Models of price dynamics and order splitting

Securities Trading: Principles and Procedures
Chapters 13 and 14

Outline

- Statistical models of security prices and order impacts
- Given these statistical models, what are the best order-splitting strategies.
- The risk-return trade-off in order splitting.

Statistical models

- The basic models are constructed by starting with a simple model and adding on the features that we need.
- Random-walk model
- Random-walk + drift (“short-term alpha”)
- Impact model: Random-walk + drift + order-price impact

 3

Random-walk model

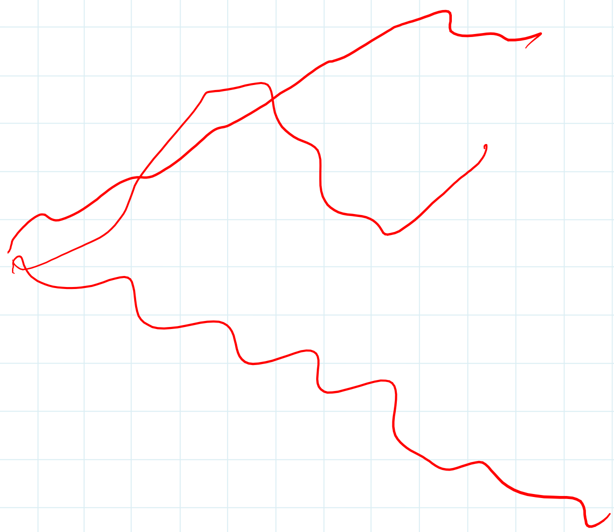
- Let t represent time (minutes, seconds, milliseconds, ticks …)
- p_t is the price at the end of interval t (at the end of the minute, second, …)
  - Usually p_t is the bid-ask midpoint (BAM), but it might be the last sale price.
- $p_t = p_{t-1} + u_t$
  - where u_t is some random disturbance or prediction error that reflects “new information”
    - In expectation this disturbance is zero: $Eu_t = 0$.
    - The standard deviation of u_t is σ_u.

 4

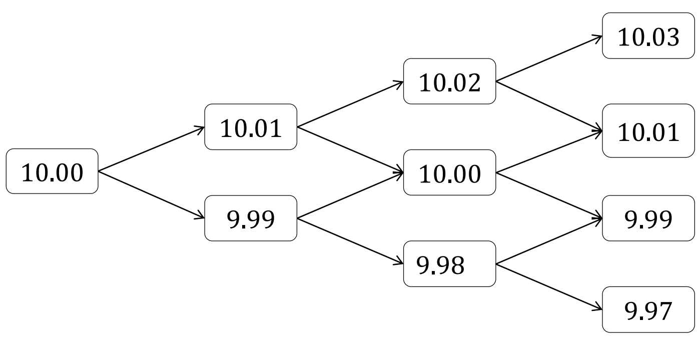
$P_1 = P_0 + u_1$

$P_2 = P_1 + u_2$

$P_3 = P_2 + u_2$



Recall the example used to analyze implementation shortfall of limit vs. market orders.

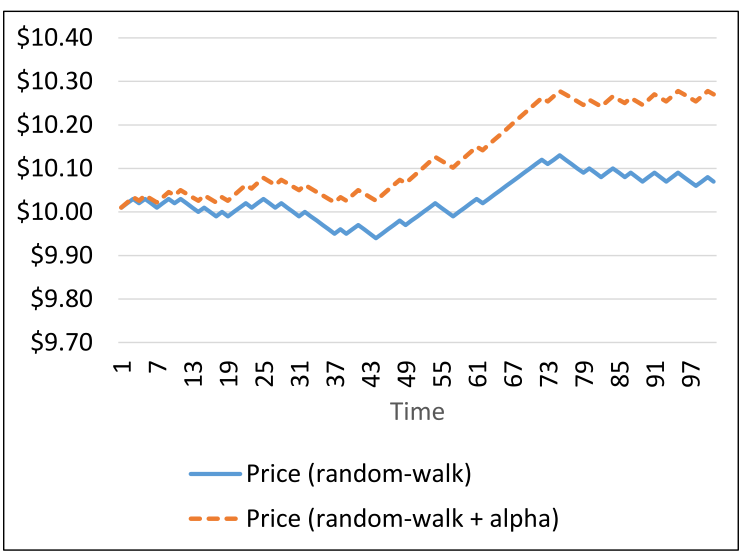


- At each step, $u_t = \pm 0.01$ with equal probability.
- $Eu_t = \frac{1}{2} \times (0.01) + \frac{1}{2} \times (-0.01) = 0$
- $\sigma_u = \sqrt{\frac{1}{2} \times (0.01)^2 + \frac{1}{2} \times (-0.01)^2} = 0.01$

Random-walk with drift (“short-term alpha”)

- The basic model ($p_t = p_{t-1} + u_t$) has no trend (on average)
- A trader might believe that there *is* a predictable trend (from momentum, over-shooting, consensus forecast error, etc.)
- $p_t = \alpha + p_{t-1} + u_t$
  - α denotes the trend (either up or down), such as, “+$0.01 per minute”
  - “alpha” is used in many finance contexts to denote superior (or, if negative, inferior) performance.
  - Recall the Security Market Line (SML) from Foundations.
- This model is used extensively in option pricing

Simulated random walk (with/without $\alpha = 0.002$ per tick)

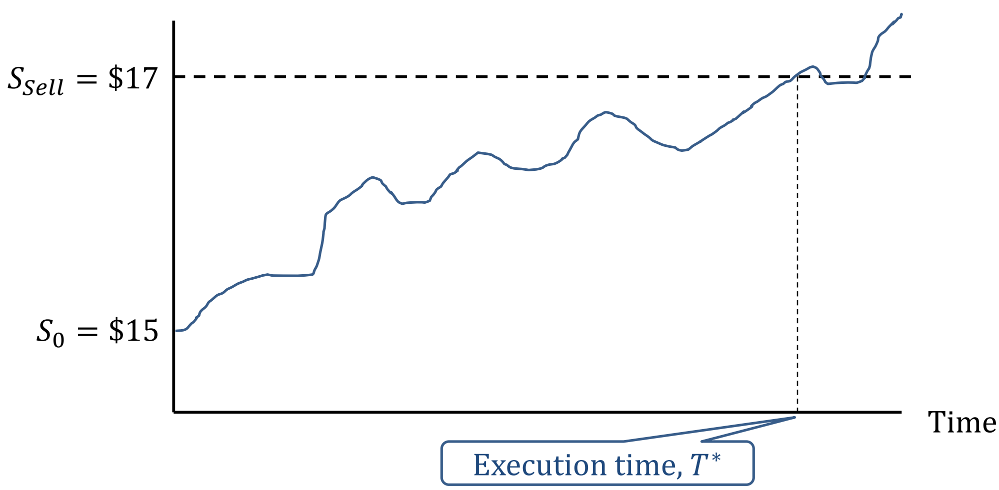


- Non-zero alpha is difficult to detect visually and statistically.

The random-walk with drift: Limit order execution times

- Suppose that the current stock price is S_0 and we want to put in a limit order to sell at some price $S_{Sell} > S_0$.
- Example: $S_0 = \$15$ and $S_{Sell} = \$17$
- How long do we think it will take for the order to execute?

The situation



A simple result

- If $p_t = \alpha + p_{t-1} + u_t$, then $ET^* = \frac{S_{Sell} - S_0}{\alpha}$
- Example
  - t measures minutes and $\alpha = \$0.01$ $per\ minute$.
  - Then $ET^* = \frac{17.00 - 15.00}{0.01} = \frac{2.00}{0.01} = 200\ minutes$
- Notes
  - The expected time to execution does not depend on volatility.
  - If $\alpha = 0$ then ET^* is infinite.
    - You might get an execution, but don't count on it.
- Embedded problem: $S_0 = 20, \alpha = -\$0.03$ $per\ minute$. We put in a limit order to buy at \$19.10. How long do we expect to wait? (Answer in online notes)

Embedded problem

- $S_0 = 20, \alpha = -\$0.03$ $per\ minute$. We put in a limit order to buy at \$19.10. How long do we expect to wait? (Answer in online notes)
- $ET^* = \frac{20.00 - 19.10}{0.03} = \frac{0.90}{0.03} = 30\ minutes$

What's useful in predicting short-term alpha?

- Advance knowledge of news announcements.
- Current/recent price changes in other stocks that are in the same industry.
- Current/recent changes in the market index.

 13

predictable drift

The impact model: random-walk + drift + order/price impact

random walk unpredictable

- $p_t = p_{t-1} + \alpha + \lambda S_t + u_t$
- S_t is the net number of shares actively purchased in interval t.
  - the number of shares that lifted the ask less the number of shares that hit the bid.
  - Example: $S_t = -100 \rightarrow$ "100 shares were sold, net"
- $\lambda > 0$ is the impact coefficient.
  - As λ increases, each trade has a larger impact.
- "Purchases drive the price up; sales drive the price down."

 14

Interpretation of S_t

- In principle S_t is the net purchase computed over all trades in interval t.
- Including our own trades and trades of others.
- $S_t = S_t^{own} + S_t^{others}$
- For forecasting and analysis, we want to use the best available prediction of S_t^{others}.
- Often trading strategies are analyzed assuming that our expectation of others' trades is $ES_t^{others} = 0$

Interpretation of λ

- The model says that order-price impact is permanent.
- Order price impact arises from the market's belief that orders might be informed.
- If we are uninformed, our trades will still move the market, but eventually the effect of our trades will vanish.

Analysis of order splitting with the impact model

- A two-period example.
- Objective: buy S_{Total} shares over two periods (1 and 2) at the lowest possible expected cost.
- $S_{Total} = S_1 + S_2$
  - where S_1 = shares purchased in period 1 and S_2 = shares purchased in period 2.
- The total cost is $C = p_1 S_1 + p_2 S_2$
- It is now time 0. We know p_0, but we don't know p_1 or p_2.
- Use the impact model to forecast p_1 and p_2.

 17

Special case where $\alpha = 0$.

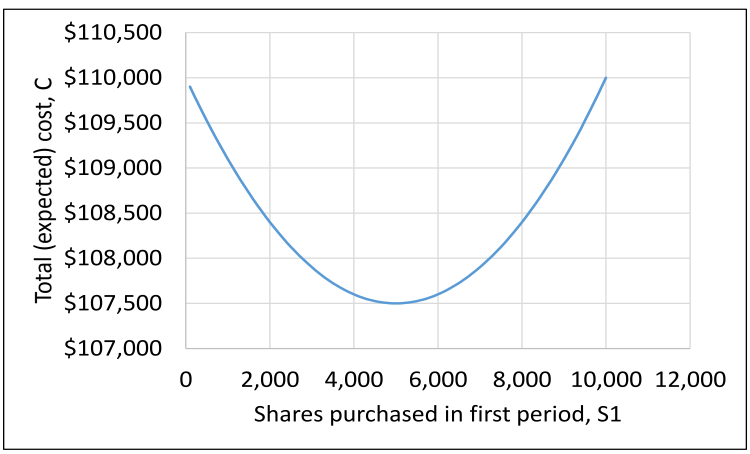
- Compute ahead:
  - $p_1 = p_0 + \lambda S_1 + u_1$
  - $p_2 = p_1 + \lambda S_2 + u_2 = p_0 + \lambda S_1 + \lambda S_2 + u_1 + u_2$
- Simplify by setting $u_1 = u_2 = 0$
  - "Looking ahead, we expect u_1 and u_2 to be zero."
  - $p_1 = p_0 + \lambda S_1$
  - $p_2 = p_0 + \lambda S_1 + \lambda S_2$
- Plug these into C as forecasts.

 18

Reworking the cost

- $C = p_1 S_1 + p_2 S_2 = S_1(p_0 + \lambda S_1) + S_2(p_0 + \lambda S_1 + \lambda S_2)$
  - where S_1 and S_2 are the "unknowns"
- Remember that $S_{Total} = S_1 + S_2$, so $S_2 = S_{Total} - S_1$
- $C = \lambda S_1^2 - S_1 \lambda S_{Total} + S_{Total}\,(p_0 + \lambda S_{Total})$
  - This is one equation in one unknown (S_1)
  - Formally, this is the expected cost, "EC"

Plot of C with parameters $S_{Total} = 10{,}000\ shares; p_0 = \$10; \lambda = \$0.0001/share$



- The total expected cost has a minimum at $S_1 = 5{,}000\ shares$ (and $S_2 = 5{,}000\ shares$)

Formally, to find the minimum, set $\frac{dC}{dS_1} = 0$

- $C = \lambda S_1^2 - S_1 \lambda S_{Total} + S_{Total}\,(p_0 + \lambda S_{Total})$
- $\frac{dC}{dS_1} = 2\lambda S_1 - \lambda S_{Total} = 0$ implies the optimal $S_1^* = \frac{S_{Total}}{2}$.
- In general, with $\alpha = 0$, and trading over T periods,

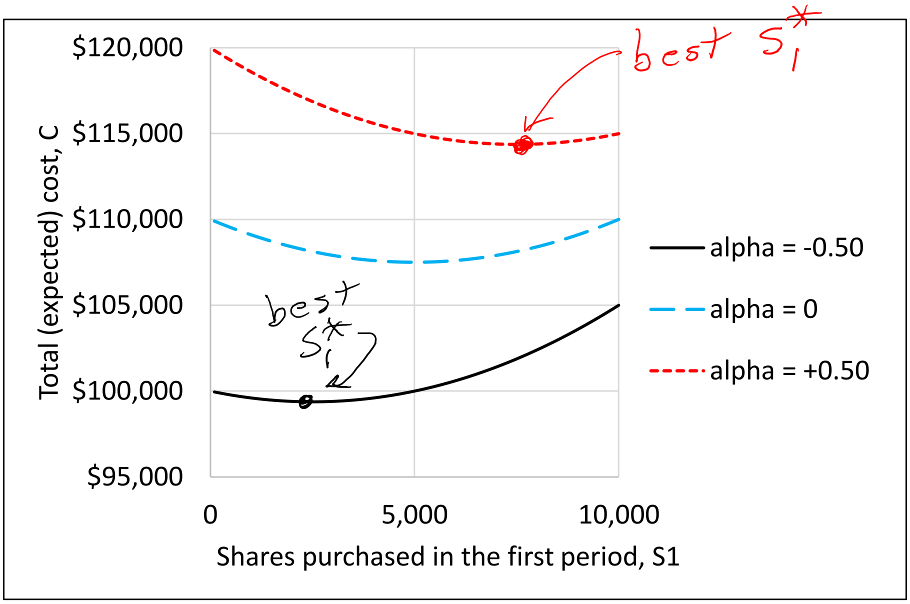
$$S_i^* = \frac{S_{Total}}{T}$$

 21

What if $\alpha \neq 0$ (in the two-period problem)?

- Modified optimum: $S_1^* = \frac{\alpha + \lambda S_{Total}}{2\lambda}$
- With $\alpha > 0$, there is positive drift, so S_1^* rises.
  - Future purchases will be more expensive.
- With $\alpha < 0$, there is negative drift.
  - The price is dropping: buy later.

 22

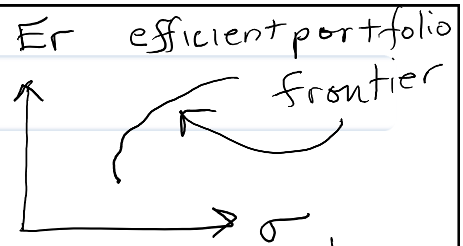


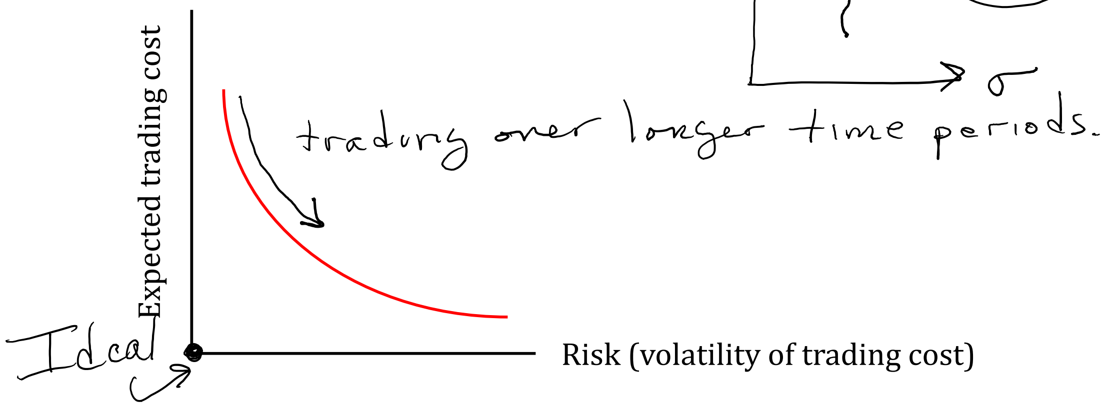
Trading Horizon (T), expected cost, and variance of cost (risk)

T	EC^*	$Var(C^*)$
1	$p_0 S_{Total} + \lambda S_{Total}^2$	$S_{Total}^2 \sigma_u^2$
2	$p_0 S_{Total} + \frac{3\lambda S_{\text{Total}}^2}{4}$	$\frac{5}{4} S_{Total}^2 \sigma_u^2$
3	$p_0 S_{Total} + \frac{2\lambda S_{\text{Total}}^2}{3}$	$\frac{14 S_{Total}^2}{9} \sigma_u^2$
	… $\downarrow$	$\uparrow$

- As T increases, expected trading cost drops and risk rises.

The "efficient trading frontier"





- (By analogy with the efficient portfolio frontier.)

 25

Extensions and modifications

- More complex order impacts that have both permanent and temporary (transient) effects.
- Addition of other securities (stocks in the same industry, a market-wide basket, and so on).
- Time variation in parameters.
  - $\alpha, \lambda,$ and/or σ_u depend on time of day.

 26

Order price impact and manipulation

- "Manipulation"
  - There are no universally accepted definitions in economics or law.
- Most definitions suggest something like
  - Trading to deliberately move the price, to establish an *artificial* price, a price that does not reflect *true* supply and demand.
  - Most manipulations involve deception.

- These definitions aren't precise
  - It is almost impossible to trade *without* moving the price.
  - Many accepted strategies attempt to obscure the trader's true information, intentions and plans.
- The follow illustrates certain possible manipulations based on order price impact.
- For a given impact function can an uninformed trader execute a series of profitable buys and sells based on the price movements that his orders generate?
- *Note: many of the schemes discussed here are illegal. They are presented to facilitate discussion of what features make a market prone to manipulation, so that, to the greatest extent possible, these features may be avoided in actual securities markets.*

Recall the impact function used to analyzer order splitting.

- $p_t = p_{t-1} + \alpha + \lambda S_t + u_t$
  - S_t is the net number of shares actively purchased in interval t.
  - $\lambda > 0$ is the impact coefficient.
- We will look at strategies that start "flat" (with no position) and end flat.
  - Example: buy 10 shares using 10 orders of one share, then sell using 10 orders of one share, OR two orders of five shares, OR …
- We'll assume that $\alpha = 0$.
  - If $\alpha > 0$, just buy. Sell after the price has gone up.
  - If $\alpha < 0$, …
- We'll also ignore risk ($u_t = 0$)
- $p_t = p_{t-1} + \lambda S_t$

Attempted manipulation 1

- Starting at p_0, buy 5 shares slowly, one at a time.
  - $p_1 = p_0 + \lambda \times 1 = p_0 + \lambda$
  - $p_2 = (p_0 + \lambda) + \lambda \times 1 = p_0 + 2\lambda$
  - …
  - $p_5 = p_0 + 5\,\lambda$
  - Average purchase price is $\frac{p_1 + p_2 + \cdots + p_5}{5} = p_0 + 3\lambda$
- Now sell the shares, one at a time
  - $p_6 = p_5 - \lambda \times 1 = p_0 + 5\,\lambda - \lambda = p_0 + 4\,\lambda$
  - $p_7 = p_6 - \lambda \times 1 = p_0 + 4\,\lambda - \lambda = p_0 + 3\,\lambda$
  - …
  - $p_{10} = p_9 - \lambda \times 1 = p_0 + \lambda - \lambda = p_0$
  - Average sale price is $\frac{p_6 + p_7 + \cdots + p_{10}}{5} = p_0 + 2\lambda$
- The average profit per share is $-(p_0 + 3\lambda) + (p_0 + 2\lambda) = -\lambda$ (a loss)
- The receipts don't cover the expenditure. The manipulation doesn't work.

Attempted manipulation 2

- Buy the five shares slowly (as before)
  - Average price is $p_0 + 3\lambda$
- Sell the shares all at once:
  - $p_6 = p_5 - \lambda \times 5 = p_0 + 5\lambda - 5\lambda = p_0$
- Manipulation profits are $-(p_0 + 3\lambda) - p_0 = -3\lambda$
- This, too, leads to a loss.

"Theorem"

" λS_τ "

λ doesn't change

- If the price impact function is linear and constant over time, profitable manipulation isn't possible.
- Huberman, Gur, & Stanzl, Werner. (2004). Price Manipulation and Quasi-Arbitrage. *Econometrica*, 72(4), 1247-1275. doi: 10.2307/3598784
- Are there non-linear or time-varying price impact functions that allow for manipulation.
- *Yes*

Time variation in impact

- Suppose that initially $\lambda = 1$, and we know that it will drop to $\lambda = 0.1$.
- Then we can buy two units
  - $p_1 = p_0 + \lambda \times 1 = p_0 + 1$
  - $p_2 = p_1 + \lambda \times 1 = p_0 + 2$
  - Average share price is $p_0 + 1.5$
- ... and sell them when $\lambda = 0.1$
  - $p_3 = p_2 - \lambda \times 1 = p_2 - 0.1 = p_0 + 1.9$
  - $p_4 = p_3 - \lambda \times 1 = p_0 + 1.8$
  - Average share price is $p_0 + 1.85$
- Manipulation profits are $-(p_0 + 1.5) + p_0 + 1.85 = 0.35 > 0$

Asymmetry in the impact function

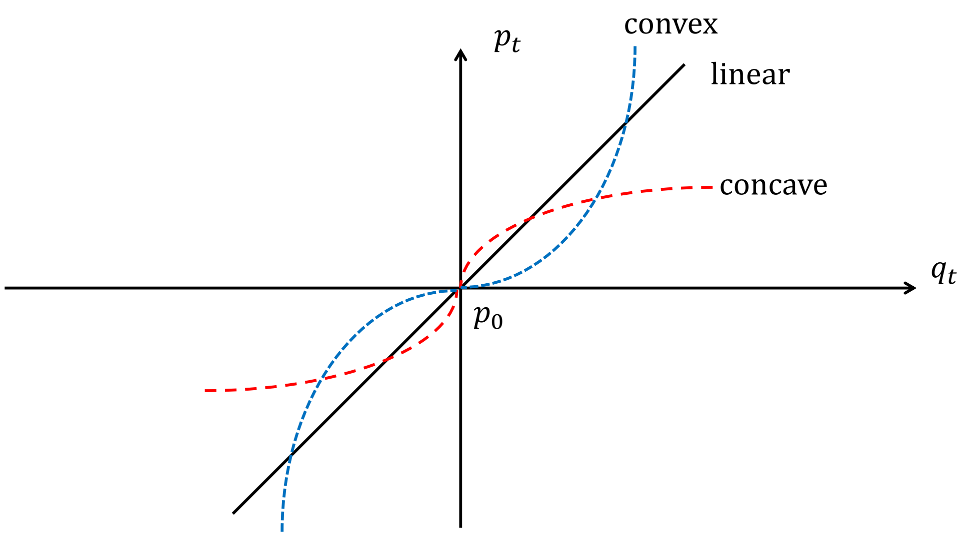
- Suppose that λ for buys is $\lambda_{Buy} = 0.1$ and λ for sells is $\lambda_{Sell} = 1$.
- We (short) sell two shares
  - $p_1 = p_0 - \lambda_{Sell} \times 1 = p_0 - 1$
  - $p_2 = p_1 - \lambda_{Sell} \times 1 = p_0 - 2$
  - Average price is $p_0 - 1.5$
- Now we cover our short sales
  - $p_3 = p_2 + \lambda_{Buy} \times 1 = p_2 - 2 + .1 = p_0 - 1.9$
  - $p_4 = p_3 + \lambda_{Buy} \times 1 = p_0 - 1.8$
  - Average price is $p_0 - 1.85$
- Manipulation profits are $+(p_0 - 1.5) - (p_0 - 1.85) = 0.35 > 0$

General structure of manipulations

- To establish the position, first trade to maximize the price impact.
  - This doesn't necessarily mean "buy"; sometimes the initial position is short.
- To unwind the position, trade to minimize the price impact.
- Time variation
- Asymmetry

 35

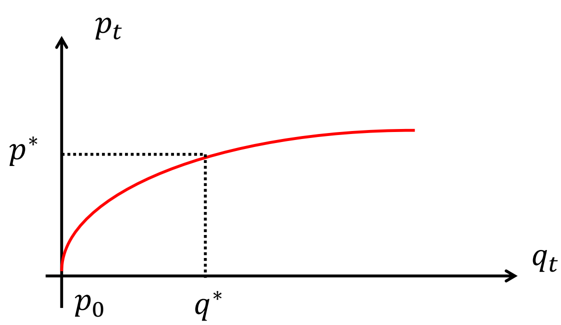
Nonlinearities in the impact function …



 36

The concave case, for purchases

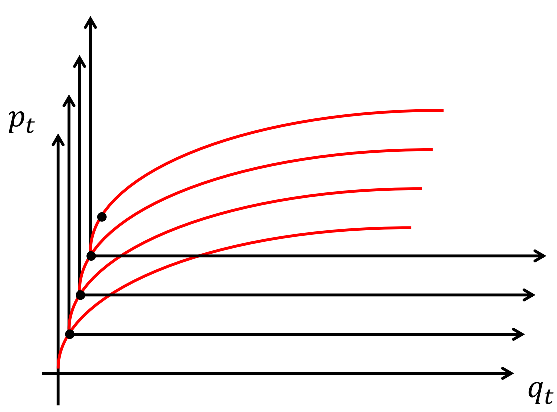
- Suppose that $p_t = \begin{cases} p_{t-1} + \lambda\sqrt{S_t} \text{ for buy orders}, S_t > 0 \\ p_{t-1} - \lambda\sqrt{-S_t} \text{ for sell orders}, S_t < 0 \end{cases}$



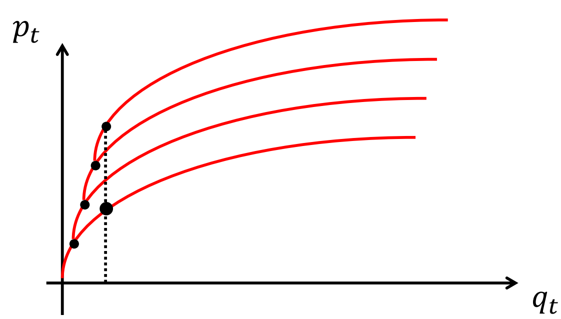
The average price for a purchase of q^* shares is p^*/q^*.
This is lower for large traders.

 37

A series of small trades vs. one large trade



Small trades: high impact



Large trade: lower impact

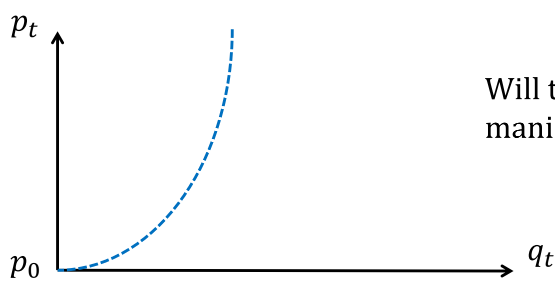
 38

Concave example

- Suppose that $p_t = \begin{cases} p_{t-1} + \lambda\sqrt{S_t} \text{ for buy orders, } S_t > 0 \\ p_{t-1} - \lambda\sqrt{-S_t} \text{ for sell orders, } S_t < 0 \end{cases}$
- Buy 8 units with eight 1-unit trades
- Sell 8 units with two 4-unit trades

- Purchases
  - $p_1 = p_0 + \lambda \times \sqrt{1} = p_0 + \lambda$
  - …
  - $p_8 = p_7 + \lambda \times \sqrt{1} = p_0 + 8\,\lambda$
  - Average purchase price is $p_0 + 4.5\,\lambda$
- Sales
  - $p_9 = p_8 - \lambda \times \sqrt{4} = p_0 + 6\lambda$
  - $p_{10} = p_9 - \lambda \times \sqrt{4} = p_0 + 4\lambda$
  - Average sale price is $p_0 + 5\,\lambda$
- Profits per share are $(p_0 + 5\lambda) - (p_0 + 4.5\,\lambda) = 0.5\,\lambda > 0$

The convex case, for purchases



Will the same buy-small, sell-large manipulation work?

- Suppose that $p_t = \begin{cases} p_{t-1} + \lambda\, S_t^2 \text{ for buy orders}, S_t > 0 \\ p_{t-1} - \lambda\, S_t^2 \text{ for sell orders}, S_t < 0 \end{cases}$
- We'll buy eight shares with two trades of 4 shares.
- Sell eight shares with eight sales of 1 share.

- Purchases
  - $p_1 = p_0 + \lambda \times 4^2 = p_0 + 16\,\lambda$
  - $p_2 = p_1 + \lambda \times 4^2 = p_0 + 32\,\lambda$
  - Average purchase price is $p_0 + 24\,\lambda$
- Sales
  - $p_3 = p_2 - \lambda \times 1 = p_0 + 31\,\lambda$
  - $p_4 = p_3 - \lambda \times 1 = p_0 + 30\,\lambda$
  - …
  - $p_{10} = p_9 - \lambda \times 1 = p_0 + 24\,\lambda$
  - Average sale price is $p_0 + 27.5\,\lambda$
- Manipulation profits are $3.5\,\lambda$ per share

 43

- Is manipulation really possible when the price impact function is non-linear, buy-sell asymmetric, or time varying?
  - Other costs (like bid-ask spread, commissions) might reduce profits.
  - There are risks:
    - We don’t know for sure what the price impact function looks like.
    - Prices change for reasons other than incoming orders.
- What is the empirical evidence?

 44

The square-root "law"

- Many practitioners and academics believe that price impact goes up with the square root of order size.
  - Example: $p_t = p_{t-1} + \lambda \sqrt{S_t}$
- This seems to fit many samples of financial data. Typically:
  - A broker looks at the price impact of all its customer orders.
  - A hedge fund looks at the price impact of all of its own orders.

Spoofing and layering

- Spoofing: entering a bid or offer that is not intended for execution.
- Layering: entering *large* bids/offers not intended for execution *priced away* from the market.
- Priced away: a buy limit order priced below the bid or a sell limit order priced above the offer.
- Why?

BATS book in PBR (Petrobras) on April 27, 2015

BATS

PBR

Market Quality Statistics

PETROLEO BRASILEIRO SA PETROBR SPONSORED ADR

Orders Accepted 239,828 | Total Volume 3,755,485

	TOP OF BOOK		LAST 10 TRADES		
	SHARES	PRICE	TIME	PRICE	SHARES
ASKS	12,558	9.55	15:04:12	9.51	300
	11,558	9.54	15:04:12	9.51	100
	14,158	9.53	15:04:12	9.51	200
	13,858	9.52	15:04:12	9.51	100
	24,808	**9.51**	15:04:12	9.51	400
BIDS	11,200	**9.50**	15:04:03	9.51	57
	16,810	9.49	15:04:03	9.51	300
	17,830	9.48	15:04:03	9.51	300
	13,130	9.47	15:04:03	9.51	200
	10,630	9.46	15:04:03	9.51	600

Last updated 15:04:19

- Large quantities at the best bid and offer and away from the best bid an offer.
- Conveys the sense of a liquid market.

47

BATS book in PRK National

Book Viewer | BZX | BYX | EDGX | EDGA

BATS

PRK

Market Quality Statistics

PARK NATL CORP COM

Orders Accepted 10,091 | Total Volume 636

20,000

	TOP OF BOOK		LAST 10 TRADES		
	SHARES	PRICE	TIME	PRICE	SHARES
ASKS	100	85.92	14:07:53	84.60	35
	100	85.68	13:19:47	84.60	100
	100	85.44	13:19:43	84.69	100
	100	85.32	13:15:43	85.00	100
	100	84.99	12:54:20	85.09	1
BIDS	100	84.69	12:19:33	85.29	100
	100	84.24	10:49:13	86.54	100
	100	84.00	10:06:52	86.01	100
	100	83.88			
	100	83.76			

Last updated 15:07:24

- Suppose we suddenly place an order to sell 20,000 shares at 85.35.
- What inferences would the market draw?
- What action might ensue?

48

US v. Sarao, 2015, US District Court, Northern District of Illinois Eastern Division

- Layering case brought by the Commodities Futures Trading Commission (civil suit) and the Department of Justice (criminal prosecution).
- Read up to item 25, p. 13, ("SARAO's Responses to Queries …"), especially items
  - 5,6 (Overview of the investigation)
  - 13-24 (Layering schemes, overview of Sarao's activity)