Bond Option Models

Outline

- From short rates to bond prices
- The simple Black, Derman, Toy model
- Calibration to current the term structure
- Short Rate Dollar Duration
- Short Rate Duration

Readings

- Tuckman and Serrat, Chapters 8-10

Building the Bond Option Model

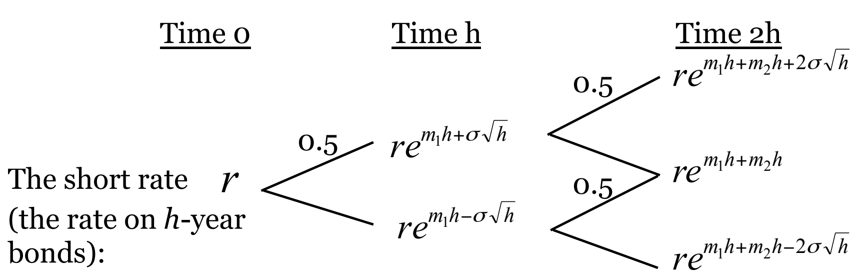
1. Start with a model (recipe) for the tree of one-period rates ("short rates") and risk-neutral probabilities.

   For example, Black-Derman-Toy, Ho and Lee, …
2. Build the tree of bond prices from the tree of short rates using the risk-neutral pricing equation (RNPE)

   Bond price = disc. factor x [p x up payoff + (1-p) x down payoff]
3. Build the tree of derivative prices from the tree of bond prices by pricing by replication.

   Repl. cost = disc. factor x [p x up payoff + (1-p) x down payoff]
4. Calibrate the model parameters (drift, volatility) to make the model prices match observed bond prices and option prices.

Example of a Model of the Evolution of Short Rates for Pricing Bond Options: Black-Derman-Toy with Constant Volatility



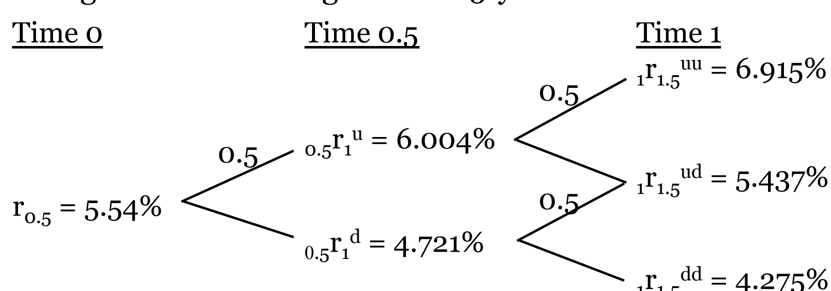
- Each date the short rate changes by a multiplicative factor:
  up factor = $e^{mh+\sigma\sqrt{h}}$,
  down factor = $e^{mh-\sigma\sqrt{h}}$
- The exponential function is always positive, which guarantees that interest rates are always positive in this model.

Description of the Model

- The parameter h is the amount of time between dates in the tree, in years. For example, in a semi-annual tree, $h = 0.5$. In a monthly tree, $h = 1/12 = 0.08333$.
- Each value in the tree represents the short rate or interest rate for a zero with maturity h.
- **For simplicity, this model sets the risk-neutral probability of moving up or down at each date equal to 0.5.**
- The mean-rate-change parameters m_1, m_2,... are not random, but they vary over time. In practice, they are calibrated to make the model bond prices match the observed current term structure.
- The *proportional volatility* σ, is constant here – this is typically calibrated to an option price.
- In the full-blown BDT model, σ also varies each period to allow the model to fit multiple option prices.
- In the limit, as h->0, the distribution of the future instantaneous short rate is lognormal, i.e., its log is normally distributed.

Model Calibrated to Our Given Term Structure and Historical Volatility

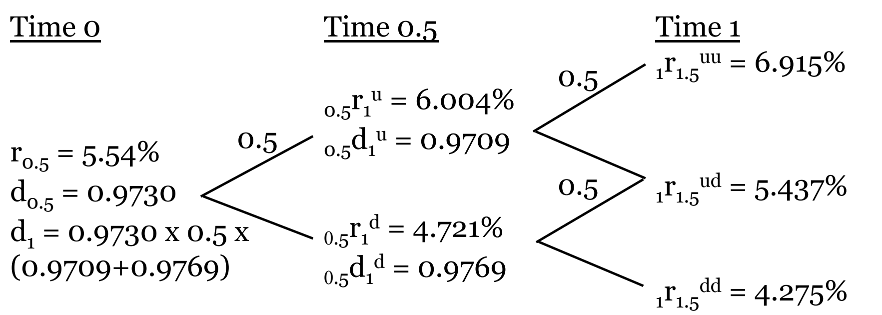
- To calibrate a semi-annual model (h=0.5) to the time 0 term structure $r_{0.5}$=5.54%, r_1=5.45%, $r_{1.5}$=5.47%, and incorporate a historical volatility estimate of σ=0.17, it turns out we need m_1=-0.0797 and m_2=0.0422, as we shall see.
- This gives the following tree of 0.5-year rates:



For example, at time 0.5, up, the 6-month zero rate is $0.0554e^{-0.0797\text{x}0.5+0.17\sqrt{0.5}}$ =0.06004

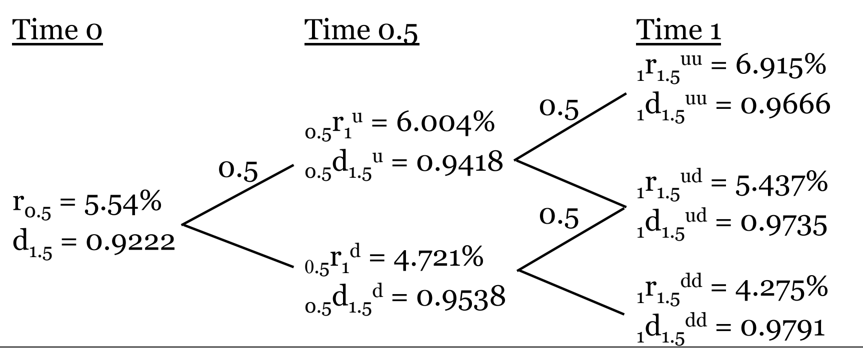
Building the Price Tree from the Rate Tree

- Next, we derive the tree prices of zeroes from the tree of short rates:
- **Zero maturing at time 0.5**: Time 0 price
  $d_{0.5} = 1/(1+0.0554/2) = 0.973047$
- **Zero maturing at time 1:** Time 0.5 possible prices
  ${}_{0.5}d_1^u = 1/(1+0.06004/2) = 0.9709$, ${}_{0.5}d_1^d = 1/(1+0.04721/2) = 0.9769$
- Time 0 price of the zero maturing at time 1
  $d_1 = 0.973047\ [0.5 \times 0.9709 + 0.5 \times 0.9769] = 0.9476$



Price Tree for the Zero Maturing at Time 1.5

- Time 1 possible prices:
  ${}_1d_{1.5}^{uu} = 1/(1+0.06915/2) = 0.9666$, ${}_1d_{1.5}^{ud} = 1/(1+0.05437/2) = 0.9735$
  ${}_1d_{1.5}^{dd} = 1/(1+0.04275/2) = 0.9791$,
- Time 0.5 possible prices:
  ${}_{0.5}d_{1.5}^u = 0.9709\ [0.5 \times 0.9666 + 0.5 \times 0.9735] = 0.9418$
  ${}_{0.5}d_{1.5}^d = 0.9769\ [0.5 \times 0.9735 + 0.5 \times 0.9791] = 0.9538$
- Time 0 price:
  $d_{1.5} = 0.973047\ [0.5 \times 0.9418 + 0.5 \times 0.9538] = 0.9222$



Resulting Zero Price Tree

- At each node, the prevailing prices of outstanding zeros are listed, in ascending order of maturity.
- For instance, the price of a 1-year zero at time 0.5. state up, is ${}_{0.5}d_{1.5}^{\mathrm{u}} = 0.941787$.

Time 0	Time 0.5	Time 1
		0.966581
	0.970857 0.941787	
0.973047 0.947649 0.922242		0.973533
	0.976941 0.953790	
		0.979071

- Then, from the zero price tree, we can build the tree of prices of any bond option by using the RNPE working back from the last period back to time 0. Or equivalently, by replication.
- We can build the tree out as many periods as needed to model any given bond option.

Model as a Tree of Future Possible Term Structures

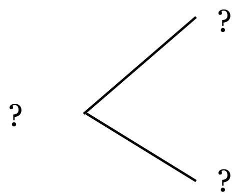
- We can envision the model as a tree of future possible term structures.
- For example, at each node, the prevailing term structure of zero rates is listed, in ascending order of maturity.
- For instance, the 1-year zero rate at time 0.5, state up, is ${}_{0.5}r_{1.5}^{\mathrm{u}} = 6.089\%$.

Time 0	Time 0.5	Time 1
		6.915%
	6.004% 6.089%	
5.54% 5.45% 5.47%		5.437%
	4.721% 4.788%	
		4.275%

Class Problems

1) Build a tree of 0.5-year rates out to time 0.5 using h=0.5, ${}_0r_{0.5}$=2%, m_1=0.01, and σ=0.20.

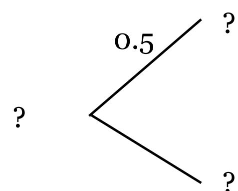
Time 0 | Time 0.5



Class Problems

2) What is the price of the 0.5-year zero at each node?

Time 0 | Time 0.5



3) What is the price of the 1-year zero at time 0?

4) What is the 1-year zero rate at time 0?

Model Calibration

- Notice that if we increase the parameter m_1, then
  - both time 0.5 six-month rates will rise,
  - both time 0.5 six-month zero prices will fall,
  - so the time 0 price of the zero maturing at time 1 will fall,
  - so the time 0 1-year rate will rise.
- *Thus, the time 0 1-year rate in the tree is determined by the value of m_1.*
- *To calibrate the tree, numerically solve for the value of m_1 that makes the model 1-year rate match the observed 1-year rate.*
- Similarly, the time 0 1.5-year rate in the tree is determined by the value of m_2, so we *calibrate the value of m_2 to make the model time 0 1.5-year rate match the observed 1.5-year rate.*
- And so on for each successive zero maturing at 2, 2.5, 3, ...
- Similarly, the volatility parameter σ could be set by solving for the value that makes the model price of a given option match the observed price (an "implied volatility).

Limitations of This Model

- Only one volatility parameter
  - The model may not be able to fit the prices of options with different maturities simultaneously.
  - The full-blown Black-Derman-Toy model allows the proportional volatility parameter to vary over time to match prices of options with different maturities, allowing for a term structure of volatilities.
- Independent interest rate change over time.
  - Some feel that rates should be mean reverting. This would mean down moves would be more likely at higher interest rates.
  - For example, there is a Black-Karasinski model that introduces mean reversion in the interest rate process.

Limitations of This Model…

- Only a One-Factor Model
  - Each period one factor (the short rate) determines the prices of all bonds.
  - This means that each period all bond prices move together. Their returns are perfectly correlated. There is no possibility that some bond yields could rise while others fall.
  - To allow for this possibility the model would require additional factors, or sources of uncertainly, which would expand the dimensions of the state-space. For example, in a two-factor model, each period you could move up or down and right or left, so there would be four possible future states.
- Large investment banks and derivatives dealers often have their own proprietary models.

Short Rate Dollar Duration

- Recall: dollar duration ≈ -Δprice/Δrates
- For securities like options that don't have fixed cash flows, there may not be an explicit price-rate function to differentiate.
- But the binomial tree gives prices at the end of the first period (time h) in two different interest rate environments, which we can use to estimate dollar duration with respect to the short rate with the following formula:

	Time 0.5
Asset price	K_u
Short rate	${}_{0.5}r_1^{\,u}$
	K_d
	${}_{0.5}r_1^{\,d}$

$$\text{short rate \$dur} = -\frac{K_u - K_d}{{}_{0.5}r_1^{\,u} - {}_{0.5}r_1^{\,d}}$$

Examples

Recall the zero prices in our binomial tree below:

Time 0	Time 0.5
	${}_{0.5}r_1^u =$ 6.004%
	0.970857
0.973047	0.941787
0.947649	
0.922242	
	${}_{0.5}r_1^d = 4.721\%$
	0.976941
	0.953790

Calculate the SR dollar duration for $1 par of each of these zeroes:

- $1 par of 0.5- year zero:

$$-\frac{1-1}{0.06004-0.04721}=0$$

- $1 par of 1- year zero:

$$-\frac{0.970857-0.976941}{0.06004-0.04721}=0.4742$$

- $1 par of 1.5- year zero:

$$-\frac{0.941787-0.953790}{0.06004-0.04721}=0.9356$$

SR $Duration and Hedging

- For ordinary bonds, the short rate dollar duration is like ordinary dollar duration after one period of time., i.e., 0.5-years later in our example.
- In practice, the time steps are much shorter, so the effect of the passage of time is small and the short rate dollar duration is very close to traditional dollar duration.
- The point of this risk measure is that it applies not only to bonds, but also to bond derivatives.
- It is also the right risk measure to use to calibrate a hedge.
- I.e., a position with zero SR $dur is riskless here.

SR $Duration for an Option

- Recall the issuer's 1-year call embedded in the 1.5-year bond from last lecture.

Time 0

0.973047
0.922242
C = 0.157
($N_{0.5}$=-22.97, $N_{1.5}$=24.41)

Time 0.5

0.941787
C_u = 0.015

0.953790
C_d = 0.308

- **Class Problems**

1) Calculate the SR $duration for 24.41 par of the underlying 1.5-year zero:

2) Calculate the SR $duration of the call:

Short Rate Duration

- For securities with positive prices, we can define short rate duration = short rate $duration/price
- This is essentially the effective duration of the security.
- Analogous to the traditional duration, this measures interest rate risk per dollar invested.
- It is essentially the same as the traditional duration of the security at time h.
- If the time step h is very small, it is virtually the same as the traditional duration.
- But this measure applies to derivatives as well.
- **Class Problem**: What is the SR duration of the call?