Forward Contracts and Forward Rates

Outline and Readings

- Outline
  - Forward Contracts
  - Forward Prices
  - Forward Rates
  - Information in Forward Rates
- Reading
  - Tuckman and Serrat, Chapters 2 and 13

Forward Contracts

- A *forward contract* is an agreement to buy an asset at a future *settlement date* at a *forward price* specified today.
  - No money changes hands today.
  - The pre-specified forward price is exchanged for the asset at settlement date.
- By contrast, an ordinary transaction that settles immediately is called a *spot* or *cash* transaction, and the price is called the *spot* price or *cash* price.

Motivation – Hedging? Speculating?

- Hedging: Suppose today, time 0, you know you will need to do a transaction at a future date, time t.
  - One thing you can do is wait until time *t* and then do the transaction at prevailing market price, i.e., do a *spot* transaction in the *future.*
  - Alternatively, you can try to lock in the terms of the transaction today, i.e., arrange a *forward* transaction *today*.
- Speculating: A long or short position in a forward contract by itself is a bet on the price of the underlying asset that does not involve paying cash up front.

Financial Forward Contract as a Portfolio

- On the settlement date t, the long side pays F and takes delivery of a financial asset worth V_t

```
                 V_t - F
|-----------------|
0                 t
```

- What is the PV of this contract? It is a portfolio:
  - long one unit of the underlying (excluding any payments it makes before the settlement date)
  - short F par of t-year zeros
- So PV forward contract = $-F \times d_t + V_0^{\text{ex pv of interim pmts}}$
- On the settlement date the contract is worth zero.
- To make contract worth zero, $F = (V_0^{\text{ex pv of interim pmts}})(1+r_t/2)^{2t}$

  i.e., forward price = spot price + interest to the settlement date

Bond Forward Contract as a Portfolio of Zeroes

- If the underlying is a bond, for example, a zero maturing at time T, the forward contract is a portfolio of zeroes:

```
                 -$F               +$1
|-----------------|-----------------|
0                 t                 T
```

- What is the PV of this contract?
- It is a portfolio:

  Long \$1 par of T-year zeros

  Short \$F par of t-year zeros
- So its present value is $V = -F \times d_t + 1 \times d_T$

Forward Price of Zero Maturing at T for Settlement at t.

- At t=0 the contract "costs" zero.
- The forward price is negotiated to make that true.
- What is the forward price that makes the contract worth zero?

```
                    -$F           +$1
|----------------|----------------|
0                t                T
```

$V = -F \times d_t + 1 \times d_T = 0$

→ $F = d_T / d_t = d_T\,(1+r_t/2)^{2t}$ = spot price + interest to the settlement date.

- We'll call this forward price F_t^T.

Class Problems

Recall the spot prices of $1 par of the 0.5-, 1-, and 1.5-year zeroes for our classroom examples are 0.9730, 0.9476, and 0.9222.

1) What is the no-arbitrage forward price of the 1-year zero for settlement at time 0.5?

2) What is the no-arbitrage forward price of the 1.5-year zero for settlement at time 1?

Class Problem

- Suppose a firm has an old forward contract on its books.
- The contract commits the firm to buy, at time t=0.5, \$1000 par of the zero maturing at time T=1.5 for a price of \$950.
- At inception, the contract was worth zero, but now markets have moved. What is the value of this contract to the firm now?

Forward Contract on a Zero as a Forward Loan

- Just as we can think of the spot purchase of a zero as lending money, we can think of a forward purchase of a zero as a *forward loan*.
- The forward lender agrees today to lend F_t^T on the settlement date t and get back \$1 on the date T.
- Define the *forward rate*, f_t^T, as the interest rate earned from lending F_t^T for T-t years and getting back \$1:

$$F_t^T = \frac{1}{(1+f_t^T/2)^{2(T-t)}} \qquad f_t^T = 2\left(\left(\frac{1}{F_t^T}\right)^{\frac{1}{2(T-t)}} - 1\right)$$

- This is the same transaction, just described in terms of lending or borrowing at rate instead of buying or selling at a price.

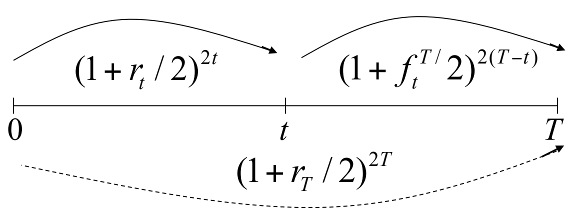
Arbitrage Argument in Terms of Rates: New Riskless Lending Possibilities

- Consider the lending possibilities when a forward contract for lending from time t to time T is available.
- Now there are two ways to lend risklessly from time 0 to time T:
  1) Lend at the current spot rate r_T (i.e., buy a T-year zero). A dollar invested at time 0 would grow risklessly to $(1+r_T/2)^{2T}$.
  2) Lend risklessly to time t (i.e., buy a t-year zero) and roll the time t payoff into the forward contract to time T. A dollar invested at time 0 would grow risklessly to $(1+r_t/2)^{2t} \text{x} (1+f_t^T/2)^{2(T-t)}$.

No Arbitrage Forward Rate

In the absence of arbitrage, the two ways of lending risklessly to time T must be equivalent:

$$(1+r_t/2)^{2t} \times (1+f_t^T/2)^{2(T-t)} = (1+r_T/2)^{2T}$$



Example: The forward rate from time t = 0.5 to time T = 1 must satisfy

$$(1+0.0554/2)^1 \times (1+f_{0.5}^1/2)^1 = (1+0.0545/2)^2$$

$$\Rightarrow f_{0.5}^1 = 5.36\%$$

No Arbitrage Forward Rate…

$$(1+r_t/2)^{2t}\times(1+f_t^T/2)^{2(T-t)} = (1+r_T/2)^{2T}$$

$$\Rightarrow (1+f_t^T/2)^{2(T-t)} = \frac{(1+r_T/2)^{2T}}{(1+r_t/2)^{2t}}$$

$$\Rightarrow f_t^T = 2[(\frac{(1+r_T/2)^{2T}}{(1+r_t/2)^{2t}})^{1/[2(T-t)]} - 1]$$

Class Problem:
The 1.5-year zero rate is $r_{1.5}$ = 5.47%. What is the forward rate from time t = 0.5 to time T=1.5?

Connection Between Forward Prices and Forward Rates

Of course, this is the same as the no arbitrage equations we saw before:

$$(1+f_t^T/2)^{2(T-t)} = \frac{(1+r_T/2)^{2T}}{(1+r_t/2)^{2t}} \Leftrightarrow F_t^T = \frac{d_T}{d_t}$$

Example: The implied forward rate for a loan from time 0.5 to time 1 is 5.36%. This gives a discount factor of 0.9739, which we showed before is the synthetic forward price to pay at time 0.5 for the zero maturing at time 1.

$$\frac{1}{(1+f_t^T/2)^{2(T-t)}} = \frac{(1+r_t/2)^{2t}}{(1+r_T/2)^{2T}} = \frac{d_T}{d_t} = F_t^T$$

$$\frac{1}{(1+0.0536/2)^1} = \frac{(1+0.0554/2)^1}{(1+0.0545/2)^2} = \frac{0.9476}{0.9730} = 0.9739$$

Summary: One No Arbitrage Equation, Three Economic Interpretations:

(1) Forward price = Spot price + Interest to the settlement date

$$F_t^T = d_T \times (1 + r_t/2)^{2t}$$

(2) Present value of forward contract cash flows at inception = 0:

$$-d_t \times F_t^T + d_T \times 1 = 0$$

(3) Lending short + Rolling into forward loan = Lending long:

$$(1 + r_t/2)^{2t} \times (1 + f_t^T/2)^{2(T-t)} = (1 + r_T/2)^{2T}$$

Using the relations between prices and rates,

$$d_t = \frac{1}{(1 + r_t/2)^{2t}} \quad \text{and} \quad F_t^T = \frac{1}{(1 + f_t^T/2)^{2(T-t)}} \quad \text{or} \quad f_t^T = 2\left(\left(\frac{1}{F_t^T}\right)^{\frac{1}{2(T-t)}} - 1\right)$$

we can verify that these equations are all the same. Other arrangements:

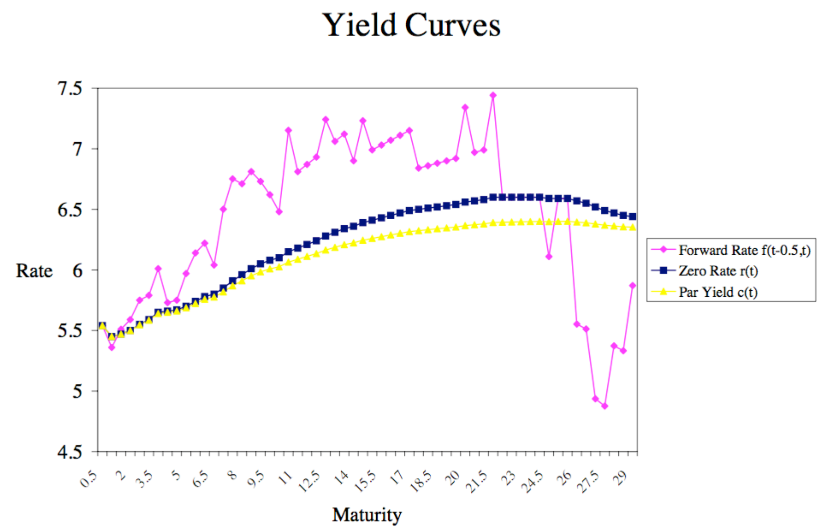
$$F_t^T = \frac{d_T}{d_t} \qquad (1 + f_t^T/2)^{2(T-t)} = \frac{(1 + r_T/2)^{2T}}{(1 + r_t/2)^{2t}}$$

Spot Rates as Averages of Forward Rates

- Rolling money through a series of short-term forward contracts is a way to lock in a long term rate and therefore synthesizes an investment in a long zero. Here are two ways to lock in a rate from time 0 to time t:

$$(1 + r_{0.5}/2) \times (1 + f_{0.5}^1/2) \times \cdots \times (1 + f_{t-0.5}^t/2) = (1 + r_t/2)^{2t}$$

- The growth factor *$(1+r_t/2)$* is the geometric average of the $(1+f/2)$'s and so the interest rate r_t is approximately the average of the forward rates.
- Recall the example
  - The spot 6-month rate is 5.54% and the forward 6-month rate is 5.36%.
  - Their average is equal to the 1-year rate of 5.45%.



Zero rates are averages of the one-period forward rates up to their maturity, so while the zero curve is rising, the marginal forward rate must be above the zero rate, and while the zero curve is falling, the marginal forward rate must be below the zero rate.

Forward Rates vs. Future Spot Rates

- The forward rate is the rate you can fix today for a loan that starts at some future date.
- By contrast, you could wait around until that future date and transact at whatever is the prevailing spot rate.
- Is the *forward rate* related to the random *future spot rate*?
- For example, **is the forward rate equal to people's expectation of the future spot rate**?

The Pure Expectations Hypothesis

- The "Pure Expectations Hypothesis" says that **the forward rate is equal to the expected future spot rate**.
- It turns out that's roughly equivalent to the hypothesis that **expected returns on all bonds over a given horizon are the same**, as if people were risk-neutral.
- For example, if the forward rate from time 0.5 to time 1 equals the expected future spot rate over that time, then the expected one-year rate of return from rolling two six-month zeroes is equal to the one-year rate of return from holding a one-year zero:

$$E({}_{0.5}\tilde{r}_1) = f_{0.5}^1$$

$$\Rightarrow E\{(1+r_{0.5}/2)(1+{}_{0.5}\tilde{r}_1/2)\} = (1+r_{0.5}/2)(1+f_{0.5}^1/2)$$

$$\Rightarrow E\{(1+r_{0.5}/2)(1+{}_{0.5}\tilde{r}_1/2)\} = (1+r_1/2)^2$$

Example in which the Pure Expectations Hypothesis Holds: Upward-Sloping Yield Curve

Time 0	Time 0.5	0.5-yr horizon		1-yr horizon	
	Zero rate	ROR on 0.5-yr z.	ROR on 1-yr z.	ROR on 0.5-yr z.	ROR on 1-yr z.
	${}_{0.5}r_1^u$=6.50%	5.00%	4.008%	5.749%	5.25%
${}_0r_{0.5}$=5.00% ${}_0r_1$=5.25%					
	${}_{0.5}r_1^d$=4.50%	5.00%	6.003%	4.750%	5.25%
Expected:	5.50%	5.00%	5.005%	5.249%	5.25%

Forward rate
$f_{0.5}^1$=5.50%

If the pure expectations hypothesis holds, then an upward-sloping yield curve indicates rates are expected to rise.

Example in which the Pure Expectations Hypothesis Holds: Downward-Sloping Yield Curve

Zero Rates			Rates of Return over Various Horizons			
Time 0	Time 0.5		0.5-Year ROR		1-Year ROR	
			0.5-yr zero	1-yr zero	0.5-yr zero	1-yr zero
	5.860%	(w.p. 50%)	5.540%	5.041%	5.700%	5.450%
5.540% 5.450%						
	4.860%	(w.p. 50%)	5.540%	6.042%	5.200%	5.450%
Expected:	5.360%		5.540%	5.541%	5.450%	5.450%
Forward rate: 5.360%						

If the pure expectation hypothesis holds, then the downward slope of the yield curve indicates that rates are expected to fall.

Problem with the Pure Expectations Hypothesis: Expected Rates of Returns Differ Across Bonds

- As we have seen, both theory and evidence suggest that different maturity bonds have different expected rates of return because their returns have different risk properties (variance, covariance with other risks, etc.).
- So the "pure expectations hypothesis" is only a conceptual benchmark, not general enough to describe actual bond pricing.
- The evidence suggests that longer maturity bonds have higher expected returns. Or equivalently, forward rates are higher than expected future spot rates.
- This is consistent with a yield curve that is typically upward sloping.
- The difference between a forward rate and the corresponding expected future spot rate is sometimes called the **term premium**. It is the longer bond risk premium in yield terms.

Example in which Longer Bonds Have Higher Expected Returns

Time 0	Time 0.5	0.5-yr horizon		1-yr horizon	
	Zero rate	ROR on 0.5-yr z.	ROR on 1-yr z.	ROR on 0.5-yr z.	ROR on 1-yr z.
	${}_{0.5}r_1^u$=6.00%	5.00%	4.503%	5.499%	5.25%
${}_0r_{0.5}$=5.00%, ${}_0r_1$=5.25%					
	${}_{0.5}r_1^d$=4.00%	5.00%	6.508%	4.499%	5.25%
Expected:	5.00%	5.00%	5.505%	4.999%	5.25%

Forward rate
$f_{0.5}^1$=5.50%

Here, the yield curve is upward-sloping, not because rates are expected to rise, but because longer bonds are priced to offer a higher expected return.

Some Evidence

Results of regressions of future spot rates on past forward rates,

$${}_{t+j}r_{t+j+1} - {}_tr_{t+1} = a + \beta({}_tf_{t+j}{}^{t+j+1} - {}_tr_{t+1}) + \varepsilon_{t,j}$$

for j=1, 2, 3, 4 years, sample period 1980-2006.
The Pure Expectation Hypothesis would imply α=0 and β=1.

Country	j	α	Std. err.	β	Std. err.	R^2
US	1	-0.30	0.33	0.11	0.26	0.21
	2	-0.70	0.82	0.25	0.42	1.16
	3	-1.45	1.12	0.72	0.37	8.39
	4	-2.25	1.09	1.22	0.25	21.17
UK	1	-0.19	0.26	0.49	0.23	9.34
	2	-0.74	0.52	1.00	0.27	26.17
	3	-1.01	0.66	1.18	0.31	34.26
	4	-1.45	0.66	1.40	0.33	46.28
Germany	1	-0.36	0.32	0.48	0.18	6.30
	2	-1.01	0.51	0.98	0.26	19.14
	3	-1.77	0.51	1.39	0.33	35.44
	4	-2.46	0.45	1.62	0.29	49.86

From Boudoukh, Richardson, Whitelaw, 2007, The information in long forward rates: Implications for exchange rates and the forward premium anomaly.