Convexity

Outline

- Curvature of the Price-Yield Relation
- Dollar Convexity
- Convexity
- Barbells vs. Bullets

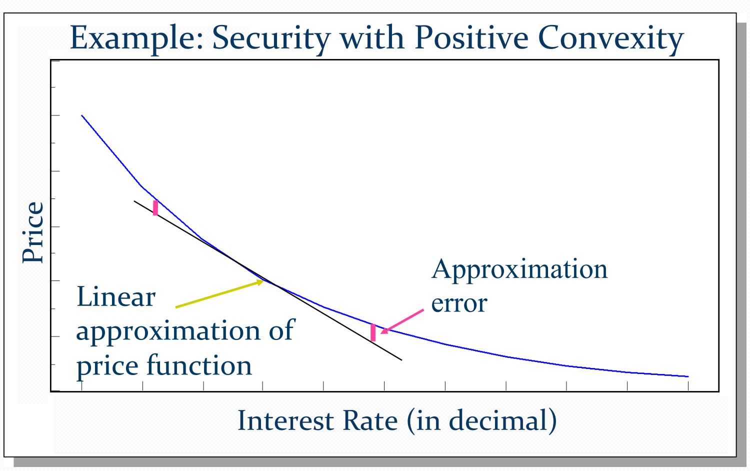
Readings

- Tuckman and Serrat, Chapter 4

Convexity

- Convexity is a measure of the curvature of the value of a security or portfolio as a function of interest rates.
- Duration is related to the slope, i.e., the first derivative.
- Convexity is related to the curvature, i.e. the second derivative of the price function.
- Using convexity together with duration gives a better approximation of the change in value given a change in interest rates than using duration alone.

Price-Rate Function



Correcting the Duration Error

- The price-rate function is nonlinear.
- Duration and dollar duration use a linear approximation to the price rate function to measure the change in price given a change in rates.
- The error in the approximation can be substantially reduced by making a convexity correction.

Dollar Convexity

- Think of bond prices, or bond portfolio values, as functions of interest rates.
- The Taylor Theorem says that if we know the first and second derivatives of the price function (at current rates), then we can approximate the price impact of a given change in rates.

$$f(x) - f(x_0) \approx f'(x_0) \times (x - x_0) + 0.5 \times f''(x_0) \times (x - x_0)^2$$

★ The first derivative is minus dollar duration.

★ Call the second derivative *dollar convexity*.

- Then change in price ≈ -$duration x change in rates

  + 0.5 x $convexity x change in rates squared

Dollar Convexity of a Portfolio

If we assume all rates change by the same amount, then

the dollar convexity of a portfolio is the sum of the dollar convexities of its securities.

Sketch of proof:

$$\sum f_i(x) - f_i(x_0) \approx \left(\sum f_i'(x_0)\right) \times (x - x_0) + 0.5 \times \left(\sum f_i''(x_0)\right) \times (x - x_0)^2$$

I.e., Δ portfolio value ≈ - (sum of dollar durations) x Δr

+ 0.5 x (sum of dollar convexities) x $(\Delta \text{rates})^2$

⇒Portfolio dollar duration = sum of dollar durations

⇒Portfolio dollar convexity = sum of dollar convexities

Convexity

- Just as dollar duration describes dollar price sensitivity, dollar convexity describes curvature in dollar performance.
- To get a scale-free curvature measure, i.e., curvature per dollar invested, we define

$$\text{convexity} = \frac{\text{dollar convexity}}{\text{price}}$$

⇒*The convexity of a portfolio is the average convexity of its securities, weighted by present value:*

$$\text{convexity} = \frac{\sum \text{price}_i \times \text{convexity}_i}{\sum \text{price}_i} = \text{pv wtd average convexity}$$

- Just like dollar duration and duration, dollar convexities add, convexities average.

Dollar Formulas for $1 Par of a Zero

For $1 par of a t-year zero-coupon bond

$$\text{price} = d_t(r_t) = \frac{1}{(1+r_t/2)^{2t}}$$

$$\text{dollar duration} = -d_t'(r_t) = \frac{t}{(1+r_t/2)^{2t+1}}$$

$$\text{dollar convexity} = d_t''(r_t) = \frac{t^2+t/2}{(1+r_t/2)^{2t+2}}$$

For $$N$ par, these would be multiplied by N.

Percent Formulas for Any Amount of a Zero

$$\text{duration} = \frac{\text{dollar duration}}{\text{price}} = \frac{N \times t/(1+r_t/2)^{2t+1}}{N \times 1/(1+r_t/2)^{2t}} = \frac{t}{1+r_t/2}$$

$$\text{convexity} = \frac{\text{dollar convexity}}{\text{price}} = \frac{N \times (t^2+t/2)/(1+r_t/2)^{2t+2}}{N \times 1/(1+r_t/2)^{2t}} = \frac{t^2+t/2}{(1+r_t/2)^2}$$

- These formulas hold for any par amount of the zero – they are scale-free.
- The duration of the t-year zero is approximately t.
- The convexity of the t-year zero is approximately t^2.
- (If we defined price as $d_t = e^{-rt}$, and differentiated w.r.t. this r, then the duration of the t-year zero would be exactly t and the convexity of the t-year zero would be exactly t^2.)

Example: Dollar Convexity of a Coupon Bond

The dollar convexity of a coupon bond is the sum of the dollar convexities of its individual cash flows (zeroes):

$$\frac{c}{2}\left[\frac{0.25+0.5/2}{(1+r_{0.5}/2)^3}+\frac{1+1/2}{(1+r_1/2)^4}+\frac{2.25+1.5/2}{(1+r_{1.5}/2)^5}+\ldots+\frac{T^2+T/2}{(1+r_T/2)^{2T+2}}\right]+\frac{T^2+T/2}{(1+r_T/2)^{2T+2}}.$$

Class Problems

Calculate the price, dollar duration, and dollar convexity of $1 par of the 20-year zero if r_{20} = 6.50%.

Class Problems

Suppose r_{20} rises to 7.50%.

1) Approximate the price change of $1,000,000 par using only dollar duration.

2) Approximate the price change of $1,000,000 par using both dollar duration and dollar convexity.

3) What is the exact price change?

Class Problems

Suppose r_{20} falls to 5.50%.

1) Approximate the price change of $1,000,000 par using only dollar duration.

2) Approximate the price change of $1,000,000 par using both dollar duration and dollar convexity.

3) What is the exact price change?

Sample Risk Measures

Duration and convexity for $1 par of a 10-year, 20-year, and 30-year zero.

Maturity	Rate	Price	$Duration	Duration	$Convexity	Convexity
10	6.00%	0.553676	5.375493	9.70874	54.7987	98.9726
20	6.50%	0.278226	5.389364	19.37046	107.0043	384.5951
30	6.40%	0.151084	4.391974	29.06977	129.8015	859.1356

For zeroes,

- duration is roughly equal to maturity,
- convexity is roughly equal to maturity squared.

Barbells and Bullets

- Consider two portfolios with the same duration:
  - A *barbell* invested in a long-term zero and a short-term zero
  - A *bullet* invested in an intermediate-term zero
- The barbell will have more convexity.
- Stylized example to illustrate the point simply:
- Consider a bullet portfolio invested 100% in a 20-year zero.
  - Its duration is ~20. Its convexity is $\sim 20^2 = 400$.
- Consider a barbell portfolio invested 50% in 10-year zero and 50% in a 30-year zero.
  - Its duration is $\sim 0.5 \times 10 + 0.5 \times 30 = 20$.
  - Its convexity is $\sim 0.5 \times 10^2 + 0.5 \times 30^2 = 500$.
- More generally, for a given duration, the more disperse the cash flows, the greater the convexity. **For example, a coupon bond will have greater convexity than the same-duration zero.**

Exact Example

Consider a barbell consisting of $25,174 par value of the 10-year zero and $91,898 par value of the 30-year zero. Let's see that this barbell has the same market value and duration as $100,000 par of the 20-year zero:

Maturity	Rate	Price	$Duration	Duration	$Convexity	Convexity
10	6.00%	0.553676	5.375493	9.70874	54.7987	98.9726
20	6.50%	0.278226	5.389364	19.37046	107.0043	384.5951
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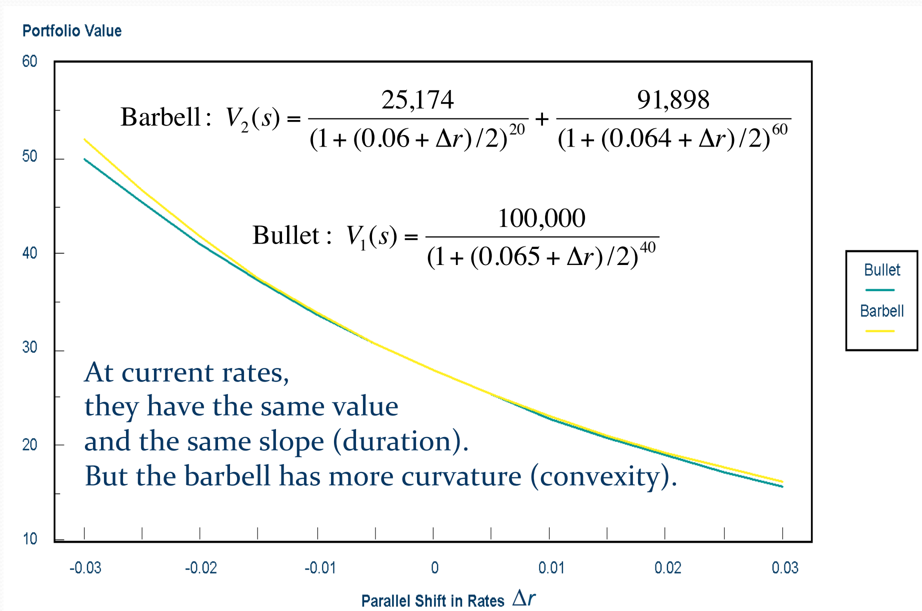
1) What is the market value of the barbell portfolio?
   25,174 x 0.553676 + 91,898 x 0.151084 = 13,938 + 13,884 = 27,822
2) What is the dollar duration of the barbell portfolio?
   25,174 x 5.375 + 91,898 x 4.392 = 538,926
3) What is the duration of the barbell? 538,926/27,822 = 19.37
4) What is the dollar convexity of the portfolio?
   25,174 x 54.799 + 91,898 x 129.502 = 13,280,485
5) What is its convexity? 13,280,485/27,822 = 477

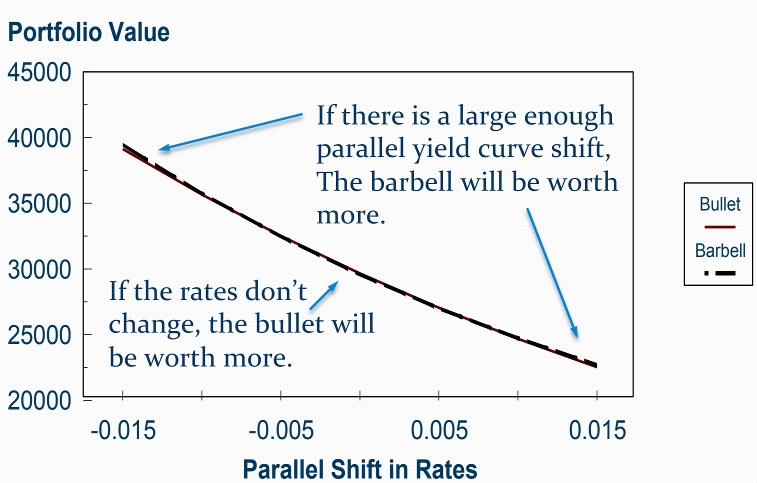
Value of Barbell and Bullet



Does the Barbell Always Outperform the Bullet?

- If there is an immediate parallel shift in interest rates, either up or down, then the barbell will outperform the bullet.
- If the shift is not parallel, anything could happen.
- If the rates on the bonds stay exactly the same, then in this example, as time passes, the bullet will actually outperform the barbell:
  - the bullet will return 6.5%
  - the barbell will return about 6.2%, the market value-weighted average of the 6% and 6.4% on the 10- and 30-year zeroes.

Value of Barbell and Bullet: One Year Later



Convexity and the Shape of the Yield Curve?

- If the yield curve were flat and made parallel shifts, more convex portfolios would always outperform less convex portfolios, and there would be arbitrage.
- So to the extent that market movement is described by parallel shifts, bullets must have higher yield to start with, to compensate for lower convexity.
- This would explain why the term structure is often hump-shaped, dipping down at very long maturities where convexity is greatest relative to duration—investors may give yield to buy convexity.
- Some evidence suggests that the yield curve is more curved when volatility is higher and convexity is worth more.