \in [0, T]$  a.s., then  $X_{\nu,t}^{x,c,p} = X_t^{x,c,p}$  a.s.

### Intuition for Proof

$$dX_\nu^{x,c,p} = (r_\nu X_\nu^{x,c,p} - c) dt + X_\nu^{x,c,p} p \sigma(\theta_\nu + dB_t) \quad (14)$$

$$= (r X_\nu^{x,c,p} - c) dt + X_\nu^{x,c,p} p \sigma(\theta + dB_t) + X_\nu^{x,c,p} (\zeta(\nu_t) + p \nu_t) dt. \quad (15)$$

Since  $\zeta(\nu_t) + p \nu_t \geq 0$ , it follows that the drift of  $X_\nu^{x,c,p}$  is at least as great as the drift of  $X_\nu^{x,c,p}$ .

**Proposition 2** Let  $x > 0$  be given and suppose  $\exists \hat{\nu} \in \mathcal{D}_0$  s.t. the optimal trading strategy  $p_{\hat{\nu},t}$  for the unconstrained problem in the  $\nu$ -market satisfies  $p_{\hat{\nu},t} \in K$  and  $\zeta(\hat{\nu}_t) + p_{\hat{\nu},t} \hat{\nu}_t = 0$  for a.e.  $t \in [0, T]$  a.s.

Then  $(c_{\hat{\nu},t}, p_{\hat{\nu},t})$  is optimal for the constrained problem in the original market,  $V(x; K) = V_{\hat{\nu}}(x)$ , and  $\hat{\nu}$  minimizes  $V_\nu(x)$ .

**Proof** On one hand, from Proposition 1,  $V(x; K) \leq V_\nu(x)$  for all  $\nu$ , including  $\hat{\nu}$ .

On the other hand,  $p_{\hat{\nu},t}$  is feasible for the constrained problem in the original market and  $X_T^{x,c_{\hat{\nu}},p_{\hat{\nu}}} = X_{\hat{\nu},T}^{x,c_{\hat{\nu}},p_{\hat{\nu}}}$ , so

$$V(x; K) \geq E\left\{\int_0^T U(c_{\hat{\nu},t}, t) dt + U(X_T^{x,c_{\hat{\nu}},p_{\hat{\nu}}}, T)\right\} \quad (16)$$

$$= E\left\{\int_0^T U(c_{\hat{\nu},t}, t) dt + U(X_{\hat{\nu},T}^{x,c_{\hat{\nu}},p_{\hat{\nu}}}, T)\right\} \quad (17)$$

$$= V_{\hat{\nu}}(x) \quad (18)$$

Therefore,  $V(x; K) = V_{\hat{\nu}}(x)$  and  $\hat{\nu}$  minimizes  $V_\nu(x)$ .

**Interpretation**  $V_\nu$  is like a Lagrangian for the constrained optimization problem in the original market,  $-\nu_t$  is like a Lagrange multiplier for the constraint  $p_t \in K$ , and  $\zeta(\nu_t) + p_t \nu_t = 0$  is like a complementary slackness condition.

## Equivalent Optimality Conditions

Let  $x > 0$ ,  $(\hat{c}, \hat{p}) \in \mathcal{A}(x; K)$ , and  $\hat{X}_t \equiv X_t^{x, \hat{c}, \hat{p}}$ . Consider first the statement

(A) Optimality of  $(\hat{c}, \hat{p})$  for the original constrained problem:

$$V(x; K) = E\left\{\int_0^T U_1(\hat{c}_t, t) dt + U_2(\hat{X}_T)\right\}. \quad (19)$$

Next, consider the following statements in terms of a process  $\hat{\nu} \in \mathcal{D}_0$ .

(B) Financeability of  $(c_{\hat{\nu}}, p_{\hat{\nu}})$ : The optimal trading strategy in the  $\hat{\nu}$ -market,  $p_{\hat{\nu}, t}$ , is s.t.  $(c_{\hat{\nu}, t}, p_{\hat{\nu}, t}) \in \mathcal{A}(x; K)$  and  $\zeta(\hat{\nu}_t) + p_{\hat{\nu}, t} \hat{\nu}_t = 0$  for a.e.  $t \in [0, T]$  a.s.

(C) Minimality of  $\hat{\nu}$ :  $V_{\hat{\nu}}(x) \leq V_{\nu}(x)$  for every  $\nu \in \mathcal{D}$ .

**Theorem 1** Conditions (B) and (C) are equivalent and imply (A) with  $(\hat{c}, \hat{p}) = (c_{\hat{\nu}, t}, p_{\hat{\nu}, t})$ . Conversely, (A) implies the existence of a process  $\hat{\nu} \in \mathcal{D}_0$  that satisfies (B) and (C) with  $p_{\hat{\nu}} = \hat{p}$ , provided that  $U_1$  and  $U_2$  satisfy the following conditions:

- (a) the map  $x \mapsto xU'_i(x)$  is nondecreasing on  $(0, \infty)$ ,  $i = 1, 2$ , and
- (b)  $\exists \beta \in (0, 1), \gamma \in (0, \infty)$  s.t.  $\beta U'_i(x) \geq U'_i(\gamma x) \forall x \in (0, \infty)$ ,  $i = 1, 2$ .

For the proof, see Karatzas and Shreve (1998), pp. 276-282, 335-347.

## Jensen's Alpha

Jensen's (1968) CAPM alpha was designed to evaluate how much a mutual fund is outperforming the market, or how much its return is in excess of the return required by the CAPM given its risk, i.e., the intercept in the following time-series regression:

$$R_p - R_f = \alpha + \beta(R_m - R_f) + \varepsilon_p. \quad (20)$$

The concept is used widely and has been generalized to Fama-French 4-factor alpha, and could be applied to a security return as well as to a fund return:

$$R_p - R_f = \alpha + \beta_1 RMRF + \beta_2 SMB + \beta_3 HML + \beta_4 WML + \varepsilon_p. \quad (21)$$

- ▶ The point is that a security's alpha, or abnormal return, depends on the asset pricing model adopted.
- ▶ One might say that in a complete market with a unique sdf where investors can trade freely, no security or portfolio has alpha, all payoffs are priced fairly.

## Subjective Alphas for Constrained Investors

- ▶ But if investors face constraints, or hold suboptimal portfolios, they may perceive other securities to have alpha, in the sense that they would improve utility by tilting toward positive alpha securities and away from negative alpha securities.
- ▶ The solution methodology here gives a way to formalize that idea.
- ▶ Note that the solution to the constrained problem is the solution to the unconstrained problem in the  $\hat{\nu}$ -market, in which the investor sets marginal utility proportional to the  $\hat{\nu}$ -sdf  $M_{\hat{\nu}}$ .
- ▶ Thus,  $M_{\hat{\nu}}$  can be regarded as the investor's subjective sdf – if a security's actual price equals its expected subjectively discounted payoff, the investor will not want to buy more or less. Otherwise the investor will want to tilt toward or away.

- ▶ Assuming zero dividend for ease of exposition, we can define security  $k$ 's alpha as the appreciation rate of  $M_{\hat{\nu}}S_k$ . From Itô's lemma,

$$\frac{dM_{\hat{\nu}}S_k}{M_{\hat{\nu}}S_k} = [r - r_{\hat{\nu}} + \sigma_k(\theta - \theta_{\hat{\nu}})] dt + (\sigma_k - \theta_{\hat{\nu}}) dB \quad (22)$$

$$= [-\zeta(\hat{\nu}) - \sigma_k\sigma^{-1}\hat{\nu}] dt + (\sigma_k - \theta_{\hat{\nu}}) dB \quad (23)$$

$$= [-\zeta(\hat{\nu}) - \hat{\nu}_k] dt + (\sigma_k - \theta_{\hat{\nu}}) dB . \quad (24)$$

So, for example, in the case of incomplete markets,

- $\zeta(\hat{\nu}) = 0$ ,
- $\hat{\nu}_k < 0$  for securities the investor would like to buy,
- and  $\hat{\nu}_k > 0$  for securities the investor would like to sell,

and  $-\hat{\nu}_k$  gives a measure of the alpha the investor holding the constrained optimal portfolio perceives security  $k$  to have.

- ▶ The presence of differential portfolio constraints and preferences creates a role for fund managers to add value by relaxing those constraints where alpha is high enough to justify their fees.



## Problem

Consider a complete, standard continuous-time financial market with a single risky asset and assume that  $\sigma_t > 0$  is bounded and bounded away from zero,  $r_t$  is bounded below, and  $\theta_t$  is bounded. Suppose an investor with initial wealth  $x$  derives utility from consumption plan  $\{c_t\}$  and terminal wealth  $W$  equal to

$$\mathbb{E} \int_0^T \log c_t dt + \log W .$$

Suppose the investor is required to keep the proportion of his wealth  $p_t$  that is invested in the risky asset in the range  $[\alpha, \beta]$  where  $-\infty < \alpha \leq 0 \leq \beta < \infty$ . Solve for the optimal proportion process  $p_t$ . Also, give the optimal consumption process in feedback form.