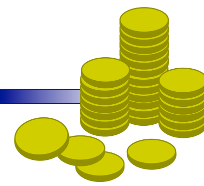
Coupon Bonds and Zeroes



Outline

- Coupon bonds
- Zero-coupon bonds (“zeroes”)
- Replicating coupon bonds from zeroes
- Law of one price between bond prices and zero prices
- Zero prices implied by bond prices
- Zero rates
- Zero rates implied by bond prices
- Bid and ask prices and rates
- Yield curve/term structure of interest rates

Reading

- Tuckman and Serrat, Chapters 1 and 2

Coupon Bonds

- In practice, the most common form of debt instrument is a coupon bond. In the U.S and in many other countries, coupon bonds pay coupons every six months and par value at maturity.
- The quoted coupon rate is annualized. That is, if the quoted coupon rate is c, and bond maturity is time T, then each $1 par value (quantity) of the bonds pays out cash flows

c/2	c/2	c/2	…	1 + c/2
0.5 years	1 year	1.5 years	…	T years

- For N par value, the bond cash flows are

Nc/2	Nc/2	Nc/2	…	N(1 + c/2)
0.5 years	1 year	1.5 years	…	T years

Class Problem

The current "long bond," the newly issued 30-year Treasury bond, is the 3%-coupon bond maturing August 15, 2048.

What are the cash flows of $100,000 par of this bond?
(Dates and amounts.)

…

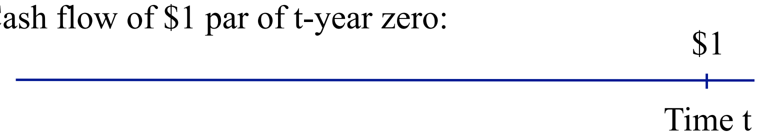
…

U.S. Treasury Notes and Bonds

- Institutionally speaking, the prices of government bonds form the basis for the pricing in fixed income markets.
- All other fixed income instruments, including derivatives, are priced in relation to the prices of these benchmark bonds.
- The underlying analytics rely on the argument that
  1. any fixed income instrument can be replicated with a (possible very complicated) portfolio or dynamic trading strategy using these bonds, and
  2. then it can be priced at its replication cost, by no arbitrage.
- The US Treasury auctions new 2-, 3-, 5-, 7-year notes monthly, and 10-year notes and 30-year bonds quarterly. See http://www.wsj.com/mdc/public/page/2_3020-treasury.html for a listing of outstanding Treasury bonds.

Zeroes

- For the analytics of fixed income valuation and risk management, it is convenient to unpack the original government coupon bonds into individual **zero-coupon bonds**, **or zeroes**—bonds with a single cash flow equal to face value at maturity.
- Cash flow of $1 par of t-year zero:



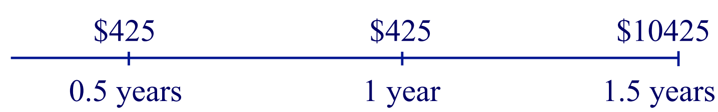
- In the US, Treasury zeroes called STRIPS are actually traded in a secondary market.
- In general, however, we derive the zero prices from coupon bond prices mathematically, using the **Law of One Price**.
- It is easy to trade across the Treasury coupon bond and STRIPS markets and eliminate arbitrage opportunities, so the actual STRIPS and implied zero prices match very closely in practice.

Zero Prices or "Discount Factors"

- Let d_t denote the price today of the t-year zero, the asset that pays off $1 in t years.
- I.e., d_t is the price of a t-year zero as a fraction of par value.
- This is also called the t-year "discount factor."
- Current prices of traded UST STRIPS, in percent of par value, are available at http://www.wsj.com/mdc/public/page/2_3020-tstrips.html?mod=mdc_bnd_pglnk

A Coupon Bond as a Portfolio of Zeroes

Consider: $10,000 par of a one and a half year, 8.5% Treasury bond makes the following payments:



Note that this is the same as a portfolio of three different zeroes:

–$425 par of a 6-month zero
–$425 par of a 1-year zero
–$10425 par of a 1 1/2-year zero

No Arbitrage and The Law of One Price

- **The Law of One Price** *Two assets which offer exactly the same cash flows must sell for the same price.*
- Why? If not, then one could buy the cheaper asset and sell the more expensive, making a profit today with no cost in the future.
- This would be an **arbitrage opportunity**, which could not persist in equilibrium in a frictionless market
- However, when there are "limits to arbitrage" such as transaction costs, capital constraints, or barriers to trading across markets, then violations of the law of one price can persist.
- Example: The A-H premium for dual-listed Chinese stocks.

Real-Life Example:
Valuing a Coupon Bond Using Zero Prices

Let's value $10,000 par of a 1.5-year 8.5% Treasury coupon bond based on the zero prices (discount factors) in the table below.

These discount factors come from actual STRIPS prices listed in the WSJ on 11/15/1995. We will use these discount factors for many examples throughout the course.

Years to Maturity	Discount Factor	Bond Cash Flow	Value
0.5	0.9730	$425	$414
1.0	0.9476	$425	$403
1.5	0.9222	$10425	$9614
			Total $10430

On the same day, the WSJ reported that the 1.5-year 8.5%-coupon bond was priced at 104 10/32 or 104.3125 percent of par, which is within 1/32 of its no-arbitrage price.

An Arbitrage Opportunity

- What if the 1.5-year 8.5% coupon bond were worth only 104% of par value?
- You could buy, say, \$1 million par of the bond for \$1,040,000 and sell the cash flows off individually as zeroes for total proceeds of \$1,043,000, making \$3000 of riskless profit.
- Similarly, if the bond were worth 105% of par, you could buy the portfolio of zeroes, reconstitute them, and sell the bond for riskless profit.

The Law of One Price for Coupon Bonds and Zeroes

By the L.O.O.P., if a bond has coupon c and maturity T, then, in terms of the zero prices d_t, its price *P(c, T)* per \$1 par must be the same as the price of a portfolio of zeroes with the the same cash flows:

$$\boxed{P(c,T) = (c/2) \times d_{0.5} + (c/2) \times d_1 + \ldots + (1 + c/2) \times d_T}$$

Class Problems

Suppose instead, the discount factors are:

$d_{0.5}$=0.9996 , d_1=0.9984 , and $d_{1.5}$=0.9964.

1) What would be the price of $100 par of an 8.5%-coupon, 1.5-year bond?

2) What would be the price of $100 par of a 2%-coupon, 1-year bond?

Fundamental Theorem of Bond Math

- More generally, consider a debt instrument with fixed cash flows (as opposed to a debt instrument with random cash flows, such as an option or mortgage-backed security).
- If it pays cash flows K_1, K_2, …, K_n, at times t_1, t_2, …, t_n, it is the same as the portfolio of

  K_1 t_1-year zeroes + K_2 t_2-year zeroes + … + K_n t_n-year zeroes
- Therefore, the **Law of One Price** requires that the asset's value V relates to the zero prices, or discount factors, as follows:

$$V = K_1 \times d_{t_1} + K_2 \times d_{t_2} + ... + K_n \times d_{t_n}$$

$$\text{or} \quad V = \sum_{j=1}^{n} K_j \times d_{t_j}$$

Deriving Zero Prices from Coupon Bond Prices

- If we know the prices coupon bonds with maturities every 0.5 years, then we can derive the implied prices of the zeroes.
- Real-Life Example: On the same trading day as above, the three coupon bonds below were priced as follows:

Coupon Bond	Years to Maturity	Coupon	Price in 32nds	Price in Decimal
#1	0.5	4.250%	99-13	99.40625
#2	1.0	4.375%	98-31	98.96875
#3	1.5	8.50%	104-10	104.31250

- We derive the implied zero prices $d_{0.5}$, d_1, and $d_{1.5}$ using the information above to give 3 equations in the 3 unknown "d"s:

1. $(1+0.0425/2)d_{0.5} = 0.9940625 \Rightarrow d_{0.5}=0.973$
2. $(0.04375/2)d_{0.5} + (1+0.04375/2)d_1 = 0.9896875 \Rightarrow d_1=0.948$
3. $(0.085/2)d_{0.5} + (0.085/2)d_1 + (1+0.085/2)d_{1.5} = 1.043125$ $\Rightarrow d_{1.5}=0.922$, which closely match the actual STRIPS prices.

Class Problems

1) Suppose the price of the 4.25%-coupon, 0.5-year bond is 99.50. What is the implied price of a 0.5-year zero per $1 par?

2) Suppose the price of the 4.375%-coupon, 1-year bond is 99. What is the implied price of a 1-year zero per $1 par?

Market Frictions

- In practice, prices of Treasury STRIPS and Treasury bonds don't fit the pricing relationship exactly
  - transaction costs and search costs in stripping and reconstituting
  - bid/ask spreads
- Note: The terms "bid" and "ask" are from the viewpoint of the dealer.
  - The dealer buys at the bid and sells at the ask, so the bid price is always less than the ask.
  - The customer sells at the bid and buys at the ask.

Interest Rates

- People try to summarize information about bond prices and cash flows by quoting interest rates.
- Buying a zero is lending money--you pay money now and get money later
- Selling a zero is borrowing money--you get money now and pay later
- A bond transaction can be described as
  - buying or selling at a given price, or
  - lending or borrowing at a given rate.
- The convention in U.S. bond markets is to use **semi-annually compounded interest rates**.

Annual vs. Semi-Annual Compounding

At 10% per year, *annually* compounded, \$100 grows to \$110 after 1 year, and \$121 after 2 years:

$$100 \times 1.10 = 110$$

$$100 \times (1.10)^2 = 121$$

10% per year *semi-annually* compounded really means 5% every 6 months. At 10% per year, *semi-annually* compounded, \$100 grows to \$110.25 after 1 year, and \$121.55 after 2 years:

$$100 \times (1.05)^2 = 110.25$$

$$100 \times (1.05)^4 = 121.55$$

Annual vs. Semi-Annual Compounding...

After T years, at annually compounded rate r_A, P grows to

$$F = P(1 + r_A)^T$$

Present value of F to be received in T years with annually compounded rate r_A is

$$P = \frac{F}{(1 + r_A)^T}$$

In terms of the semi-annually compounded rate r, the formulas become

$$F = P(1 + r/2)^{2T} \qquad P = \frac{F}{(1 + r/2)^{2T}}$$

$\boxed{\text{The key: } (1 + r/2)^2 = 1 + r_A}$

An (annualized) semi-annually compounded rate of r per year really means r/2 every six months.

Zero Rates

- If you buy a t-year zero and hold it to maturity, you lend at rate r_t where r_t is defined by

$$d_t \times (1 + r_t/2)^{2t} = 1,$$

$$\text{or } d_t = \frac{1}{(1 + r_t/2)^{2t}},$$

$$\text{or } r_t = 2 \times ((\frac{1}{d_t})^{\frac{1}{2t}} - 1)$$

- Call r_t the t-year zero rate or t-year discount rate.

Class Problems: Rate to Price

- According to market convention, zero prices are quoted using rates. Sample STRIPS rates from the 11/15/95 WSJ:
  - 0.5-year rate: 5.54%
  - 1-year rate: 5.45%

1) What is the 0.5-year zero price?

2) What is the 1-year zero price?

Class Problems: Price to Rate

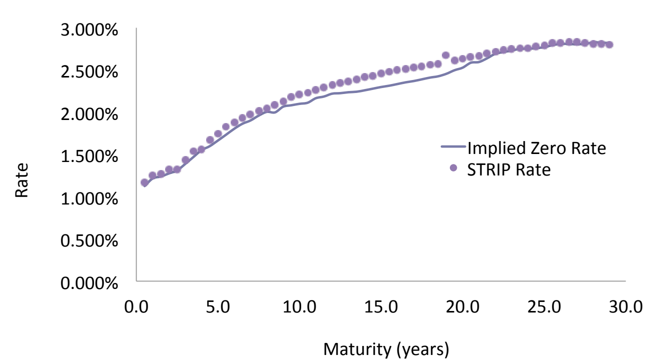
On 8/16/17 the 10-year zero bid price was 0.7853 and the ask price was 0.7861.

1. What was the 10-year zero bid rate?

2. What was the 10-year zero ask rate?

Yield Curve of US Treasury Zero Rates 9/6/2017

- A "yield curve" summarizes the pricing of bonds of different maturity by plotting yields or zero rates for different maturities. It depicts the "term structure of interest rates."
- This graph plots the zero rates implied by Treasury coupon bond prices (line), and the actual traded Treasury STRIPS rates (dots).
- It shows that the Law of One Price holds very closely across the US Treasury coupon bond and STRIPS markets.



Value of a Stream of Cash Flows in Terms of Zero Rates

- Recall that any asset with fixed cash flows can be viewed as a portfolio of zeroes.
- So its price must be the sum of its cash flows multiplied by the relevant zero prices:

$$V = \sum_{j=1}^{n} K_j \times d_{t_j}$$

- Equivalently, the price is the sum of the present values of the cash flows, discounted at the zero rates for the cash flow dates:

$$V = \sum_{j=1}^{n} \frac{K_j}{(1+r_{t_j}/2)^{2t_j}}$$

Example

\$10,000 par of a 1.5-year, 8.5% Treasury bond makes the following payments:

\$425	\$425	\$10425
0.5 years	1 year	1.5 years

Using STRIPS rates from the 11/15/95 WSJ to value these cash flows, the bond price is:

$$V = \frac{\$425}{(1+0.0554/2)^1} + \frac{\$425}{(1+0.0545/2)^2} + \frac{\$10425}{(1+0.0547/2)^3}$$

$$= \$10430$$