

# Bonds with Embedded Options and Option Pricing by Replication



## Outline

- Bonds with Embedded Options (callable bonds, fixed-rate mortgages, mortgage-backed securities, capped floaters, structured notes, structured deposits,...)
- Option Pricing by Replication
- Option Pricing with Risk-Neutral Probabilities
- Volatility Effects on Options and Callable Bonds

## Reading

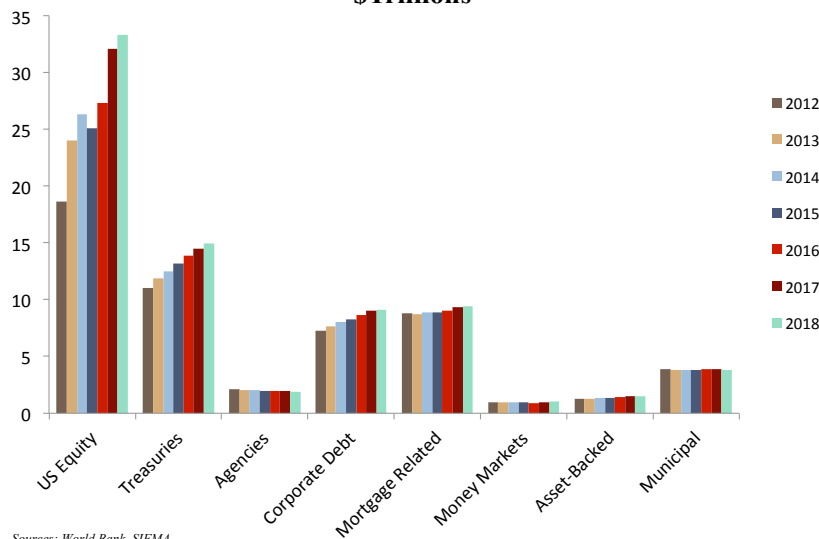
- Tuckman and Serrat, Chapter 7

## Bonds with Embedded Options

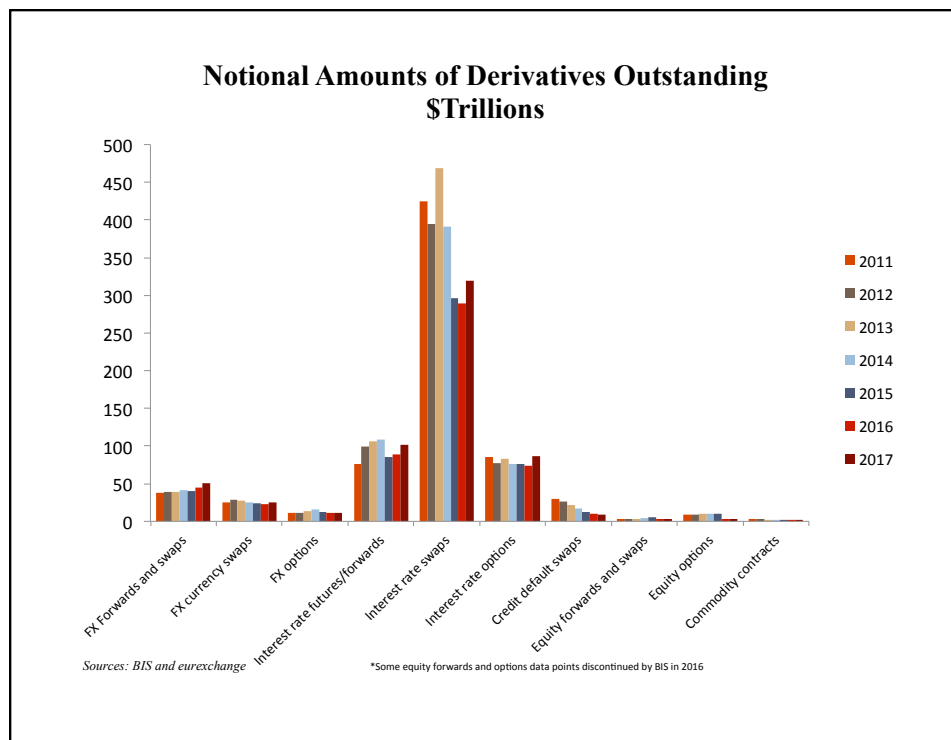


- Loans and bonds issued by households, firms, and government agencies frequently contain options that give these borrowers the flexibility to pay off their debt early or insurance against rising interest rates.
- Important examples are prepayment options in fixed rate mortgages and interest rate caps in floating rate mortgages. Short positions in these options are passed through to mortgage-backed securities investors.
- Similarly, corporate and government agency bonds frequently contain call provisions that allow these issuers to pay off their bonds early, at a pre-determined price.
- Embedded options complicate the valuation and risk management of callable bonds and mortgage-backed securities, which represent a large component of US debt markets.
- The need to hedge these options has driven the development of wholesale fixed income option markets.

**Market Value of US Debt and Equity Markets**  
\$Trillions



Sources: World Bank, SIFMA



## Bonds Minus Bond Options



- In most cases, the embedded option benefits the borrower, so that the added flexibility or insurance can reduce the risk of default.
- In these cases, we can view the bond with an embedded option as a straight bond minus a kind of option on that bond.
- For example, a bond that is callable by the issuer at par, is equivalent to a noncallable bond with the same coupon and maturity minus a call option on that noncallable bond with a strike price of par:
- Callable Bond = NC Bond – Call on that NC Bond
- A prepayable fixed rate mortgage is a callable bond with an amortizing principle.

## Fixed Income Options



- The theory of option pricing developed by Black, Scholes, and Merton for equity options has been extended to fixed income options and is one of the most widely used financial theories in practice.
- Models for pricing callable bonds, mortgage-backed securities, and fixed income options are readily available on Bloomberg.
- Although the financial engineering for fixed income options becomes extremely elaborate, the core principle of **pricing by replication** still underlies all option pricing models.
- The next few lectures illustrate the logic of this theory and the basics of fixed income option models.
- Then we'll apply these models to callable bonds and mortgage-backed securities, unpacking them into basic bonds and options, and clarifying their dynamics.

## One-Period Model of a Callable Bond

- Consider a \$100 par of a 1-year semi-annual 5.5%-coupon bond that is callable at par at time 0.5.
- The logic of option pricing can be seen with a one-period example.
- There are two trading dates, time 0 and time 0.5.
- At time 0.5, there is a high interest rate state (the “up state”) and a low interest rate state (the “down state”).
- Calibrating the model to the term structure from the class examples, ( $r_{0.5}=5.54\% \Rightarrow d_{0.5}=0.973047$  and  $r_1=5.45\% \Rightarrow d_1=0.947649$ ) as well as to historical interest rate volatility, gives the following model of zero prices, illustrated with a one-period binomial tree:

	<u>Time 0</u>	<u>Time 0.5</u>
\$1 par zero maturing at time 0.5	0.973047	1 0.970857
\$1 par zero maturing at time 1	0.947649	1 0.976941

### One-Year Semi-Annual 5.5% Bond Callable at Par

- If the bond were non-callable, then at time 0.5 its ex-coupon price would go to 99.76 in the up state and 100.38 in the down state.
- However, at time 0.5, the issuer has the option to call the bond for 100. The issuer's option is a call on the noncallable bond with a strike price equal to par.

	<u>Time 0</u>	<u>Time 0.5</u>
\$1 par zero maturing at time 0.5	0.973047	1
\$1 par zero maturing at time 1	0.947649	0.970857
Ex-coupon price of \$100 par of the noncallable 5.5%-coupon bond maturing at time 1.	$2.75 \times 0.9730$ $102.75 \times 0.9476$ $= 100.047$	$102.75 \times 0.9709$ $= 99.76$ 0
Issuer's call option	?	99.76
Ex-coupon price of \$100 par of the 5.5% bond maturing at time 1, callable at par at time 0.5, assuming the issuer maximizes the option value.	?	1 0.976941 $102.75 \times 0.9769$ $= 100.38$
*"Ex-coupon price" means the price excluding the coupon paid at the current date.		0.38 100

### Option Pricing by Replication

- The issuer's call option can be replicated with a portfolio of the "underlying asset," which can be taken to be the zero maturing at time 1 here, and a "riskless asset," the zero maturing at time 0.5 here.
- Consider a portfolio containing  $N_1 = \$62.574$  par of the zero maturing at time 1, and  $N_{0.5} = -60.75$  par of the zero maturing at time 0.5.
- *This portfolio has the same time 0.5 payoff as the call:*

	<u>Time 0</u>	<u>Time 0.5</u>
\$1 par zero maturing at time 0.5	0.973047	1
\$1 par zero maturing at time 1	0.947649	0.970857
Issuer's call option	?	0
Call-replicating portfolio	?	$-60.75 \times 1$ $+62.574 \times 0.970857 = 0$
• Class problem: What is the time 0 replication cost of the call?		1 0.976941 0.38 $-60.75 \times 1$ $+62.574 \times 0.9769 = 0.38$

### Class Problem: What is the Price of the Callable Bond?

	<u>Time 0</u>	<u>Time 0.5</u>
		1
\$1 par zero maturing at time 0.5	0.973047	0.970857
\$1 par zero maturing at time 1	0.947649	102.75*0.9709 = 99.76
Ex-coupon price of \$100 par of the non-callable 5.5% bond maturing at time 1.	2.75*0.9730 102.75*0.9476 = 100.047	0 99.76
Issuer's call option	?	1 0.976941
Ex-coupon price of \$100 par of the 5.5% bond maturing at time 1, callable at time 0.5 at par.	?	102.75*0.9769 = 100.38 0.38 100

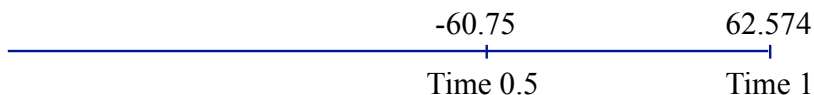
### Constructing the Replicating Portfolio

- To construct the replicating portfolio (the holdings  $N_{0.5}$  and  $N_1$ ), solve 2 equations, one to match the up-state payoff and one to match the down-state payoff, for the 2 unknown N's.
  - Up-state payoff:  $N_{0.5} * 1 + N_1 * 0.970857 = 0$
  - Down-state payoff:  $N_{0.5} * 1 + N_1 * 0.976941 = 0.38$
- The solution is  $N_{0.5} = -60.75$  par amount of zeroes maturing at time 0.5 and  $N_1 = 62.574$  par amount of the zero maturing at time 1.
- This is replication is important for a dealer who has to hedge a short position in the option (or swaption, in practice) using bonds.

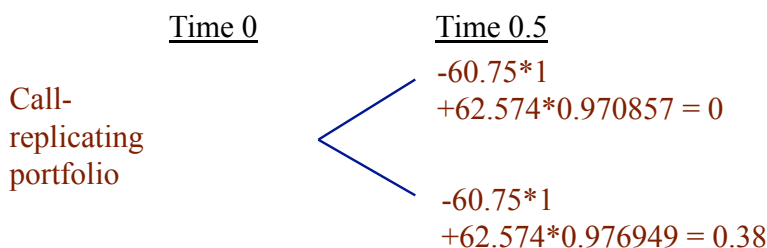
	<u>Time 0</u>	<u>Time 0.5</u>
		1
\$1 par zero maturing at time 0.5	0.9730	0.9709
\$1 par zero maturing at time 1	0.9476	0
Issuer's call option	?	1 0.9769 0.38

### New View of Portfolio “Payoff”

- Consider again the call-replicating portfolio:
  - $N_{0.5} = -60.75$  par amount of zeroes maturing at time 0.5 and
  - $N_1 = 62.574$  par amount of the zero maturing at time 1.
- Old static “buy-and-hold” picture of portfolio payoff:

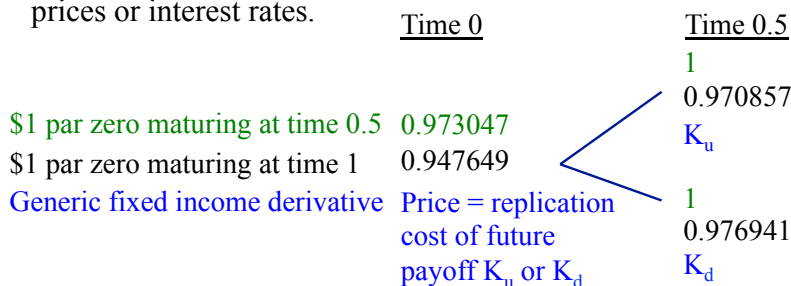


- New dynamic “marked-to-market” picture of portfolio payoff:



### Fixed Income “Derivatives”

- We use the term “derivative” to generalize the idea of an option to any instrument whose payoff depends on the the future price of some “underlying asset.” E.g., equity derivative, currency derivative, fixed income derivative, etc.
- A fixed income derivative is any instrument whose payoff depends on future bond prices or interest rates, including options, swaps, forwards, futures, etc.
- Callable bonds and MBS can also be viewed as fixed income derivatives—their payoffs are random, but linked to future bond prices or interest rates.



## Two-Step Method to Compute Replication Cost

1. First construct the replicating portfolio for the fixed income derivative (the holdings  $N_{0.5}$  and  $N_1$ ), solve 2 equations, one to match the up-state payoff and one to match the down-state payoff, for the 2 unknown N's.

- Up-state payoff:  $N_{0.5} * 1 + N_1 * {}_{0.5}d_1^u = K_u$

- Down-state payoff:  $N_{0.5} * 1 + N_1 * {}_{0.5}d_1^d = K_d$

2. Then price the derivative at its replication cost  $N_{0.5} * {}_0d_{0.5} + N_1 * {}_0d_1$

★ This method gives both the derivative price and the hedging strategy.

	<u>Time 0</u>		<u>Time 0.5</u>
			1
\$1 par zero maturing at time 0.5	${}_0d_{0.5}$	}	${}_{0.5}d_1^u$
\$1 par zero maturing at time 1	${}_0d_1$		$K_u$
Generic fixed income derivative	Price = replication cost of future payoff $K_u$ or $K_d$	}	1
			${}_{0.5}d_1^d$
			$K_d$

## One-Step Math Trick to Compute Replication Cost Using “Risk-Neutral Probabilities”

- In our examples so far, the derivative payoffs are functions of the time 0.5 price the zero maturing at time 1.

- So the **underlying asset is the zero maturing at time 1** and the **riskless asset is the zero maturing at time 0.5**.

- First, consider the “Risk-Neutral Pricing Equation” (RNPE)  
**derivative price (replication cost) =  $d_{0.5} [p \times K_u + (1-p) \times K_d]$**   
 for so-called “risk-neutral probabilities”  $p$  and  $1-p$ .

	<u>Time 0</u>		<u>Time 0.5</u>
			1
\$1 par zero maturing at time 0.5	${}_0d_{0.5}$	}	${}_{0.5}d_1^u$
\$1 par zero maturing at time 1	${}_0d_1$		$K_u$
Generic fixed income derivative	Price = replication cost of future payoff $K_u$ or $K_d$	}	1
			${}_{0.5}d_1^d$
			$K_d$



### “Risk-Neutral Pricing” Method: Main Result

- Consider the “Risk-Neutral Pricing Equation” (RNPE)  
 $\text{derivative price (replication cost)} = d_{0.5} [p \times K_u + (1-p) \times K_d]$
- **Proposition:** *The “p” that makes the RNPE hold for the underlying asset also makes the RNPE hold for all of its derivatives.*
- I.e., let  $p^*$  be the p that solves  

$${}_0d_1 = d_{0.5} [p \times {}_{0.5}d_1^u + (1-p) \times {}_{0.5}d_1^d]. \quad (\Rightarrow p^* = \frac{d_1 - {}_{0.5}d_1^d}{{}_{0.5}d_1^u - {}_{0.5}d_1^d})$$
- Then the RNPE with p set equal to  $p^*$ ,  
 $\text{derivative price (replication cost)} = d_{0.5} [p^* \times K_u + (1-p^*) \times K_d]$ ,  
 holds for all the derivatives of the underlying.

	<u>Time 0</u>		<u>Time 0.5</u>
\$1 par zero maturing at time 0.5	${}_0d_{0.5}$	}	1
\$1 par zero maturing at time 1	${}_0d_1$	}	${}_{0.5}d_1^u$
Generic fixed income derivative	Price = replication cost	}	$K_u$
		}	1
		}	${}_{0.5}d_1^d$
		}	$K_d$

### “Risk-Neutral Pricing” Method: Examples

- The model we’re using here (a version of the Black-Derman-Toy model) calibrates the binomial tree so that  $p^*$  always equals 0.5:
- For the underlying:  $0.9476 = 0.9730 [0.5 \times 0.9709 + 0.5 \times 0.9749]$

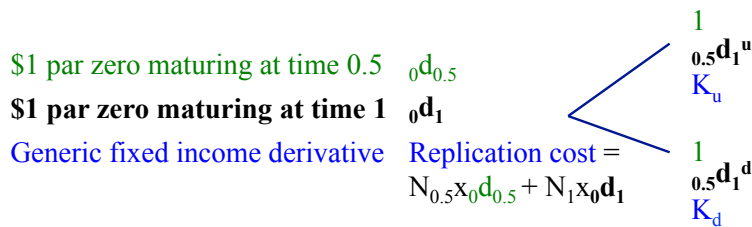
#### Class Problems

1. Verify that the RNPE with  $p=0.5$  holds for the **issuer’s call option**:
2. Verify that the RNPE with  $p=0.5$  holds for the **callable bond**:

	<u>Time 0</u>		<u>Time 0.5</u>
\$1 par zero maturing at time 0.5	0.973047	}	1
\$1 par zero maturing at time 1	0.947649	}	0.970857
Issuer’s call option	0.185	}	0
Callable bond	99.862	}	99.76
		}	1
		}	0.976941
		}	0.38
		}	100

### “Risk-Neutral Pricing” Method: Proof

- $p^*$  is constructed to solve the RNPE for the **underlying asset**:  
(Eqn. 1)  ${}_0d_1 = {}_0d_{0.5} [p^* \times {}_{0.5}d_1^u + (1-p^*) \times {}_{0.5}d_1^d]$ .
- Any  $p$ , including  $p^*$ , solves the RNPE for the **riskless asset**:  
(Eqn. 0.5)  ${}_0d_{0.5} = {}_0d_{0.5} [p \times 1 + (1-p) \times 1]$ .
- Therefore,  $p^*$  solves the RNPE for any portfolio of the underlying and riskless assets:  
 $N_{0.5} \times$  (Eqn. 0.5) +  $N_1 \times$  (Eqn. 1)  $\Rightarrow$   
 $(N_{0.5} \times {}_0d_{0.5} + N_1 \times {}_0d_1) = {}_0d_{0.5} [p^* \times (N_{0.5} \times 1 + N_1 \times {}_{0.5}d_1^u) + (1-p^*) \times (N_{0.5} \times 1 + N_1 \times {}_{0.5}d_1^d)]$ .
- Since all derivatives are priced as portfolios of the underlying and the riskless assets, the same  $p^*$  solves the RNPE for them too:  
**Derivative replication cost** =  $N_{0.5} \times {}_0d_{0.5} + N_1 \times {}_0d_1 = {}_0d_{0.5} [p^* \times K_u + (1-p^*) \times K_d]$ .



### Expected Returns with RN Probs



- We can rearrange the risk-neutral pricing equation, price = discounted “expected” payoff, as “expected” return = the riskless rate (unannualized)

$$V = d_{0.5} [p \times K_u + (1 - p) \times K_d], \text{ or}$$

$$V = \frac{p \times K_u + (1 - p) \times K_d}{1 + r_{0.5}/2}$$

$$\Leftrightarrow \frac{p \times K_u + (1 - p) \times K_d}{V} = 1 + r_{0.5}/2$$

- Thus, with the risk-neutral probabilities, all assets have the same expected return--equal to the riskless rate.
- This is why we call them "risk-neutral" probabilities.

## Risk-Neutral vs. True Probabilities



- The risk-neutral probabilities are not the same as the *true probabilities* of the future states.
- Notice that pricing derivatives did not involve the true probabilities of the up or down state actually occurring.
- This is because the pricing was by exact replication and did not involve any risk taking.
- Now let's suppose that the true probabilities are 0.4 chance the up state occurs and 0.6 chance the down state occurs.
- As we'll see, this will give the underlying asset an expected return higher than the riskless rate, i.e., a positive risk premium, which is consistent with theory and the evidence we have seen.

## Riskless Rate



Recall that the un-annualized return on an asset over a given horizon is

$$\frac{\text{future value}}{\text{initial value}} - 1$$

For the riskless 6-month zero the un-annualized return over the next 6 months is

$$\frac{1}{0.973047} - 1 = 2.77\%$$

with *certainty*, regardless of which state occurs.

This is the un-annualized *riskless rate* for this horizon.

Of course, the annualized semi-annually compounded ROR is 5.54%, the quoted zero rate.

### True Expected Return on Underlying Risky Asset



- The return on the 1-year zero over the next 6 months will be either

$$\frac{0.970857}{0.947649} - 1 = 2.45\% \text{ with probability } 0.4, \text{ or}$$

$$\frac{0.976941}{0.947649} - 1 = 3.09\% \text{ with probability } 0.6.$$

- The expected return on the 1-year zero over the next 6 months is  $0.4 \times 2.45 + 0.6 \times 3.09 = 2.83\%$ .
- This is higher than the return of 2.77% on the riskless asset—a 6 basis point risk premium.

### True Expected on the Call



- What is the expected rate of return on the call over the next 6 months?
- The possible returns are:

$$\frac{0}{0.185} - 1 = -100\% \text{ with probability } 0.4, \text{ or}$$

$$\frac{0.381}{0.185} - 1 = 106\% \text{ with probability } 0.6.$$

- The true expected return on the call is  $0.4 \times -100\% + 0.6 \times 106\% \sim 23\%$ .
- Why so high? It is equivalent to a highly levered portfolio of the underlying risky asset.
- The replicating portfolio is  $N_{0.5} = -60.75$  and  $N_1 = 62.574$ , so the the portfolio pv weights are  $w_{0.5} = -31914\%$  and  $w_1 = 32014\%$  (320-fold leverage), so the 6 bp risk premium is levered 320-fold.

### Risk-Neutral Expected Returns



- Using the risk-neutral probabilities to compute expected (unannualized) returns sets all expected returns equal to the riskless rate.

Asset	Unannualized Up Return ("prob"=0.5)	Unannualized Down Payoff ("prob"=0.5)	"Expected" Unannualized Return
0.5-Year Zero	$1/0.9730 - 1 = 2.77\%$	$1/0.9730 - 1 = 2.77\%$	2.77%
1-Year Zero	$0.970857/0.947649 - 1 = 2.45\%$	$0.976941/0.947649 - 1 = 3.09\%$	2.77%
Call	$0/0.185 - 1 = -100\%$	$0.381/0.185 - 1 = 105.54\%$	2.77%

- Because once the expected returns on the underlying riskless and risky assets are the same, the expected return on all portfolios of them is the same.
- Regardless of how the portfolio is weighted or levered.

### Volatility Effects in Options and Callable Bonds

If we re-calibrate the binomial tree to increase the volatility of future interest rates to  $\sigma=0.25$ , leaving the current term structure the same,

- the prices of **zeros** and the **noncallable bond will stay the same**,
- the price of the **issuer's call option will rise 0.07 from 0.185 to 0.256**,
- and the price of the **callable bond will fall 0.07 from 99.86 to 99.79**.

★ This creates demand to hedge volatility risk.

	<u>Time 0</u>	<u>Time 0.5</u>
		1
\$1 par zero maturing at time 0.5	0.973047	0.96945
\$1 par zero maturing at time 1	0.947649	$102.75 \times 0.96945 = 99.61$
Ex-coupon price of \$100 par of the non-callable 5.5% bond maturing at time 1.	$2.75 \times 0.9730 = 2.67375$ $102.75 \times 0.9476 = 97.34775$ = 100.047	0
Issuer's call option	0.256 = $0.9730 \times 0.5 \times (0 + 0.525)$	1
Ex-coupon price of \$100 par of the 5.5% bond maturing at time 1, callable at par at time 0.5.	$99.79 = 100.47 - 0.256 = 0.9730 \times [2.75 + 0.5 \times (99.61 + 100)]$	$102.75 \times 0.97835 = 100.525$ 100