TreesDemo

Empirical Market Microstructure

(2006, Oxford University Press)

Companion *Mathematica* notebook

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This notebook illustrates the use of the Tree package to build, display and analyze trees used in sequential trade models.

```
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```

Notebook evaluated Monday 4 June 2007 20:12:55

A tree is represented as a multilevel list where the entries are expressions corresponding to nodes. For example:

- `{root}` is a tree with one node (named '`root`')
- `{root, a1, a2}` is a tree where `root` can branch to `a1` or `a2`.
- In `{root, {a1, a11, a12}, a2}`, the node at `a1` can branch to `a11` or `a12`.

More formally, a tree is `{node}` where nodes may be iteratively defined as `node := {node1, ..., nodeN}`. You will usually build a tree by adding nodes that have meaningful symbols. Transition probabilities are contained in a multilevel list with the same structure as the tree, with the root node = 1; Total probabilities are similarly structured.

Initialization

```
SetDirectory[
  "c:/Active/Empirical Market Microstructure/Mathematica/Spring 2007"];
```

```
<< Trees.m
```

Example using a simple decision tree

Here is a simple label tree for a sequential trade model in which value can be either low or high:

```
LabelTree = {V, VLo, VHi}
```

```
{V, VLo, VHi}
```

```
ShowTree[LabelTree]
```

```
V VLo
  VHi
```

```
LabelTree = LabelTree /. x : VLo | VHi :→ {x, Inf, U}
```

```
{V, {VLo, Inf, U}, {VHi, Inf, U}}
```

```
ShowTree[LabelTree]
```

```
V VLo Inf
      U
  VHi Inf
      U
```

```
LabelTree = LabelTree /. {VLo, Inf, a_} :→ {VLo, {Inf, S}, a} /.
    {VHi, Inf, a_} :→ {VHi, {Inf, B}, a} /. U → {U, B, S}
```

```
{V, {VLo, {Inf, S}, {U, B, S}}, {VHi, {Inf, B}, {U, B, S}}}
```

```
ShowTree[LabelTree]
```

```
V VLo Inf S
      U   B
          S
  VHi Inf B
      U   B
          S
```

Transitional probabilities

Now we'll construct the tree of transitional probabilities. Since this has the same form as the label tree, it is easiest to start with the label tree and replace the names of the nodes with the transition probabilities. To start, we put in the probabilities of uninformed buys and sells:

```
V VLo Inf S
      U   B
          S
  VHi Inf B
      U   B
          S
```

```
PrTree = LabelTree /. {U, B, S} → {U, 1 / 2, 1 / 2}
```

$\left\{V,\ \left\{VLo,\ \{Inf,\ S\},\ \left\{U,\ \frac{1}{2},\ \frac{1}{2}\right\}\right\},\ \left\{VHi,\ \{Inf,\ B\},\ \left\{U,\ \frac{1}{2},\ \frac{1}{2}\right\}\right\}\right\}$

```
ShowTree[PrTree]
```

V	VLo	Inf	S
		U	$\frac{1}{2}$
			$\frac{1}{2}$
	VHi	Inf	B
		U	$\frac{1}{2}$
			$\frac{1}{2}$

```
PrTree = PrTree /. S → 1 /. B → 1 /. Inf → μ /. U → 1 - μ /. VLo → δ /. VHi → 1 - δ /. V → 1
```

$\left\{1,\ \left\{\delta,\ \{\mu,\ 1\},\ \left\{1-\mu,\ \frac{1}{2},\ \frac{1}{2}\right\}\right\},\ \left\{1-\delta,\ \{\mu,\ 1\},\ \left\{1-\mu,\ \frac{1}{2},\ \frac{1}{2}\right\}\right\}\right\}$

```
ShowTree[PrTree]
```

1	δ	μ	1
		$1-\mu$	$\frac{1}{2}$
			$\frac{1}{2}$
	$1-\delta$	μ	1
		$1-\mu$	$\frac{1}{2}$
			$\frac{1}{2}$

```
ShowTree[LabelTree, PrTree]
```

V 1	VLo δ	Inf μ	S 1
		U $1-\mu$	B $\frac{1}{2}$
			S $\frac{1}{2}$
	VHi $1-\delta$	Inf μ	B 1
		U $1-\mu$	B $\frac{1}{2}$
			S $\frac{1}{2}$

Total probabilities

The `PrTotal` function takes the transitional probabilities and returns a tree with total probabilities:

V 1	VLo δ	Inf μ	S 1
		U $1-\mu$	B $\frac{1}{2}$
			S $\frac{1}{2}$
	VHi $1-\delta$	Inf μ	B 1
		U $1-\mu$	B $\frac{1}{2}$
			S $\frac{1}{2}$

```
PrTotal[PrTree] // ShowTree
```

1	δ	$\delta\,\mu$	$\delta\,\mu$
		$\delta\,(1-\mu)$	$\frac{1}{2}\,\delta\,(1-\mu)$
			$\frac{1}{2}\,\delta\,(1-\mu)$
	$1-\delta$	$(1-\delta)\,\mu$	$(1-\delta)\,\mu$
		$(1-\delta)\,(1-\mu)$	$\frac{1}{2}\,(1-\delta)\,(1-\mu)$
			$\frac{1}{2}\,(1-\delta)\,(1-\mu)$

```
BasicTree = BuildTree[LabelTree, PrTree]
```

Trees\`Private\`Tree$\Big[$\{V, \{VLo, \{Inf, S\}, \{U, B, S\}\}, \{VHi, \{Inf, B\}, \{U, B, S\}\}\},

$\left\{1, \left\{\delta, \{\mu, 1\}, \left\{1-\mu, \frac{1}{2}, \frac{1}{2}\right\}\right\}, \left\{1-\delta, \{\mu, 1\}, \left\{1-\mu, \frac{1}{2}, \frac{1}{2}\right\}\right\}\right\},$

$\left\{1, \left\{\delta, \{\delta\,\mu, \delta\,\mu\}, \left\{\delta\,(1-\mu), \frac{1}{2}\,\delta\,(1-\mu), \frac{1}{2}\,\delta\,(1-\mu)\right\}\right\},\right.$

$\left.\left\{1-\delta, \{(1-\delta)\,\mu, (1-\delta)\,\mu\}, \left\{(1-\delta)\,(1-\mu), \frac{1}{2}\,(1-\delta)\,(1-\mu), \frac{1}{2}\,(1-\delta)\,(1-\mu)\right\}\right\}\right\}\Big]$

```
ShowTree[BasicTree]
```

V 1 1	VLo δ δ	Inf μ $\delta\,\mu$	S 1 $\delta\,\mu$
		U $1-\mu$ $\delta\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
			S $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
	VHi $1-\delta$ $1-\delta$	Inf μ $(1-\delta)\,\mu$	B 1 $(1-\delta)\,\mu$
		U $1-\mu$ $(1-\delta)\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$
			S $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$

Joint and conditional probabilities

The `Pr` function may be used to obtain the total probability of a particular node:

V 1 1	VLo δ δ	Inf μ $\delta\,\mu$	S 1 $\delta\,\mu$
		U $1-\mu$ $\delta\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
			S $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
	VHi $1-\delta$ $1-\delta$	Inf μ $(1-\delta)\,\mu$	B 1 $(1-\delta)\,\mu$
		U $1-\mu$ $(1-\delta)\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$
			S $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$

```
Pr[BasicTree, VLo]
```

δ

```
Pr[BasicTree, S]
```

$$\frac{1}{2}\,(1-\delta)\,(1-\mu)+\frac{1}{2}\,\delta\,(1-\mu)+\delta\,\mu$$

```
ShowTree[BasicTree]
```

V 1 1	VLo δ δ	Inf μ $\delta\,\mu$	S 1 $\delta\,\mu$
		U $1-\mu$ $\delta\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
			S $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
	VHi $1-\delta$ $1-\delta$	Inf μ $(1-\delta)\,\mu$	B 1 $(1-\delta)\,\mu$
		U $1-\mu$ $(1-\delta)\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$
			S $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$

```
Pr[BasicTree, S] + Pr[BasicTree, B]
```

$(1-\delta)\,(1-\mu)+\delta\,(1-\mu)+(1-\delta)\,\mu+\delta\,\mu$

```
% // Simplify
```

1

```
Pr[BasicTree, VLo, B] // Simplify
```

$-\frac{1}{2}\,\delta\,(-1+\mu)$

```
Pr[BasicTree, VLo, Inf, B]
```

0

```
Pr[BasicTree, VLo, Inf, S]
```

$\delta\,\mu$

```
Pr[BasicTree, VLo, B] / Pr[BasicTree, B]
```

$$\frac{\delta\,(1-\mu)}{2\left(\frac{1}{2}\,(1-\delta)\,(1-\mu)+\frac{1}{2}\,\delta\,(1-\mu)+(1-\delta)\,\mu\right)}$$

```
% // Simplify
```

$$\frac{\delta-\delta\,\mu}{1+\mu-2\,\delta\,\mu}$$

Additional routines

Checking conformance of tree structures

Sometimes when you're building up label transitional probability trees, it's difficult to verify that they have the same list structure. You can do this verification with `CheckConformance`:

$$\frac{\delta-\delta\,\mu}{1+\mu-2\,\delta\,\mu}$$

```
CheckConformance[LabelTree, PrTree]
```

```
True
```

For example, here's a different label tree, one that doesn't conform to `PrTree`.

```
BadLabelTree = LabelTree /. Inf -> {Inf}
```

```
{V, {VLo, {{Inf}, S}, {U, B, S}}, {VHi, {{Inf}, B}, {U, B, S}}}
```

In this case, `CheckConformance` highlights the first

```
CheckConformance[BadLabelTree, PrTree]
```

```
CheckConformance. First list mismatch at 1

Tree 1 {2, 2, 1}

{{Inf}, S}

Tree 2 {2, 2}
```

$\left\{\delta, \{\mu, 1\}, \left\{1-\mu, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

```
$Aborted
```

$\left\{\delta, \{\mu, 1\}, \left\{1-\mu, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

```
Tree 2 {2, 2}

{{Inf}, S}

Tree 1 {2, 2, 1}

CheckConformance. First list mismatch at 1

CheckConformance. First list mismatch at 1

Tree 1 {2, 2, 1}

{{Inf}, S}

Tree 2 {2, 2}
```

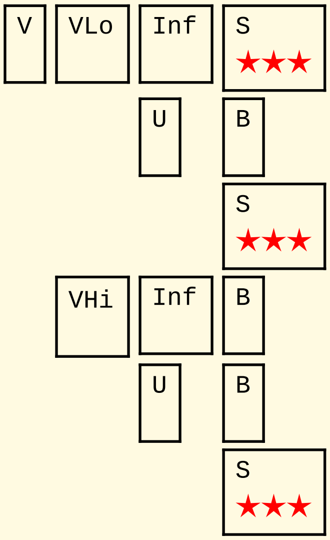
$\left\{\delta, \{\mu, 1\}, \left\{1-\mu, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

```
$Aborted
```

Marked trees

Sometimes we'd like to display a tree with certain nodes prominently marked. Here is the `LabelTree`, with `S` nodes marked:

```
ShowTree[LabelTree, MarkedTree[LabelTree, S]]
```



Here is the tree with "buy" nodes marked:

```
MarkedTree[BasicTree, B] // ShowTree
```

V 1 1	VLo δ δ	Inf μ $\delta\,\mu$	S 1 $\delta\,\mu$
		U $1-\mu$ $\delta\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$ ★★★
			S $\frac{1}{2}$ $\frac{1}{2}\,\delta\,(1-\mu)$
	VHi $1-\delta$ $1-\delta$	Inf μ $(1-\delta)\,\mu$	B 1 $(1-\delta)\,\mu$ ★★★
		U $1-\mu$ $(1-\delta)\,(1-\mu)$	B $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$ ★★★
			S $\frac{1}{2}$ $\frac{1}{2}\,(1-\delta)\,(1-\mu)$