Pricing Multi-Period Options with Dynamic Trading Strategies

Outline

- Two-Period Bond Market Model
- One-Factor Assumption
- 1.5-Year Callable Bond with Embedded 1-Year Option
- Pricing with a Dynamic Trading Strategy
- Pricing with Risk-Neutral Probabilitues

Reading

- Tuckman and Serrat, Chapter 7

Option Pricing with Dynamic Replication

- The one-period binomial option model illustrates most of the logic of option pricing, but it's not rich enough to capture the full spectrum of payoff distributions and pricing dynamics seen in practice.
- Extending to a two-period model illustrates the finance logic of replication and pricing using dynamic trading strategies.
- It also highlights additional economic issues such as the number of risk factors to be considered.

Two-Period Model of a Callable Bond

- Consider a $100 par of a 1.5-year semi-annual 5.5% coupon bond that is callable at par at time 1.
- Suppose there are three trading dates, time 0, 0.5, and 1.
- We'll extend the one-period model to time 1, and model the prices of zeroes maturing at time 0.5, 1, and 1.5.
- New problem: Now there are two bonds with a risky time 0.5 value, the zero maturing at time 1 and the zero maturing at time 1.5. Should we let them move separately?
- In a "two-factor" model, each bond could have its own tree.
- For simplicity, we'll use a "one-factor" interest rate model that puts all bonds on the same binomial tree. This will imply that all bond returns are perfectly correlated, which is adequate for many purposes, but will not work well in some cases.

"One-Factor Model" of the Bond Market: One Binomial Tree for the Whole Bond Market

	Time 0	Time 0.5	Time 1
		1	1
		0.970857	0.966581
Zero mat. time 0.5	0.973047	0.941787	
Zero mat. time 1	0.947649		1
Zero mat. time 1.5	0.922242		0.973534
		1	
		0.976941	1
		0.953790	0.979071

- Each period the whole bond market can move into one of two states.
- Either all rates move up, or all rates move down: a single coin flip each period.
- In a one-factor model, any one bond can be replicated as a portfolio of the other two bonds. Our tree-building recipe ensures prices fit together so there are no arbitrage opportunities in the bond prices.

What if We Allowed for Multiple Risk Factors?

If we let each risky zero price move independently of the other, on its own tree (2 coin flips each period), we would already have 4 possible states at time 0.5, and computational complexity would grow fast, but this could be accommodated with enough computer power.

	Time 0	Time 0.5
Zero mat. time 1	0.947649	0.970857
		0.976941
Zero mat. time 1.5	0.922242	0.941787
		0.953790

1.5-Year 5.5%-Coupon Bond Callable at Time 1

	Time 0	Time 0.5 (Up)	Time 0.5 (Down)	Time 1 (Up-Up)	Time 1 (Up-Down)	Time 1 (Down-Down)
$1 par zero mat. 0.5	0.973047					
$1 par zero mat. 1	0.947649	0.970857	0.976941			
$1 par zero mat. 1.5	0.922242	0.941787	0.953790	0.966581	0.973534	0.979071
$100 par noncallable 5.5% bond mat. 1.5	100.04	99.44	100.69	99.32	100.03	100.60
Issuer's call option				0	0.03	0.60
$100 par 5.5% bond callable at par at time 1				99.32	100	100

- The prices of coupon bonds in the tree are always "ex-coupon."
- The noncallable bond can always be priced as a package of the zeroes.

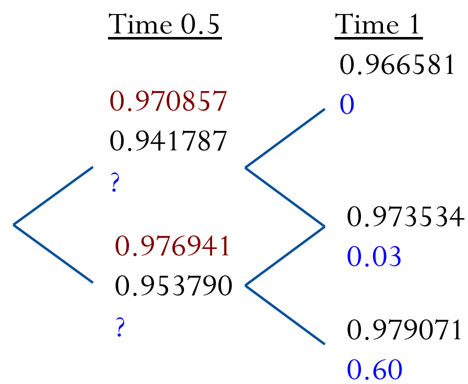
Issuer's Call Option: Time 0.5 Up

	Time 0	Time 0.5 (Up)	Time 0.5 (Down)	Time 1 (Up-Up)	Time 1 (Up-Down)	Time 1 (Down-Down)
$1 par zero mat. 0.5	0.973047					
$1 par zero mat. 1	0.947649	0.970857	0.976941			
$1 par zero mat. 1.5	0.922242	0.941787	0.953790	0.966581	0.973534	0.979071
Issuer's call option	?	?	?	0	0.03	0.60

- Class Problem: For time 0.5 in the up state, construct the portfolio N_1^u par of the z. mat. at 1 and $N_{1.5}^u$ par of the z. mat. at 1.5 that replicates the call payoff in the up-up and up-down states at time 1.

Issuer's Call Option: Time 0.5 Up

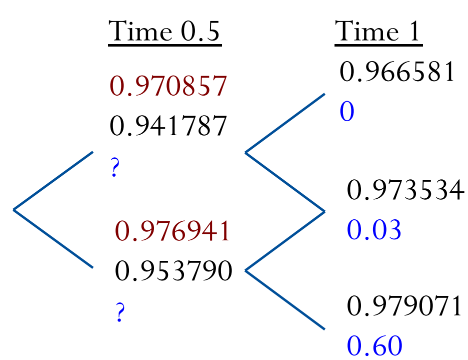
	Time 0
$1 par zero mat. 0.5	0.973047
$1 par zero mat. 1	0.947649
$1 par zero mat. 1.5	0.922242
Issuer's call option	?



- Class Problem:
  What is the time 0.5-up replication cost of the issuer's call?

Issuer's Call Option: Time 0.5 Down

	Time 0
$1 par zero mat. 0.5	0.973047
$1 par zero mat. 1	0.947649
$1 par zero mat. 1.5	0.922242
Issuer's call option	?



- Similarly, for the time 0.5 down state, the portfolio N_1^d par of the z. mat. 1 and $N_{1.5}^d$ par of the z. mat. 1.5 that replicates the call payoff in the up-down and down-down states at time 1 is given by
  1. Match up payoff: $N_1^d \times 1 + N_{1.5}^d \times 0.973534 = 0.03$
  2. Match down payoff: $N_1^d \times 1 + N_{1.5}^d \times 0.979071 = 0.60$

=> $N_1^d = -100$, $N_{1.5}^d = 102.75$ (The call always finishes in the money from here, so by now it's just a forward contract on the underlying NC bond with forward price 100) => rep. cost = 100x0.9769 + 102.75x0.9538 = 0.308

Issuer's Call Option: Time 0

	Time 0	Time 0.5	Time 1
		0.970857 0.941787 0.015	0.966581 0
$1 par zero mat. 0.5	0.973047		0.973534 0.03
$1 par zero mat. 1	0.947649	0.976941	
$1 par zero mat. 1.5	0.922242	0.953790	0.979071
Issuer's call option	?	0.308	0.60

- At time 0, the problem is to find a portfolio that matches the option value at time 0.5, when the portfolio can be rebalanced. No "buy-and-hold" portfolio of zeroes can replicate the call.
- We have three zeroes to work with. All portfolios that match the call price at time 0.5 have the same time 0 price, by the built-in no-arb. property
- Use $N_{0.5}$ par of the z. mat. 0.5 and $N_{1.5}$ par of the z. mat. 1.5:
  1. Match up payoff: $N_{0.5} \times 1 + N_{1.5} \times 0.941787 = 0.015$
  2. Match down payoff: $N_{0.5} \times 1 + N_{1.5} \times 0.95379 = 0.308$

=> $N_{0.5}=-22.97$, $N_{1.5}=24.41$, cost=-22.97x0.9730+24.41x0.9222=0.157

Summary of Dynamic Trading Strategy for Call

	Time 0	Time 0.5	Time 1
		0.970857 0.941787 $N_1^u=-4.25$, $N_{1.5}^u=4.397$ $C_u=0.015$	0.966581 0
$1 par zero mat. 0.5	0.973047		
$1 par zero mat. 1	0.947649		0.973534 0.03
$1 par zero mat. 1.5	0.922242		
Issuer's call option	$N_{0.5}=-22.97$, $N_{1.5}=24.41$ $C=0.157$	0.976941 0.953790 $N_1^d= -100$, $N_{1.5}^d=102.75$ $C_d=0.308$	0.979071 0.60

Pricing the Call with Risk-Neutral Probabilities

- The tree is calibrated so that the risk-neutral probabilities are always 0.5.
- The RNPE is always

  price = riskless discount factor x (0.5 x up payoff + 0.5 x down payoff)

	Time 0	Time 0.5	Time 1
		0.970857 0.941787 C_u=0.9709x 0.5(0+0.03) =0.015	0.966581 0
$1 par zero mat. 0.5	0.973047		
$1 par zero mat. 1	0.947649		
$1 par zero mat. 1.5	0.922242		0.973534 0.03
Issuer's call option	C=0.9730x 0.5x(0.015+0.308) =0.157	0.976941 0.953790 C_d=0.9769x 0.5(0.03+0.60) =0.308	
			0.979071 0.60

1.5-Year 5.5%-Coupon Bond Callable at Time 1

- Class Problem: Fill in the prices of the callable bond in the tree.

	Time 0	Time 0.5	Time 1
		0.970857 0.941787 99.44 0.015 ?	0.966581 99.32 0 99.32
$1 par zero mat. 0.5	0.973047		
$1 par zero mat. 1	0.947649		
$1 par zero mat. 1.5	0.922242		0.973534 100.03 0.03 100
$100 par noncallable 5.5% bond mat. 1.5	100.04		
Issuer's call option	0.157	0.976941 0.953790 100.69 0.308 ?	
$100 par 5.5% bond callable at par at time 1	?		
			0.979071 100.60 0.60 100