TradingStrategiesI

Empirical Market Microstructure

(2006, Oxford University Press)

Companion *Mathematica* notebook

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This notebook covers the Bertsimas and Lo order splitting model (Chapter 15, section 15.1)

```
Text @ Style["Notebook evaluated " <> DateString["DateTime"], "Subtitle"]
```

Notebook evaluated Monday 4 June 2007 20:35:42

Initializations

```
<< Notation`
```

The following commands define symbolizations that are convenient for labeling things.

```
Symbolize[Anything_Rule]; Symbolize[Anything_Rules];
```

Basic problem (Section 15.1.1)

Model dynamics

```
mRule = m_t_ :> (m_t-1 /. m_0 → 0) + λ s_t + μ + ϵ_t;
pRule = p_t_ :> m_t + γ (s_t /. s_0 → 0);
WRule = W_t_ :> W_t-1 - (s_t-1 /. s_0 → 0) /; ! t === 0;
ϵZap = ϵ_ → 0;
```

■ Minimize the total expected purchase cost over T = 3 periods.

```
T = 3;
```

$$\text{ExpCost} = \left(\left(\sum_{t=1}^{T} p_t\, s_t\right) \text{//. \{}p_{\text{Rule}},\ m_{\text{Rule}},\ \epsilon\text{Zap\}}\right)$$

$$s_1\,(\mu + \gamma\, s_1 + \lambda\, s_1) + s_2\,(2\,\mu + \lambda\, s_1 + \gamma\, s_2 + \lambda\, s_2) + s_3\,(3\,\mu + \lambda\, s_1 + \lambda\, s_2 + \gamma\, s_3 + \lambda\, s_3)$$

$$\text{Lagrangian} = \text{ExpCost} + \delta\left(1 - \sum_{t=1}^{T} s_t\right)$$

$$s_1\,(\mu + \gamma\, s_1 + \lambda\, s_1) + s_2\,(2\,\mu + \lambda\, s_1 + \gamma\, s_2 + \lambda\, s_2) + \delta\,(1 - s_1 - s_2 - s_3) + s_3\,(3\,\mu + \lambda\, s_1 + \lambda\, s_2 + \gamma\, s_3 + \lambda\, s_3)$$

$$\text{FOC} = \text{Append}[(\partial_{s_{\#}}\ \text{Lagrangian}) == 0\ \&\ /@\ \text{Range}[T],\ (\partial_{\delta}\ \text{Lagrangian}) == 0]$$

$$\{-\delta + \mu + \gamma\, s_1 + \lambda\, s_1 + (\gamma + \lambda)\, s_1 + \lambda\, s_2 + \lambda\, s_3 == 0,$$
$$-\delta + 2\,\mu + \lambda\, s_1 + \gamma\, s_2 + \lambda\, s_2 + (\gamma + \lambda)\, s_2 + \lambda\, s_3 == 0,$$
$$-\delta + 3\,\mu + \lambda\, s_1 + \lambda\, s_2 + \gamma\, s_3 + \lambda\, s_3 + (\gamma + \lambda)\, s_3 == 0,\ 1 - s_1 - s_2 - s_3 == 0\}$$

$$\text{solutions} = \text{Solve}[\text{FOC},\ \text{Append}[\text{Table}[s_i,\ \{i,\ T\}],\ \delta]]\ //\ \text{Simplify}$$

$$\left\{\left\{\delta \to \frac{2}{3}\,(\gamma + 2\,\lambda + 3\,\mu),\ s_1 \to \frac{2\,\gamma + \lambda + 3\,\mu}{6\,\gamma + 3\,\lambda},\ s_2 \to \frac{1}{3},\ s_3 \to \frac{2\,\gamma + \lambda - 3\,\mu}{6\,\gamma + 3\,\lambda}\right\}\right\}$$

In the no-drift case:

$$\text{solutions}\ /.\ \mu \to 0\ //\ \text{Simplify}$$

$$\left\{\left\{\delta \to \frac{2}{3}\,(\gamma + 2\,\lambda),\ s_1 \to \frac{1}{3},\ s_2 \to \frac{1}{3},\ s_3 \to \frac{1}{3}\right\}\right\}$$

```
Clear[T];
```

□ *Procedure*

The following procedure automates the solution.

```
OptOrders[T_: 3, AllRules_: {}, params_: {}, PrintFlag_: True] :=
  Module[{a, Lagrangian, solutions},
    a = (Sum[p_t s_t, {t, 1, T}]) //. AllRules //. params;
    Lagrangian = a + δ (1 - Sum[s_t, {t, 1, T}]);
    FOC = Append[(∂_{s_#} Lagrangian) == 0 & /@ Range[T], (∂_δ Lagrangian) == 0];
    solutions =
     Solve[FOC, Append[Table[s_i, {i, T}], δ]] // Simplify // Chop // Flatten;
    If[PrintFlag,
     Print["Lagrangian: ", Lagrangian];
     Print["First order conditions:"]; Print[TableForm[FOC]];
     Print["Solutions:"]; Print[TableForm[solutions]];
    ];
    solutions
  ];
sol = OptOrders[3, {p_Rule, m_Rule, ϵZap}];
```

Lagrangian: $s_1 (\mu + \gamma s_1 + \lambda s_1) + s_2 (2\mu + \lambda s_1 + \gamma s_2 + \lambda s_2) + \delta (1 - s_1 - s_2 - s_3) + s_3 (3\mu + \lambda s_1 + \lambda s_2 + \gamma s_3 + \lambda s_3)$

First order conditions:

$-\delta + \mu + \gamma s_1 + \lambda s_1 + (\gamma + \lambda) s_1 + \lambda s_2 + \lambda s_3 == 0$

$-\delta + 2\mu + \lambda s_1 + \gamma s_2 + \lambda s_2 + (\gamma + \lambda) s_2 + \lambda s_3 == 0$

$-\delta + 3\mu + \lambda s_1 + \lambda s_2 + \gamma s_3 + \lambda s_3 + (\gamma + \lambda) s_3 == 0$

$1 - s_1 - s_2 - s_3 == 0$

Solutions:

$\delta \to \frac{2}{3} (\gamma + 2\lambda + 3\mu)$

$s_1 \to \frac{2\gamma+\lambda+3\mu}{6\gamma+3\lambda}$

$s_2 \to \frac{1}{3}$

$s_3 \to \frac{2\gamma+\lambda-3\mu}{6\gamma+3\lambda}$

Analysis of time structure

Find the coefficients of μ in the solutions when $T = 3$;

```
T = 3;
s_Solutions = OptOrders[T, {p_Rule, m_Rule, ϵZap}, {}, False] // FullSimplify
```

$$\left\{\delta \to \frac{2}{3} (\gamma + 2\lambda + 3\mu),\ s_1 \to \frac{1}{3} + \frac{\mu}{2\gamma + \lambda},\ s_2 \to \frac{1}{3},\ s_3 \to \frac{1}{3} - \frac{\mu}{2\gamma + \lambda}\right\}$$

```
Coefficient[#, μ] & /@ (Table[s_i, {i, T}] /. s_Solutions)
```

$$\left\{\frac{1}{2\gamma+\lambda}, 0, -\frac{1}{2\gamma+\lambda}\right\}$$

Compare with:

$$\text{Table}\left[\frac{(T+1)-2t}{2\,(2\gamma+\lambda)}, \{t, T\}\right]$$

$$\left\{\frac{1}{2\gamma+\lambda}, 0, -\frac{1}{2\gamma+\lambda}\right\}$$

With T = 4

```
T = 4;
s_Solutions = OptOrders[T, {p_Rule, m_Rule, ϵZap}, {}, False] // FullSimplify;
Coefficient[#, μ] & /@ (Table[s_i, {i, T}] /. s_Solutions)
```

$$\left\{\frac{3}{2\,(2\gamma+\lambda)}, \frac{1}{4\gamma+2\lambda}, -\frac{1}{2\,(2\gamma+\lambda)}, -\frac{3}{2\,(2\gamma+\lambda)}\right\}$$

Compare with:

$$\text{Table}\left[\frac{(T+1)-2t}{2\,(2\gamma+\lambda)}, \{t, T\}\right]$$

$$\left\{\frac{3}{2\,(2\gamma+\lambda)}, \frac{1}{2\,(2\gamma+\lambda)}, -\frac{1}{2\,(2\gamma+\lambda)}, -\frac{3}{2\,(2\gamma+\lambda)}\right\}$$

With T = 5

```
T = 5;
s_Solutions = OptOrders[T, {p_Rule, m_Rule, ϵZap}, {}, False] // FullSimplify;
Coefficient[#, μ] & /@ (Table[s_i, {i, T}] /. s_Solutions)
```

$$\left\{\frac{2}{2\gamma+\lambda}, \frac{1}{2\gamma+\lambda}, 0, -\frac{1}{2\gamma+\lambda}, -\frac{2}{2\gamma+\lambda}\right\}$$

Compare with:

$$\text{Table}\left[\frac{(T+1)-2t}{2\,(2\gamma+\lambda)}, \{t, T\}\right]$$

$$\left\{\frac{2}{2\gamma+\lambda}, \frac{1}{2\gamma+\lambda}, 0, -\frac{1}{2\gamma+\lambda}, -\frac{2}{2\gamma+\lambda}\right\}$$

```
Clear[T];
```

Exercise 15.1

Model dynamics

```
mRule = m_{t_} :> (m_{t-1} /. m_0 -> 0) + λ_t s_t + μ + ϵ_t;
pRule = p_{t_} :> m_t + γ (s_t /. s_0 -> 0);
WRule = W_{t_} :> W_{t-1} - (s_{t-1} /. s_0 -> 0) /; ! t === 0;
ϵZap = ϵ_ -> 0;
```

```
solution = OptOrders[3, {pRule, mRule, ϵZap} /. μ -> 0 /. γ -> 0] // Simplify;
```

Lagrangian: $\delta\,(1 - s_1 - s_2 - s_3) + s_1^2\,\lambda_1 + s_2\,(s_1\,\lambda_1 + s_2\,\lambda_2) + s_3\,(s_1\,\lambda_1 + s_2\,\lambda_2 + s_3\,\lambda_3)$

First order conditions:

$-\delta + 2\,s_1\,\lambda_1 + s_2\,\lambda_1 + s_3\,\lambda_1 == 0$

$-\delta + s_1\,\lambda_1 + 2\,s_2\,\lambda_2 + s_3\,\lambda_2 == 0$

$-\delta + s_1\,\lambda_1 + s_2\,\lambda_2 + 2\,s_3\,\lambda_3 == 0$

$1 - s_1 - s_2 - s_3 == 0$

Solutions:

$\delta \to \frac{2\,\lambda_1\,(\lambda_2^2+\lambda_1\,\lambda_3-4\,\lambda_2\,\lambda_3)}{\lambda_2\,(\lambda_2-4\,\lambda_3)}$

$s_1 \to 1 + \frac{2\,\lambda_1\,\lambda_3}{\lambda_2^2-4\,\lambda_2\,\lambda_3}$

$s_2 \to \frac{\lambda_1\,(\lambda_2-2\,\lambda_3)}{\lambda_2\,(\lambda_2-4\,\lambda_3)}$

$s_3 \to -\frac{\lambda_1}{\lambda_2-4\,\lambda_3}$

```
solution /. {λ1 -> 2, λ2 -> 1, λ3 -> 1/2} // TableForm
```

$\delta \to 0$

$s_1 \to -1$

$s_2 \to 0$

$s_3 \to 2$

As a check, verify that with constant λ, we obtain the original solution to the basic problem.

```
solution /. {λ_ -> λ} // TableForm
```

$\delta \to \frac{4\,\lambda}{3}$

$s_1 \to \frac{1}{3}$

$s_2 \to \frac{1}{3}$

$s_3 \to \frac{1}{3}$

Geometrically declining non-stochastic temporary impact (Section 15.1.2)

Definitions:

```
m_Rule = m_t_ :> (m_(t-1) /. m_0 → 0) + λ s_t + μ + ϵ_t;
p_Rule = p_t_ :> m_t + γ A_t;
W_Rule = W_t_ :> W_(t-1) - (s_(t-1) /. s_0 → 0) /; ! t === 0;
A_Rule = A_t_ :> s_t + θ A_(t-1) /; ! t === 0;
```

Example:

```
Collect[A_5 //. A_Rule, θ]
```

$$\theta^5 A_0 + \theta^4 s_1 + \theta^3 s_2 + \theta^2 s_3 + \theta s_4 + s_5$$

```
AllRules = {p_Rule, A_Rule, m_Rule, ϵZap};
```

Quadratic analysis

```
sol = OptOrders[2, AllRules];
```

Lagrangian:

$$s_1 (\mu + \lambda s_1 + \gamma (\theta A_0 + s_1)) + \delta (1 - s_1 - s_2) + s_2 (2 \mu + \lambda s_1 + \lambda s_2 + \gamma (\theta (\theta A_0 + s_1) + s_2))$$

First order conditions:

$$-\delta + \mu + \lambda s_1 + (\gamma + \lambda) s_1 + \gamma (\theta A_0 + s_1) + (\gamma \theta + \lambda) s_2 == 0$$
$$-\delta + 2 \mu + \lambda s_1 + \lambda s_2 + (\gamma + \lambda) s_2 + \gamma (\theta (\theta A_0 + s_1) + s_2) == 0$$
$$1 - s_1 - s_2 == 0$$

Solutions:

$$\delta \to \frac{1}{2} (\gamma (2 + \theta) + 3 (\lambda + \mu) + \gamma \theta (1 + \theta) A_0)$$
$$s_1 \to -\frac{-\gamma (-2+\theta) + \lambda + \mu + \gamma (-1+\theta) \theta A_0}{2 \gamma (-2+\theta) - 2 \lambda}$$
$$s_2 \to \frac{\gamma (-2+\theta) - \lambda + \mu + \gamma (-1+\theta) \theta A_0}{2 \gamma (-2+\theta) - 2 \lambda}$$

```
Flatten[s_1 /. sol /. μ → 0] // Apart
```

$$\frac{1}{2} - \frac{\gamma (-1 + \theta) \theta A_0}{2 (-2 \gamma + \gamma \theta - \lambda)}$$

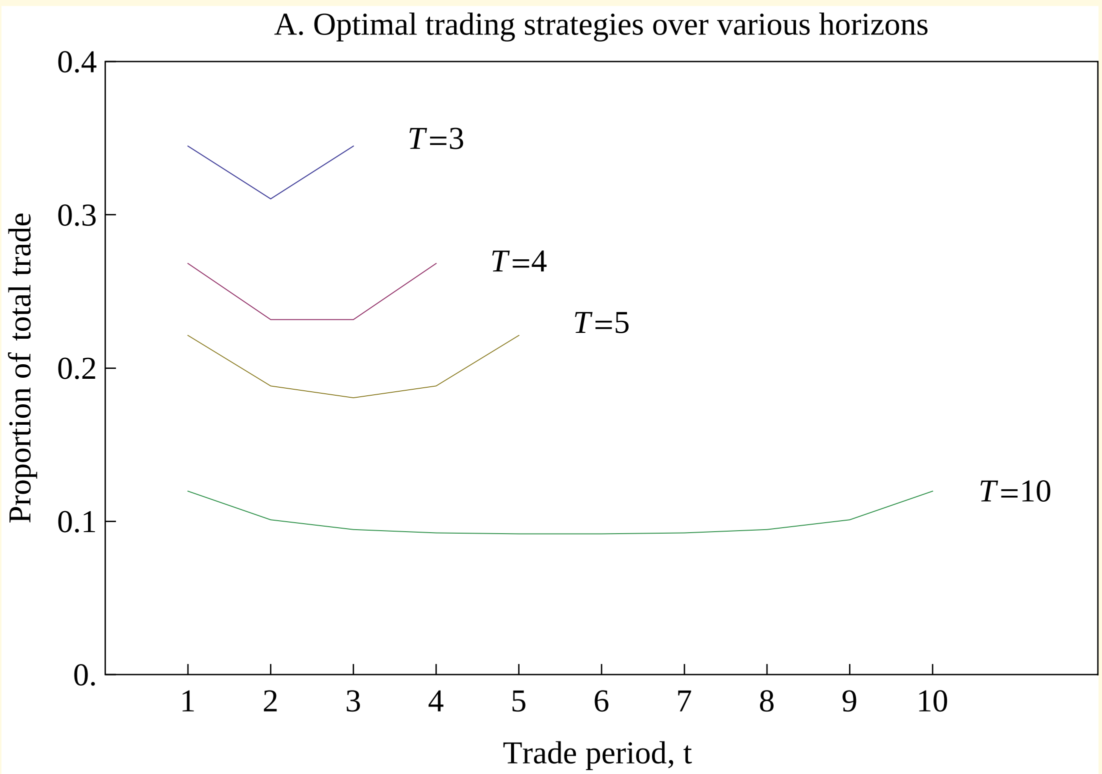
```
params = {μ → 0, γ → 1/10, λ → 1/10, θ → 1/2, A_0 → 0};
```

```
pp[T_, params_: {}] := Transpose[{Range[T],
    Table[s_i, {i, T}] /. OptOrders[T, AllRules, params, False] // Flatten}];
```

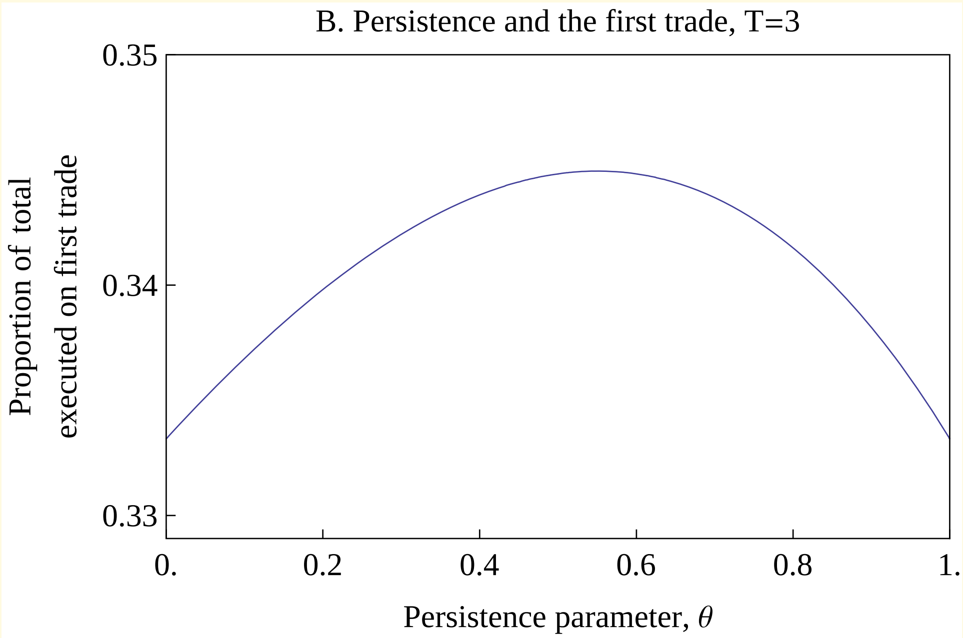
```
graphText = {Text["T=10", {11, .12}],
   Text["T=5", {6, .23}], Text["T=4", {5, .27}], Text["T=3", {4, .35}]};
```

```
baseStyle = {FontFamily → "Times", FontSize → 12};
```

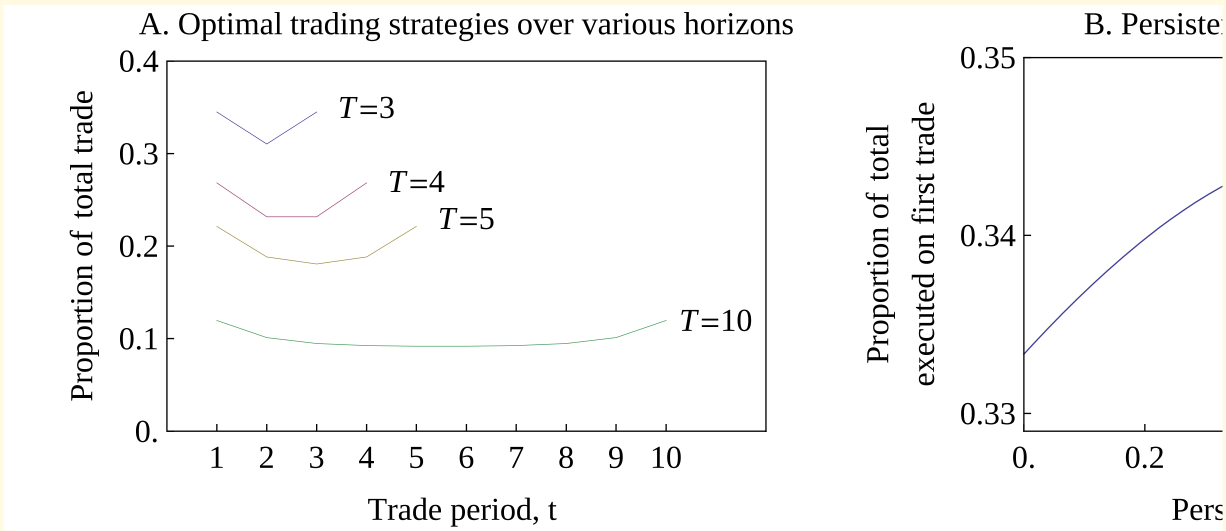
```
p1 = ListPlot[
  {pp[3, params], pp[4, params], pp[5, params], pp[10, params]}, Joined → True,
  Frame → True, FrameTicks → {Range[10], Range[0, 0.4`, 0.1`], None, None},
  FrameLabel → {"Trade period, t ", "Proportion of total trade",
    "A. Optimal trading strategies over various horizons", None},
  Background → GrayLevel[1], PlotRange → {{0, 12}, {0, 0.4`}},
  PlotStyle → {Thickness[0.001`]}, Epilog → graphText, BaseStyle → baseStyle]
```



```
p2 =
 Plot[s1 /. Flatten[OptOrders[3, AllRules, {μ → 0, γ → 1/10, λ → 1/10, A0 → 0}, False]],
  {θ, 0, 1}, Frame → True, Background → GrayLevel[1],
  PlotRange → {{0, 1}, {0.329`, 0.35`}}, FrameLabel →
   {"Persistence parameter, θ", "Proportion of total\nexecuted on first trade",
    "B. Persistence and the first trade, T=3", None},
  FrameTicks → {Range[0, 1, 0.2`], Range[0.33`, 0.35`, 0.01`], None, None},
  BaseStyle → baseStyle]
```



```
GraphicsRow[{p1, p2},
 Spacings → {Scaled[-0.05`], Scaled[0]}, Background → GrayLevel[1]]
```



■ Geometrically declining stochastic temporary impact (Section 15.1.3)

□ *Definitions:*

```
m_Rule = m_t_ :> (m_t-1 /. m_0 → 0) + λ s_t + μ + ϵ_t;
p_Rule = p_t_ :> m_t + γ A_t;
wRule = w_t_ :> w_t-1 - (s_t-1 /. s_0 → 0) /; ! t === 0;
A_Rule = A_t_ :> s_t + θ A_t-1 + u_t /; ! t === 0;
ϵZap = ϵ_ → 0;
uZap = u_ → 0;
params = {μ → 0};
```

■ "Manual" Backwards optimization

At time T the expected cost is:

```
p_T w_T /. p_Rule /. m_Rule /. s_T → w_T /. params
```

$w_T\ (\gamma A_T + m_{-1+T} + \lambda w_T + \epsilon_T)$

```
EV_T = p_T w_T /. p_Rule /. m_Rule /. A_Rule /. s_T → w_T /. ϵZap /. uZap /. params // FullSimplify
```

$w_T\ (\gamma \theta A_{-1+T} + m_{-1+T} + (\gamma + \lambda)\ w_T)$

At time T-1, the value function is:

```
(EV_T + p_{T-1} s_{T-1}) /. p_Rule /. wRule
```

$1_{-1+T}\,(\gamma\,A_{-1+T} + m_{-1+T}) + (-1_{-1+T} + w_{-1+T})\,(\gamma\,\theta\,A_{-1+T} + m_{-1+T} + (\gamma + \lambda)\,(-1_{-1+T} + w_{-1+T}))$

```
v =
 (EV_T + p_{T-1} s_{T-1}) /. params /. wRule /. p_Rule /. m_Rule /. A_Rule /. ϵZap /. uZap /. params //
  FullSimplify
```

$(-\gamma\,(-2+\theta) + \lambda)\,1_{-1+T}^2 + 1_{-1+T}\,(-\gamma\,(-1+\theta)\,\theta\,A_{-2+T} + (\gamma\,(-2+\theta) - \lambda)\,w_{-1+T}) + w_{-1+T}\left(\gamma\,\theta^2\,A_{-2+T} + m_{-2+T} + (\gamma+\lambda)\,w_{-1+T}\right)$

Optimizing:

```
sol = Flatten[Solve[0 == ∂_{s_{T-1}/.params} v /. params, s_{T-1}/.params] // Simplify][[1]]
```

$1_{-1+T} \to \dfrac{-\gamma\,(-1+\theta)\,\theta\,A_{-2+T} + (\gamma\,(-2+\theta) - \lambda)\,w_{-1+T}}{2\,\gamma\,(-2+\theta) - 2\,\lambda}$

```
s = sol[[2]] // Apart
```

$-\dfrac{\gamma\,(-1+\theta)\,\theta\,A_{-2+T}}{2\,(-2\,\gamma + \gamma\,\theta - \lambda)} + \dfrac{w_{-1+T}}{2}$

Verify that the sign of the A_{T-2} coefficient is negative:

```
Simplify[Coefficient[s, A_{T-2}] < 0, {γ > 0, 0 < θ < 1, λ > 0}]
```

True