Regime Dependent Approximations for The Single-Item Dynamic Pricing Problem

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We study the single-item dynamic pricing problem in three separate asymptotic regimes characterized by their ratio of inventory to market size. We first consider the case of customer item valuations following an exponential distribution where we are able to derive a sharp characterization of the boundaries between each of the regimes. We then proceed to the case of customer item valuations following a general distribution. In this case, we derive for each regime approximations to the optimal value function, optimal pricing policy and optimal purchasing probability policy for both the offline and the online setting. We also provide for each regime an approximation to the regret of the optimal online value function relative to the optimal offline value function. In addition to these results, we show that in the regime when the inventory to market size ratio is low, a static run-out rate policy asymptotically fails to be first-order optimal. However, a dynamic run-out rate policy is shown to achieve both first and second-order optimality. Finally, in the numerics section, we test the quality of our approximations and determine the boundaries between each of the three asymptotic regimes in the case of general customer item valuation distributions.

Key words: dynamic pricing, regret analysis, stochastic optimization, optimal control, asymptotics, market size, inventory, extreme value theory, calculus of variations

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1. Introduction

Dynamic pricing is a strategy used by firms to efficiently manage their inventory. Successfully implementing a dynamic pricing policy requires a firm to continuously update its prices in response to its current inventory levels and expected future demand. The intricate relationship between each of these quantities makes finding the optimal dynamic pricing policy a difficult problem. In fact, closed-form solutions to the single-item dynamic pricing problem are known in only a few problem instances (Gallego and Topaloglu 2019, Gallego and Van Ryzin 1994, McAfee and te Velde 2008).

One common approach to finding approximate solutions to the single-item dynamic pricing problem is to use the fluid regime of Gallego and Van Ryzin (1994). In the fluid regime, the inventory levels of the firm and the size of the market are assumed to be on the same orders of magnitude. This assumption is reasonable, however with the advent of online marketplaces smaller firms are also now capable of reaching ever larger numbers of customers. The size of such small firms could be the result of reduced production capacities or perhaps a strategic decision to limit inventory levels and consequently holding costs. Regardless of the reason, due to the differing orders of magnitude between the inventory levels of the firm and the market size, it is not clear if the approximations and insights of the fluid regime still hold.

In this paper, we simultaneously study the single-item dynamic pricing problem in three separate asymptotic regimes. The regimes are related to one another by the ratio of the firm's inventory level to its market size. Situations where the ratio is high are referred to as the small market regime. Moderate values of the ratio correspond to the classical fluid regime. The last regime where inventory levels are low relative to the market size is referred to as the large market regime.

We begin our analysis by considering the case where customers value the item being sold according to an exponential distribution. The benefit of studying this case first is that there exists a closed form expression for the optimal expected revenue of the firm. This allows us to derive straightforward approximations to the optimal value function in each of the three asymptotic regimes. We then show that there exists three distinct regions in the problem parameter space corresponding to where each of the three regime approximations are optimal. Finally, we obtain asymptotic expressions for the boundaries between the three regions and show that they may be loosely characterized by the ratio of the firm's initial inventory level to its market size.

We next proceed to the case of general item valuation distributions. Our main results of the paper provide asymptotic approximations to the optimal value function, pricing policy, and purchasing probability policy for the single-item dynamic pricing problem in each of the three asymptotic regimes mentioned above. These results are obtained for both the offline and online versions of the problem. The results that we obtain for the offline problem are apparently new. The small market and fluid results in the online setting are previously known (Gallego and Van Ryzin 1994). The large market results in the online setting are new and we demonstrate both theoretically and numerically that the large market regime occupies a significant portion of the problem parameter space. We also use our offline and online approximations in each regime to derive asymptotic expressions for the regret of the optimal policy in the online setting relative to the offline setting.

The large market regime results are new throughout the paper and deserve further discussion. Assuming customer item valuation distributions that lie in the Gumbel domain of attraction, our main results for the large market regime provide asymptotics in the offline and online setting for the optimal expected revenue, pricing policy, and purchasing probability policy. The first order approximation of the optimal expected online revenue of the firm in the large market regime differs from that in the classical fluid regime (Gallego and Van Ryzin 1994). On the other hand, the first order approximations of the optimal pricing and purchasing probability policies correspond to a dynamic run-out rate policy the same as in the fluid regime. It also turns out interestingly that in the large market regime the order of the remainder term for the optimal expected revenue and pricing policy approximations depends on the Mills ratio (or reciprocal hazard rate function) of the customer item valuation distribution. We also show that in the large market regime, a static run-out rate pricing policy fails to be first-order optimal, but a dynamic run-out rate pricing policy does still achieve first and second order optimality. This is contrary to the fluid regime where both static and dynamic run-out rate policies are first order optimal.

The remainder of the paper is organized as follows. In Section 2, we conduct a literature review. In Section 3, we present the model which is used throughout the paper. Section 4 provides the details of the small market, fluid, and large market regimes. In Section 5, we consider the special situation of customer item valuations following an exponential distribution, in which case we asymptotically characterize the boundaries between the three regimes. In Section 6, we provide the details of the extreme value theory Gumbel domain of attraction. In Section 7, we study the offline version of the model presented in Section 3 and provide approximations to its optimal value function in all three regimes. Section 8 contains our results for the model presented in Section 3. In Section 8.1, we analyze the performance of static pricing policies in the large market regime. Section 8.2 provides asymptotics on the optimal expected revenue, pricing policy, and purchasing probability policy in the small market, fluid and large market regimes. In Section 8.3, we provide examples of our results from Section 8.2 for specific customer item valuation distributions. In Section 8.4, we study the asymptotics of the regret of the optimal expected revenue with respect to its offline upper bound in all three regimes. In Section 8.5, we show how to construct asymptotically optimal dynamic pricing policies in the large market regime using our results from Section 8.2. Finally, in Section 9, we present the results of several numerical experiments. All proofs may be found in the Appendix.

2. Related Literature

The classical single-item dynamic pricing problem was originally studied by Gallego and Van Ryzin (1994), who proposed the fluid regime that scales both the customer arrival rate and initial inventory level. They showed that a static pricing policy is asymptotically optimal in this regime. There have since been many extensions of the classical model of Gallego and Van Ryzin (1994), which include multiple products (Gallego and Van Ryzin 1997, Kunnumkal and Topaloglu 2010, Chen et al. 2019, Lei and Jasin 2020), learning (Araman and Caldentey 2009, Besbes and Zeevi 2012, Keskin and Zeevi 2014, den Boer and Zwart 2015), consumer behavior (Liu and Cooper 2015, Chen and Farias 2018, Gao et al. 2018), and competition (Adida and Perakis 2010, Martínez-de Albéniz and Talluri 2011, Gallego and Hu 2014). For a thorough overview of the literature, one may refer to Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003) and Talluri and Van Ryzin (2004), and, more recently, Gallego and Topaloglu (2019) and Lobel (2021).

Several authors have studied the performance of dynamic pricing policies in the fluid regime by using a resolving heuristic for the fluid problem. It turns out that such policies improve over static ones. See, for instance, Maglaras and Meissner (2006), Chen and Farias (2013), and Jasin (2014). There are, however, some scenarios in which a dynamic policy performs worse than a corresponding static one; see Cooper (2002), Secomandi (2008), and Jasin and Kumar (2013). In this paper, we show that a static run-out rate policy is not first-order optimal in the large market regime. However, a dynamic run-out rate policy is first- and second-order optimal in the large market regime.

A growing body of literature also analyzes the regret for different classes of dynamic resource allocation problems. A notable contribution in this area is the work of Arlotto and Gurvich (2019), who propose an algorithm for the multi-secretary problem that achieves a constant regret when customer types (valuations) are discrete. Vera and Banerjee (2021), Vera et al. (2021), and Bumpensanti and Wang (2020) extend these concepts to online packing, matching, and network management problems. However, constant regret policies may not hold with a continuum of types, as shown by Bray (2024) and Besbes et al. (2022). Balseiro et al. (2023) provide conditions that generalize the constant and logarithmic regret analysis. Wang and Wang (2022) also study the regret of the network dynamic pricing problem and demonstrate that the dynamic re-optimization heuristic based on the fluid regime policies achieves a constant regret under additional assumptions.

Another related stream of literature concerns prophet inequalities and, in particular, their random-order variants, often called prophet secretary problems (Esfandiari et al. 2017), due to their close connection to the classical secretary problem. Whereas the secretary problem literature typically focuses on additive regret, the prophet inequality literature focuses on establishing bounds on the competitive ratio between the online policy and the offline optimum when selecting $q \geq 1$ out of t sequentially arriving random variables (in discrete time). For an overview of recent advances on prophet inequalities, see Correa et al. (2019). Most of the existing work on prophet inequalities analyzes static threshold policies for finite q (Arnosti and Ma 2023, Chawla et al. 2024), with exceptions such as Correa et al. (2017) and Jiang et al. (2024). To the best of our knowledge, these results are generally derived in terms of q alone and hence only yield asymptotic insights in a fluid regime where q and t are scaled together to infinity. A notable exception is Correa et al. (2021), who seem to be the first to coin the term large market regime. They maintain a fixed q while letting the total number of random variables t grow to infinity. Then, they apply extreme value theory to derive limiting competitive ratios for static threshold policies in this regime.

Our dynamic pricing problem parallels Correa et al. (2021) but differs along two key dimensions. First, we study and characterize the optimal adaptive dynamic pricing policies in continuous time, where the number of arrivals is random. Second, we characterize the asymptotic behavior of the optimal (adaptive) online policy and the offline solution. This joint characterization allows us to provide precise, distribution-dependent results on additive regret (and thus the competitive ratio), which can be of independent interest to the broader literature on prophet inequalities.

Our paper is the first to study single-item dynamic pricing problems across three asymptotic regimes and examine their boundaries. Building on prior results in the small and fluid regimes for the online problem, we add a new perspective by characterizing optimal policies and value functions in the large market regime, as well as offering new asymptotic analysis for the offline problem across all regimes. It is worth noting that den Boer (2023) implicitly investigates certain characteristics of the large market regime. However, their focus is different, examining only the probability of stocking out for a single-unit setting.

3. The Model

We consider a firm selling a single item over a finite selling horizon of length $t > 0$. The initial inventory level of the item is q units, which we assume can take any value in $\mathbb{N} = \{0, 1, 2, ...\}$. Customers arrive to purchase the item according to a Poisson process with a rate $\lambda > 0$ that is known to the firm. We assume that the nth customer to arrive has the idiosyncratic nonnegative valuation X_n for the item. Moreover, we assume that the X_n's are independent and identically distributed (i.i.d.) and denote their common distribution function by F, which we assume to have a finite mean. The firm has knowledge of the customer item valuation distribution F and the fact that the customer item valuations are i.i.d. but does not know their specific values. We denote the lower limit of the support of F by x_L, and the upper limit by x_U, which may be equal to ∞. We also assume, for the sake of simplicity, that F has a positive density f on the entirety of its support. Finally, all random variables and processes are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, P)$.

At each point in time $s \in [0, t]$ the firm sets a price for the item which we denote by $p_s \in \mathbb{R}_+$. An arriving customer is willing to purchase the item if their surplus at the time of their arrival is non-negative. That is, customer n is willing to purchase the item if X_n is greater than or equal to the price when they arrive. Thus, a customer arriving and encountering a price p has a probability $1 - F(p)$ of making a purchase. We also assume that for each customer, the marginal value of more than 1 unit of the item is negligible. Therefore, each customer will purchase at most one unit.

The firm may dynamically adjust the price of the item in response to the previous history of customer purchases and its current inventory level. Due to the Markovian nature of the system we have described, and for the sake of simplicity, we restrict attention to pricing policies which at each point in time are only dependent on the current level of inventory. Such policies can be shown to be optimal amongst the much larger class of predictable pricing policies (Brémaud 1981) but at the expense of additional machinery in our proofs. In the present paper, an $\mathbb{R}_+ \cup p_\infty$-valued pricing policy[1] $p = \{p_s, 0 \le s \le t\}$ on $(\Omega, \mathcal{F}, P)$ is said to be an admissible pricing policy if the following 2 conditions are satisfied.

I. For each $0 \le s \le t$, $p_s = p(\lambda; Q_s, t-s)$ where Q_s is the inventory level at time s, see (2) below.

II. The family of functions $\{p(\lambda; q, \cdot), q \in \mathbb{N}\}$ are measurable, with $p(\lambda; 0, \cdot) = p_\infty$.

We denote the set of admissible pricing policies by $\mathcal{V}$. Note that according to our definition, at each point in time, an admissible pricing policy depends not only on the remaining inventory level and length of the selling horizon but also on the customer arrival rate.

Consistent with the revenue management literature (Gallego and Van Ryzin 1994), we assume that all items have zero marginal cost. Moreover, we assume the firm is risk-neutral and interested in maximizing its expected revenue. Given an admissible pricing policy $p \in \mathcal{V}$ and applying point process theory (Brémaud 1981), it is straightforward to show that for an initial inventory level $q \in \mathbb{N}$, the expected revenue of the firm over the selling horizon $[0, t]$ is equal to the value function

$$J_p(\lambda; q, t) = E\left[\lambda \int_0^t p(\lambda; Q_s, t-s)(1 - F(p(\lambda; Q_s, t-s)))ds | Q_0 = q\right], \tag{1}$$

where

$$Q_s = Q_0 - N\left(\lambda \int_0^s (1 - F(p(\lambda; Q_u, t-u)))du\right) \text{ for } 0 \le s \le t, \tag{2}$$

with N being a standard rate 1 Poisson process. Note in particular that Q_s is the firm's inventory level at time s under the pricing policy p.

Now taking the supremum of (1) over all admissible pricing policies $p \in \mathcal{V}$, we obtain the optimal expected revenue of the firm given q units of initial inventory, which we denote by the optimal value function

$$J^\star(\lambda; q, t) = \sup_{p \in \mathcal{V}} J_p(\lambda; q, t). \tag{3}$$

The optimization problem (3) may be viewed as a stochastic control problem. Specifically, applying C2 and T3 of VII.2 of Brémaud (1981) and setting $\mathbb{N}_+ = \{1, 2, ...\}$, we arrive at the following HJB equations characterizing the family $\{J^\star(\lambda; q, \cdot), q \in \mathbb{N}\}$ of optimal value functions.

Theorem 1. *The family of functions $\{J^\star(\lambda; q, \cdot), q \in \mathbb{N}\}$ are the unique solution to the system of equations*

$$\begin{aligned} \frac{\partial J^\star(\lambda; q, t)}{\partial t} &= \lambda \sup_{p \in \mathbb{R}_+} \{(1 - F(p))(p - [J^\star(\lambda; q, t) - J^\star(\lambda; q-1, t)])\}, \quad t \ge 0, \ q \in \mathbb{N}_+, \\ J^\star(\lambda; q, 0) &= 0, \quad q \in \mathbb{N}_+, \\ J^\star(\lambda; 0, t) &= 0, \quad t \ge 0. \end{aligned} \tag{4}$$

[1] We assume the existence of a null price p_∞ such that $1 - F(p_\infty) = 0$.

Moreover, there exists an optimal solution $p^\star$ *to* (3) *such that for* $q \in \mathbb{N}_+$*,*

$$p^\star(\lambda; q, s) = \underset{p \in \mathbb{R}_+}{\arg\max} \{(1 - F(p))(p - [J^\star(\lambda; q, s) - J^\star(\lambda; q-1, s)])\}, \ s \geq 0. \tag{5}$$

The assumption that F has a finite mean implies that the supremum in (4) is finite as well. The existence of the arg max in (5) is then a consequence of the additional assumption that F is continuous. This in turn implies that there exists at least one optimal policy $p^\star$ in (3). That is, the supremum in (3) may be converted into an arg max.

We conclude this section by noting that since the item valuation distribution F has a positive density on its support, its inverse function $F^{-1}(\pi) = \{x \in [x_L, x_U] : F(x) = \pi\}$ is uniquely defined on $[0, 1]$. This implies as is common in the literature (Gallego and Van Ryzin 1994, Maglaras and Meissner 2006) that we may equivalently consider the decision of the firm at each point in time as that of choosing optimal purchasing probabilities instead of prices. In particular, given a desired purchasing probability $\pi_s \in [0, 1]$ at time $s \in [0, t]$, the firm may set a corresponding price $p_s = F^{-1}(1 - \pi_s)$. Also note that since the inverse of F is uniquely defined, it follows that for each optimal pricing policy $p^\star$ there uniquely corresponds an optimal purchasing probability policy $\pi^\star = 1 - F(p^\star)$. Similarly, to each optimal purchasing probability policy $\pi^\star$ there uniquely corresponds an optimal pricing policy $p^\star = F^{-1}(1 - \pi^\star)$.

4. Regime Definitions and the Inventory-to-Market Size Ratio

Given a customer arrival rate $\lambda > 0$ and selling horizon of length $t > 0$, the quantity λt is the expected number of customer arrivals over the length of the selling horizon. This quantity may be viewed as a measure of the size of the market in which the firm is selling. We therefore refer to λt as the market size. It turns out by a change-of-variables in (4) that the optimal value function satisfies $J^\star(\lambda; q, t) = J^\star(1; q, \lambda t) = J^\star(\lambda t; q, 1)$. Thus, the optimal value function is dependent only on the initial inventory level and market size (as opposed to the customer arrival rate, initial inventory level, and length of the selling horizon). We may therefore represent the optimal value function by a two-parameter function $J^\star(q, \lambda t)$ from which the three-parameter version may be recovered using the relationships above. For the sake of simplicity, we mostly refer to the two-parameter version.

In this paper, we study the single-item dynamic pricing problem (3) in the small market, fluid, and large market regimes. Each regime is loosely characterized by the ratio of the the initial inventory level of the firm to the market size, see Figure 1. The small market regime is technically defined by letting the initial inventory level q tend to ∞ while holding the market size λt fixed. The market size is, therefore, small relative to the initial inventory level. The fluid regime (Gallego and Van Ryzin 1994) is well-established in the literature and corresponds to the initial inventory level q and market size λt being the same order of magnitude. The large market regime is defined by holding the initial inventory level q fixed and letting the market size λt tend to ∞. The market size is, therefore, large relative to the initial inventory level.

The large market regime contains within it two smaller regimes, see Figure 2. These regimes are obtained by holding either λ or t fixed and letting the other parameter tend to ∞, with q always being held fixed. The case of λ and q being held fixed and letting $t \to \infty$ corresponds to the firm having a relatively long time in which to sell a limited quantity of items. On the other hand, letting $\lambda \to \infty$ with t and q being held fixed corresponds to a customer arrival rate that is high, and the firm has a fixed amount of time in which to sell a limited number of items.

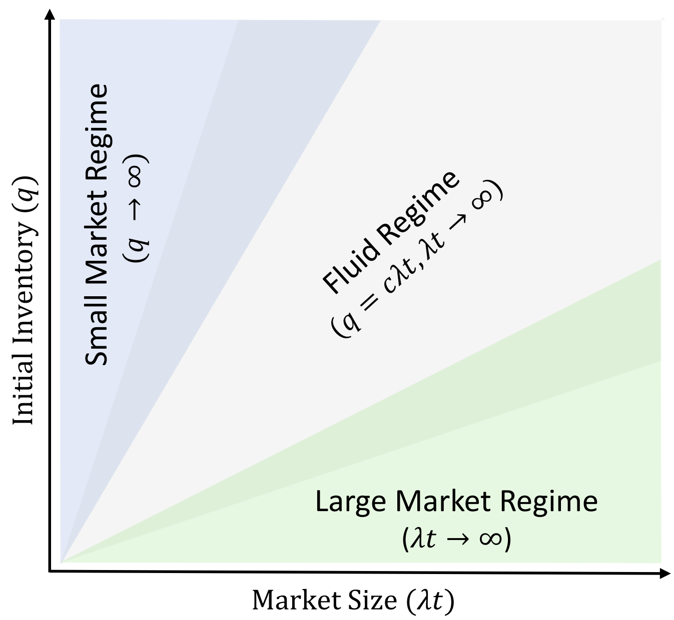

Figure 1 Relative positioning of different scaling regimes.

5. Exponentially Distributed Item Valuations

We now consider the single-item dynamic pricing problem in the specific case where customers value the item being sold according to an exponential distribution. The advantage of studying this case first is that the optimal value function is known in closed form, allowing us to obtain sharper results. For the sake of simplicity, we also assume that the customer item valuation distribution has a mean of 1. In then follows by Gallego and Van Ryzin (1994) that for each initial inventory level $q \in \mathbb{N}$ and market size $\lambda t \geq 0$,

$$J^\star(q,\lambda t) = \ln\left(\sum_{k=0}^{q} \frac{(\lambda t/e)^k}{k!}\right). \tag{6}$$

Using this explicit formula, we may derive and compare approximations to the optimal value function in the small market, fluid, and large market regimes.

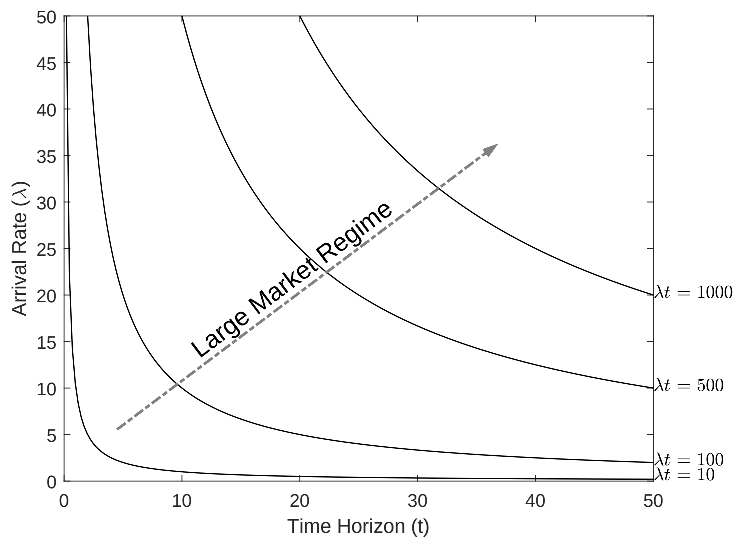

Figure 2 **Conceptual representation of the scaling in the large market regime.**

5.1. Optimal Value Function Approximations

The first step in deriving our approximations is to reexpress the optimal value function (6) as

$$J^{\star}(q,\lambda t) = \ln\left(e^{\lambda t/e}\frac{\Gamma(q+1,\lambda t/e)}{\Gamma(q+1)}\right), \tag{7}$$

where $\Gamma(\cdot)$ and $\Gamma(\cdot,\cdot)$ denote the gamma and upper incomplete gamma functions (Olver et al. 2010), respectively. We also note that both $\Gamma(q+1,\lambda t/e)$ and $\Gamma(q+1)$ are well-defined for all real-valued $q > 0$. We therefore may use the righthand side of (7) to extend $J^{\star}(q,\lambda t)$ to a function defined for $q > 0$ and $\lambda t > 0$. This simplifies the presentation of our results and also their proofs.

Our 3 approximations to $J^{\star}(q,\lambda t)$ are given in Table 1 and derived as follows.

Small Market$(q,\lambda t)$	Fluid$(q,\lambda t)$	Large Market$(q,\lambda t)$
$\dfrac{\lambda t}{e}$	$q\ln\left(\dfrac{\lambda t}{q}\right)$	$q\ln\left(\left(\dfrac{(q/e)^q}{\Gamma(q+1)}\right)^{1/q}\cdot\dfrac{\lambda t}{q}\right)$

Table 1 **Approximations to the optimal value function for each of the 3 asymptotic regimes in the case of customer item valuations following an exponential distribution with a mean of 1.**

- Small market approximation: Recall that the small market regime is defined by letting the initial inventory level q tend to ∞ while holding the market size λt fixed. Moreover (Olver et al. 2010),

$$\frac{\Gamma(q+1,\lambda t/e)}{\Gamma(q+1)} \to 1 \text{ as } q\to\infty. \tag{8}$$

Using (7) and the continuity of ln, one obtains the small market approximation in Table 1.

- Fluid approximation: The fluid regime may be defined by setting $\lambda t = (1/c)q$ for $c > 0$ and letting $q \to \infty$. Moreover, for $\gamma > 1$ one has (Nemes 2016) the asymptotic

$$\Gamma(q, \gamma q) = \frac{(\gamma q)^q e^{-\gamma q}}{q(\gamma - 1)} \left(1 + O\left(\frac{1}{q}\right)\right) \text{ as } q \to \infty.$$

The fluid approximation in Table 1 for $J^\star(q, \lambda t)$ then follows from (7) when $c < 1/e$. The approximation naturally extends to $c \geq 1/e$ too.

- Large market approximation: The large market regime is defined by holding the initial inventory level q fixed and letting the market size λt tend to ∞. Recalling (Olver et al. 2010) the asymptotic

$$e^{\lambda t/e} \frac{\Gamma(q+1, x)}{(\lambda t/e)^q} \to 1 \text{ as } \lambda t \to \infty, \tag{9}$$

using (7) and the continuity of ln, one obtains the large market approximation in Table 1.

Finally, we note that the small market and fluid approximations in Table 1 also follow from the results of Gallego and Van Ryzin (1994).

5.2. Regime Boundaries

We now ask the following question. Given a fixed initial inventory level $q > 0$ and market size $\lambda t > 0$, which of the 3 approximations in Table 1 provides the best estimate of the optimal value function $J^\star(q, \lambda t)$? We define best to be the approximation with the smallest distance in absolute value to the optimal value function. To begin to answer this question, for each $q > 0$ and $\lambda t > 0$, let Approx$^\star(q, \lambda t)$ =S, F or L, depending on whether the small market, fluid or large market approximation is optimal given q and λt. We then have the following result.

Proposition 1. *For each $q > 0$, there exists $0 < \sigma(q) < \tau(q)$ such that*

$$\text{Approx}^\star(q, \lambda t) = \begin{cases} \text{S} & \text{for } 0 < \lambda t < \sigma(q), \\ \text{F} & \text{for } \sigma(q) < \lambda t < \tau(q), \\ \text{L} & \text{for } \lambda t > \tau(q). \end{cases} \tag{10}$$

Moreover, $\sigma(q) < eq$ for $q > 0$, and $\tau(q) > eq$ for $q > 8/\pi \approx 2.55$.

Proposition 1 implies there exists 3 distinct regions of the $(q, \lambda t)$ parameter space corresponding to where each of the 3 approximations in Table 1 performs best. The function σ is the boundary between the small market and fluid regions, and τ is the boundary between the fluid and large market regions. There is no boundary between the small market and large market regions because the fluid region is between them. We next have a monotonicity result for the boundaries σ and τ.

Proposition 2. *$\sigma(q)$ is continuous and strictly increasing for $q>0$ with $\sigma(q)\to 0$ as $q\to 0$ and $\sigma(q)\to\infty$ as $q\to\infty$. Also, $\tau(q)$ is continuous for $q>0$ and strictly increasing for q sufficiently large with $\tau(q)\to\infty$ as $q\to\infty$.*

Proposition 2 implies we may uniquely define the inverse of σ, which we denote by $\sigma^{-1}(\lambda t)$. For $\lambda t>0$ sufficiently large, we may also uniquely define the inverse of τ, which we denote by $\tau^{-1}(\lambda t)$. The inverses σ^{-1} and τ^{-1} can also be used to represent the boundaries between the 3 regimes. Specifically, the following is a consequence of combining Propositions 1 and 2.

Proposition 3. *For $\lambda t>0$ sufficiently large,*

$$\text{Approx}^{\star}(q,\lambda t)=\begin{cases}\text{L} & \text{for } 0<q<\sigma^{-1}(\lambda t),\\ \text{F} & \text{for } \sigma^{-1}(\lambda t)<q<\tau^{-1}(\lambda t),\\ \text{S} & \text{for } q>\tau^{-1}(\lambda t).\end{cases} \tag{11}$$

It is unfortunately not possible to provide exact formulas for σ^{-1} and τ^{-1} (or σ and τ for that matter). However, letting $\lambda t\to\infty$ we can provide approximations to $\sigma^{-1}(\lambda t)$ and $\tau^{-1}(\lambda t)$ which are accurate up to $o((\lambda t)^{1/2})$ and $o((\lambda t)^{3/4})$, respectively. First consider σ^{-1} and recall (Olver et al. 2010) the definition of the error function

$$\text{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx \text{ for } z\in\mathbb{R}, \tag{12}$$

and let $\text{erfc}=1-\text{erf}$ denote the complementary error function. We then have the following.

Proposition 4. *The function $\sigma^{-1}(\lambda t)$ has the asymptotic expansion*

$$\sigma^{-1}(\lambda t)=\frac{\lambda t}{e}+\gamma^{\star}\sqrt{\frac{2}{e}}(\lambda t)^{1/2}+o((\lambda t)^{1/2}) \text{ as } \lambda t\to\infty, \tag{13}$$

where $\gamma^{\star}>0$ is the unique solution to

$$\frac{1}{2}\text{erfc}(-\gamma^{\star})=\exp\left(-(\gamma^{\star})^2/2\right). \tag{14}$$

We also have a similar result for the boundary τ^{-1}.

Proposition 5. *The function $\tau^{-1}(\lambda t)$ has the asymptotic expansion*

$$\tau^{-1}(\lambda t)=\frac{\lambda t}{e}-\frac{e^{-3/4}}{(2\pi)^{1/4}}\cdot(\lambda t)^{3/4}+o((\lambda t)^{3/4}) \text{ as } \lambda t\to\infty. \tag{15}$$

Figure 3 contains 4 plots of the approximations to $\sigma^{-1}(\lambda t)$ and $\tau^{-1}(\lambda t)$ derived from Propositions 4 and 5, neglecting the $o((\lambda t)^{1/2})$ and $o((\lambda t)^{3/4})$ remainder terms, respectively. Each plot captures a progressively larger portion of the $(\lambda t, q)$ parameter space. We also plot the true regime boundaries determined numerically using the expression (7) for $J^{\star}(q,\lambda t)$. The approximations of σ^{-1} and τ^{-1} are very close to their true values. Moreover, note that the small and large market regimes occupy most of the parameter space. This is discussed more in the next subsection.

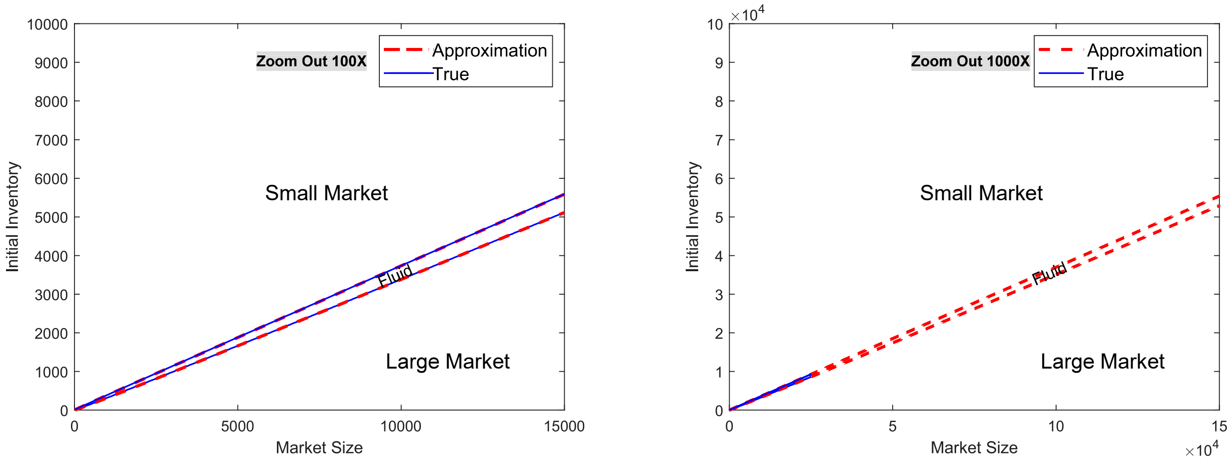

Figure 3 Boundaries between the 3 regimes in the case of item valuations with an exponential distribution with a mean of 1.

5.3. Initial Inventory-to-Market Size Ratio

Propositions 4 and 5 imply that the boundaries between the 3 regimes can be approximated by the ratio of the initial inventory level to the market size. This is also evident by viewing Figure 3. Thus, let

$$\text{IMR} = \frac{\text{initial inventory}}{\text{market size}}$$

denote the initial inventory-to-market size ratio. Intuitively, large IMR values correspond to the small market regime, while smaller values correspond to the large market regime. Values in between correspond to the fluid regime.

Table 2 uses Propositions 4 and 5 to characterize the boundaries between the 3 regimes in terms of the IMR up to $o((\lambda t)^{-1/2})$ and $o((\lambda t)^{-1/4})$, respectively. The small market regime corresponds to IMR values approximately greater than $1/e$, and the large market regime corresponds to IMR

Small Market$(q,\lambda t)$	Fluid$(q,\lambda t)$	Large Market$(q,\lambda t)$
$\text{IMR} > \frac{1}{e} + \gamma^\star \sqrt{\frac{2}{e\cdot\lambda t}}$	$\frac{1}{e} + \gamma^\star \sqrt{\frac{2}{e\cdot\lambda t}} > \text{IMR} > \frac{1}{e} - \frac{e^{-3/4}}{(2\pi\cdot\lambda t)^{1/4}}$	$\frac{1}{e} - \frac{e^{-3/4}}{(2\pi\cdot\lambda t)^{1/4}} > \text{IMR}$

Table 2 Approximate IMR boundaries between the 3 regimes.

values approximately less than $1/e$. The fluid regime occupies a small interval of IMR values in a neighborhood of $1/e$. The width of this interval shrinks to zero as the market size λt grows large. Thus, for most parameter combinations either the small market or the large market approximation is the best to use.

6. The Gumbel Domain of Attraction

We now consider the case of general customer item valuation distributions. For the remainder of the paper, for the small market and fluid regimes we make little to no additional assumptions on F beyond those in Section 3. For the large market regime, we assume that the item valuation distribution lies in the Gumbel domain of attraction of extreme value theory.

The Gumbel domain of attraction is the largest of the three extreme value domains of attraction (Embrechts et al. 2013). It includes many commonly used item valuation distributions, such as the exponential, normal, and lognormal distributions. There exist item valuation distributions in the Gumbel domain of attraction with both bounded and infinite support. That is, both $x_U < \infty$ and $x_U = \infty$ are possible. In the case of item valuation distributions with infinite support, the Gumbel domain of attraction incorporates a variety of tail behavior ranging from light (e.g., the exponential and normal distributions) to moderately heavy (e.g., the lognormal distributions). However, all distributions in the Gumbel domain of attraction have finite moments of all orders.

We further restrict attention to a class of item valuation distributions within the Gumbel domain of attraction referred to as von-Mises functions. It turns out that every distribution in the Gumbel domain of attraction is tail equivalent to a von-Mises function (Embrechts et al. 2013), so this assumption is not very restrictive.

Definition 1 (von-Mises function). An item valuation distribution F is said to be a von-Mises function if there exists an $x_0 < x_U$ and $c > 0$ such that for $x_0 < x < x_U$,

$$1 - F(x) = c \cdot \exp\left(-\int_{x_0}^{x} \frac{1}{u(t)} dt\right), \tag{16}$$

where $u(x) > 0$ for $x_0 < x < x_U$ and u is absolutely continuous on (x_0, x_U) with

$$u^{'}(x) \to 0 \text{ as } x \to x_U. \tag{17}$$

The function u in (16) is sometimes (Resnick 2013) referred to as the auxiliary function associated with F. Differentiating both sides of (16), it is straightforward to see that $u(x) = (1 - F(x))/f(x)$ for $x \in (x_0, x_U)$, where f is the density of F. Thus, u is equal to the reciprocal hazard rate function or Mills ratio of F.

If F is a von-Mises function, then we may define the norming functions b and a by setting

$$b(t) = F^{-1}(1 - 1/t) \text{ and } a(t) = u(b(t)) \text{ for } t > 1/(1 - F(x_0)). \quad (18)$$

Both functions b and a play a prominent role in the remainder of the paper. We also note (Resnick 2013) that $a(t) \in o(b(t))$ and that a is a slowly varying function.

7. The Offline Problem

In the model described in Section 3, the firm has no knowledge of the customer arrival process beyond its arrival rate λ. Nor does the firm have any information on specific customer item valuations beyond their common distribution function F and the fact that they are i.i.d. This is commonly referred to in the literature as the online setting. In this section, we study the offline version of the model in Section 3. One of the benefits of studying the offline problem is that it provides an upper bound on the optimal value function of the online problem. This upper bound is then used in Section 8.4 as a benchmark against which to compute the regret of the optimal online policy.

In the offline problem, the firm has full information on the number of customers who will arrive over the selling horizon as well as each of their item valuations. That is, the sequence of random variables $\{X_n, n = 1, 2, ..., N_t\}$ is known to the firm. The objective of the firm is to find a pricing policy $p = \{p_s, 0 \leq s \leq t\}$ that is measurable with respect to the σ-algebra generated by $\{X_n, n = 1, 2, ..., N_t\}$ and maximizes the firm's expected revenue.

Given $q \in \mathbb{N}_+$ units of initial inventory and a selling horizon of length $t > 0$, the revenue of the firm (in either the online or offline problem) is upper bounded by the $\min(q, N_t)$ highest item valuations from the sequence $\{X_1, X_2, ..., X_{N_t}\}$. Moreover, in the offline problem this upper bound may pointwise be achieved. Specifically, since the firm has full information on $\{X_n, n = 1, 2, ..., N_t\}$, it may choose to sell its items only to the customers with the top $\min(q, N_t)$ item valuations. Moreover, for each of these customers, the price can be set exactly equal to their item valuation.

In the language of order statistics, the above may be stated as follows. For each $n \in \mathbb{N}_+$ and $1 \leq m \leq n$, let $X_{m,n}$ denote the mth largest value in the sequence $\{X_1, X_2, ..., X_n\}$. Then, the optimal offline value function of the firm is given by

$$J^\star_{\text{OFF}}(\lambda; q, t) = E\left[\sum_{m=1}^{q \wedge N_t} X_{m,N_t}\right], \quad (19)$$

where we recall that N_t is a Poisson random variable with a mean of λt. It follows, as in the online case, that the optimal offline value function is dependent only on the initial inventory level q and market size λt. Thus, we may represent it by a two-parameter function $J^\star_{\mathrm{OFF}}(q, \lambda t)$.

In Proposition A3 of the appendix, an analytical expression is provided for the optimal offline value function. Specifically,

$$J^\star_{\mathrm{OFF}}(q, \lambda t) = \frac{1}{\Gamma(q)} \int_0^{\lambda t} F^{-1}\left(1 - \frac{v}{\lambda t}\right) \cdot \Gamma(q, v) \cdot dv, \tag{20}$$

for each $q \in \mathbb{N}_+$ and $\lambda t > 0$. The expression on the righthand side above holds not just for item valuation distributions in the Gumbel domain attraction but for any item valuation distribution with a finite mean. In this case, F^{-1} denotes the left-continuous inverse of F. Note also since $\Gamma(q, v)$ and $\Gamma(q)$ are well-defined for $q > 0$, we may use (20) to extend $J^\star_{\mathrm{OFF}}(q, \lambda t)$ to all $q > 0$ too.

The integral on the righthand side of (20) is complicated and cannot be simplified. It is therefore useful to construct approximations to it. For each $q > 0$, let $Q(q, \cdot) = \Gamma(q, \cdot)/\Gamma(q)$ denote the regularized upper incomplete gamma function and set

$$C_q = \int_0^\infty \ln(v) Q(q, v) dv. \tag{21}$$

In the case that $q \in \mathbb{N}_+$, it may be shown (see Lemma A3 in the appendix) that

$$C_q = \sum_{m=1}^{q} (H_{m-1} - \gamma) \; = \; q \cdot (H_q - \gamma - 1), \tag{22}$$

where H_{m-1} is the $(m-1)$st harmonic number with $H_0 = 0$, and $\gamma \approx 0.57722$ is the Euler-Mascheroni constant. Table 3 provides 3 approximations to the optimal offline value function corresponding to the small market, fluid, and large market regimes. The small market and fluid approximations hold for arbitrary item valuation distributions with a finite mean, excluding even the regularity assumptions of Section 3. The large market approximation is specific to item valuation distributions in the Gumbel domain of attraction satisfying the von-Mises conditions (16)-(17). The loose derivation of each approximation is as follows.

Small Market$(q, \lambda t)$	Fluid$(q, \lambda t)$	Large Market$(q, \lambda t)$
$\lambda t E[X]$	$\lambda t \int_0^{q/\lambda t} F^{-1}(1 - s) ds$	$q F^{-1}\left(1 - \frac{e^{C_q/q}}{\lambda t}\right)$

Table 3 Approximations to the optimal offline value function in each of the 3 asymptotic regimes.

- Small market approximation: The small market approximation in Table 3 follows from the fact (Olver et al. 2010) that $\Gamma(q,v)/\Gamma(q) \to 1$ as $q \to \infty$ for each $v > 0$, together with (20) and the dominated convergence theorem.
- Fluid approximation: For $0 < c < 1$, the fluid approximation in Table 3 follows from Proposition A5 in the appendix.
- Large market approximation: The large market approximation in Table 3 follows from Proposition A6 in the appendix.

7.1. Examples

We now present two examples of applying the approximations above.

Example 1 (Finite Right Endpoint). Consider the case where customers value the item being sold according to a distribution in the Gumbel domain of attraction with a finite right endpoint. Specifically, suppose that

$$1 - F(x) = \beta \exp\left(-\frac{\alpha}{x_U - x}\right) \text{ for } 0 \leq x < x_U, \tag{23}$$

where $\alpha > 0$ and $\beta = \exp(\alpha/x_U)$. In this case,

$$F^{-1}(p) = x_U - \frac{\alpha}{\ln(\beta/(1-p))} \quad \text{for} \quad 0 \leq p < 1. \tag{24}$$

It then follows from (20) that

$$J^\star_{\text{OFF}}(q, \lambda t) = \frac{1}{\Gamma(q)} \int_0^{\lambda t} \left(x_U - \frac{\alpha}{\ln(\beta\lambda t/v)}\right) \cdot \Gamma(q,v) \cdot dv.$$

Moreover, letting $\text{li}(\cdot)$ denote the logarithmic integral function (Olver et al. 2010) and substituting into Table 3, the 3 regime approximations to $J^\star_{\text{OFF}}(q, \lambda t)$ are given in Table 4.

Small Market$(q, \lambda t)$	Fluid$(q, \lambda t)$	Large Market$(q, \lambda t)$
$\lambda t\,(x_U - (-\alpha\beta \cdot \text{li}(1/\beta)))$	$q\left(x_U - \frac{\lambda t}{q}\left(-\alpha\beta \cdot \text{li}\left(\frac{q}{\lambda t \beta}\right)\right)\right)$	$q\left(x_U - \alpha \ln^{-1}\left(\frac{\beta\lambda t}{e^{C_q/q}}\right)\right)$

Table 4 Approximations to the optimal offline value function assuming customer item valuation distributions in the Gumbel domain of attraction following the distribution (23).

Example 2 (The Weibull Distribution). Consider next the case of items valued according to a Weibull distribution with scale parameter $\mu > 0$ and shape parameter $k > 0$. The CDF of the item valuation distribution in this case is given by $F(x) = 1 - \exp(-(x/\mu)^k)$ for $x \geq 0$. Moreover,

$$F^{-1}(p) = \mu \ln^{1/k}\left(\frac{1}{1-p}\right) \quad \text{for} \quad 0 \leq p < 1. \tag{25}$$

It then follows from (20) that

$$J^{\star}_{\text{OFF}}(q,\lambda t) = \frac{\mu}{\Gamma(q)} \int_0^{\lambda t} \ln^{1/k}\left(\frac{\lambda t}{v}\right) \cdot \Gamma(q,v) \cdot dv.$$

The 3 regime approximations to $J^{\star}_{\text{OFF}}(q,\lambda t)$ are then given in Table 5. Note that the exponential distribution with a mean of μ is a special case of the Weibull distribution where the shape parameter $k=1$. Setting $k=1$ in Table 5, we then obtain the corresponding optimal offline value function approximations for this case.

Small Market$(q,\lambda t)$	Fluid$(q,\lambda t)$	Large Market$(q,\lambda t)$
$\lambda t \mu \Gamma\left(1+\frac{1}{k}\right)$	$\lambda t \mu \Gamma\left(1+\frac{1}{k}, \ln\left(\frac{\lambda t}{q}\right)\right)$	$\mu q \ln^{1/k}\left(\frac{\lambda t}{e^{C_q/q}}\right)$

Table 5 Approximations to the optimal offline value function assuming a Weibull customer item valuation distribution with scale parameter μ and shape parameter k.

8. The Online Problem

We now consider the online setting described in Section 3. In Section 8.1, we study static pricing policies in the large market regime. Section 8.2 contains our approximations for the optimal value function, pricing policy and purchasing probability policy in the general online problem optimizing over the set of all admissible pricing policies. In Section 8.3, we provide examples using our results from Section 8.2. In Section 8.4, we provide approximations to the regret of the optimal online value function relative to the optimal offline value function. Finally, in Section 8.5, we discuss the notion of asymptotic optimality in the large market regime.

8.1. Static Pricing Policies

In this section, we restrict our attention to the class of static pricing policies in the large market regime. Such policies set a single price at the outset of the selling horizon without any knowledge of the number of customers who will arrive or their specific item valuations. The benefit of studying the case of static pricing policies first is that it provides us with a lower bound against which to compare the optimal online value function. Since we exclusively consider the large market regime, we assume in this section that F lies in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17).

Suppose now that the firm sets a single price $p \in \mathbb{R}_+$ for the entirety of the selling horizon. Then, given $q \in \mathbb{N}_+$ units of initial inventory and a selling horizon of length $t \geq 0$, the realized revenue of the firm may be written in terms of the order statistics of the customer item valuations as

$$p \sum_{m=1}^{q \wedge N_t} 1\{X_{m,N_t} > p\}. \tag{26}$$

Taking the expectation of the above and recalling that N_t is a Poisson random variable with a mean of λt, the static pricing value function of the firm is then

$$J^p_{\text{STAT}}(q, \lambda t) = pE\left[\sum_{m=1}^{q \wedge N_t} 1\{X_{m,N_t} > p\}\right]. \tag{27}$$

The optimal static pricing value function is given by

$$J^\star_{\text{STAT}}(q, \lambda t) = \sup_{p \in \mathbb{R}_+} J^p_{\text{STAT}}(q, \lambda t), \tag{28}$$

and the optimal static price is

$$p^\star_{\text{STAT}}(q, \lambda t) = \arg\max_{p \in \mathbb{R}_+} J^p_{\text{STAT}}(q, \lambda t). \tag{29}$$

One may interpret $p^\star_{\text{STAT}}(q, \lambda t)$ as the optimal static price given an initial inventory level q and market size λt.

The first-order asymptotics of the optimal static pricing value function and the optimal static price in the large market regime are as follows. Recall from (18) the definition of the norming function b.

Proposition 6. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17), then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t \to \infty} \frac{J^\star_{\text{STAT}}(q, \lambda t)}{b(\lambda t)} = q \text{ and } \lim_{\lambda t \to \infty} \frac{p^\star_{\text{STAT}}(q, \lambda t)}{b(\lambda t)} = 1. \tag{30}$$

Now recall (Resnick 2013) that b is a slowly varying function. It then follows by Proposition A6 in the appendix that

$$\lim_{\lambda t \to \infty} \frac{J^\star_{\text{STAT}}(q, \lambda t)}{J^\star_{\text{OFF}}(q, \lambda t)} = 1. \tag{31}$$

It is therefore the case that on a first-order basis, the optimal expected revenues of the static pricing problem and offline problem are asymptotically the same in the large market regime. Their second-order terms, however, differ as we show next.

For each static price $p \in \mathbb{R}_+$, initial inventory level $q \in \mathbb{N}_+$ and market size $\lambda t > 0$, the following inequality easily holds,

$$J^p_{\text{STAT}}(q, \lambda t) \leq J^\star_{\text{OFF}}(q, \lambda t). \tag{32}$$

It then follows that for each static price p we may define its regret relative to the optimal offline pricing policy by setting

$$\text{Regret}^p_{\text{STAT}}(q,\lambda t) = J^\star_{\text{OFF}}(q,\lambda t) - J^p_{\text{STAT}}(q,\lambda t) \;\geq\; 0. \tag{33}$$

The minimum static pricing regret is obtained by maximizing the static pricing value function in (33) overall prices p. We denote the resulting quantity by

$$\text{Regret}^\star_{\text{STAT}}(q,\lambda t) = J^\star_{\text{OFF}}(q,\lambda t) - J^\star_{\text{STAT}}(q,\lambda t).$$

The following is our main result on the minimum static pricing regret. It is implied by Propositions A6 and A7 in the appendix together with (A83). Recall from (18) the definition of the norming function a.

Proposition 7. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions* (16)-(17)*, then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t\to\infty} \frac{\text{Regret}^\star_{\text{STAT}}(q,\lambda t)}{a(\lambda t)} = \infty. \tag{34}$$

This result implies that on a scale of $a(\lambda t)$, the optimal expected revenue of the offline problem is asymptotically infinitely larger than that of the static pricing problem in the large market regime. Recall also that the function a may be interpreted as the reciprocal of the hazard rate function of the customer item valuation distribution F. Thus, if $x_U = \infty$, then a may diverge to ∞ in which case by the above the static pricing regret itself will diverge too.

The minimal regret in the large market regime can be significantly reduced using dynamic pricing policies. See, for instance, Section 8.5 where the family of dynamic run-out rate policies are shown to be asymptotically optimal. It turns out however that static run-out rate policies do not perform as well. For each initial inventory level $q \in \mathbb{N}_+$ and market size $\lambda t > 0$, denote by $J^{\text{RUN}}_{\text{STAT}}(q,\lambda t)$ the value function associated with the static run-out rate price $F^{-1}(1 - q/\lambda t)$. Next, recall from Section 7 the definition $Q(\cdot,\cdot) = \Gamma(\cdot,\cdot)/\Gamma(\cdot)$ of the regularized upper incomplete gamma function (Olver et al. 2010). We then have the following.

Proposition 8. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions* (16)-(17)*, then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t\to\infty} \frac{J^{\text{RUN}}_{\text{STAT}}(q,\lambda t)}{b(\lambda t)} = q\left(1 - \frac{(q/e)^q}{q!}\right). \tag{35}$$

Proposition 8 follows by Theorem 3 of Gallego and Van Ryzin (1994) together with the fact that the norming function b is slowly varying. It is straightforward to see that the righthand side of (35) is strictly less than q. It therefore follows by Proposition 6 that in the large market regime, the value function associated with a static run-out rate policy is not first-order optimal. This result is contrary to the case in the fluid regime where the value function associated with a static run-out rate policy does achieve first-order optimality (Gallego and Van Ryzin 1994). We also note that in the large market regime, a static run-out rate policy asymptotically results in a higher price than the optimal static price. In particular, by Proposition A7 in the Appendix, for each $q \in \mathbb{N}_+$,

$$\frac{F^{-1}(1-q/\lambda t) - p^\star_{\text{STAT}}(q,\lambda t)}{a(\lambda t)} \to \infty \quad \text{as } \lambda t \to \infty. \tag{36}$$

8.2. Dynamic Pricing Policies

We now consider the entire class of admissible pricing policies in the online setting of Section 3. For our fluid regime results in this section, in addition to our assumptions on the customer item valuation distribution in Section 3, we make the additional assumption that the function $r(p) = p(1-F(p))$ is unimodal on $\mathbb{R}_+$. For our large market regime results, we assume that F lies in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17).

Now note that since for each $q \in \mathbb{N}_+$ and $\lambda t > 0$ the optimal value function satisfies $J^\star(\lambda; q, t) = J^\star(1; q, \lambda t) = J^\star(\lambda t; q, 1)$, it follows from (5) that we also have $p^\star(\lambda; q, t) = p^\star(1; q, \lambda t) = p^\star(\lambda t; q, 1)$ and $\pi^\star(\lambda; q, t) = \pi^\star(1; q, \lambda t) = \pi^\star(\lambda t; q, 1)$. In other words, at each point in time, the optimal price and optimal purchasing probability only depend on the current inventory level and remaining market size. We therefore may represent these quantities by the two-parameter functions $p^\star(q, \lambda t)$ and $\pi^\star(q, \lambda t)$, respectively.

Our main results in this section provide approximations to $J^\star(q, \lambda t)$, $p^\star(q, \lambda t)$ and $\pi^\star(q, \lambda t)$ in the small market, fluid and large market regimes. Our approximations to the optimal value function in each regime are given in Table 6. Their derivation along with those of our approximations to $p^\star(q, \lambda t)$ and $\pi^\star(q, \lambda t)$ are as follows.

Small Market$(q, \lambda t)$	Fluid$(q, \lambda t)$	Large Market$(q, \lambda t)$
$\lambda t p^\star(1-F(p^\star))$	$qF^{-1}\left(1-\frac{q}{\lambda t}\right)$	$qF^{-1}\left(1-\left(\frac{q!}{(q/e)^q}\right)^{1/q}\frac{q}{\lambda t}\right)$

Table 6 Approximations to the optimal online value function in each of the 3 asymptotic regimes.

- Small market approximation: The small market regime is defined by letting the initial inventory level $q \to \infty$ and holding the market size $\lambda t > 0$ fixed. Now let

$$p^\star = \arg\max_{p \in \mathbb{R}_+} p(1-F(p)) \tag{37}$$

denote the optimal monopoly price. By the unimodality assumption at the outset of this section, $p^\star$ is unique. It is now straightforward to derive the upper bound $J^\star(q,\lambda t) \le \lambda t p^\star(1 - F(p^\star))$ for any fixed initial inventory level $q \in \mathbb{N}_+$ and market size $\lambda t > 0$. On the other hand, implementing a static price of $p^\star$ yields the lower bound

$$\lambda t p^\star(1-F(p^\star)) - J^\star(q,\lambda t) \le p^\star(1-F(p^\star)) E[\max(N_t - q, 0)]. \tag{38}$$

Expressing $E[\max(N_t - q,0)]$ as a function of $(q,\lambda t)$ in terms of the lower incomplete gamma function (Olver et al. 2010) and using the corresponding asymptotics as $q \to \infty$, it follows that $E[\max(N_t - q,0)] = O\left((\lambda t)^q/q!\right)$ as $q \to \infty$. We therefore have that in the small market regime

$$J^\star(q,\lambda t) = \lambda t p^\star(1-F(p^\star)) + O\left((\lambda t)^q/q!\right) \text{ as } q \to \infty. \tag{39}$$

Regarding the optimal price and optimal purchasing probability, one may show that for a fixed market size $\lambda t > 0$, $p^\star(q,\lambda t) \to p^\star$ and $\pi^\star(q,\lambda t) \to 1 - F(p^\star)$ as $q \to \infty$.

- Fluid approximation: The fluid regime may be defined by setting $q = c\lambda t$ for $c > 0$ and letting $\lambda t \to \infty$. If $c < 1 - F(p^\star)$, then by the results of Gallego and Van Ryzin (1994) the optimal value function has the asymptotics $J^\star(q,\lambda t) = q \cdot F^{-1}(1 - q/\lambda t) + O(\sqrt{\lambda t})$ as $\lambda t \to \infty$. Moreover, $p^\star(q,\lambda t) \to F^{-1}(1-c)$ and $\pi^\star(q,\lambda t) \to c$ as $\lambda t \to \infty$. These approximations may be extended in a natural way to $c \ge 1 - F(p^\star)$ too.
- Large market approximation: The large market regime is defined by holding the initial inventory level $q \in \mathbb{N}_+$ fixed and letting the market size $\lambda t \to \infty$. Our main result regarding the asymptotics of $J^\star(q,\lambda t)$, $p^\star(q,\lambda t)$ and $\pi^\star(q,\lambda t)$ in the large market regime is as follows.

THEOREM 2. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17), then for each $q \in \mathbb{N}_+$ as $\lambda t \to \infty$,*

$$J^\star(q,\lambda t) = qF^{-1}\left(1 - \left(\frac{q!}{(q/e)^q}\right)^{1/q} \frac{q}{\lambda t}\right) + o(a(\lambda t)) \tag{40}$$

and

$$p^\star(q,\lambda t) = F^{-1}\left(1 - \frac{q}{\lambda t}\right) + o(a(\lambda t)) \tag{41}$$

and

$$\pi^\star(q,\lambda t) = \frac{q}{\lambda t} + o(1/\lambda t). \tag{42}$$

The results of Theorem 2 are new to the literature and we discuss them below.

First note that the first order term in (40) for the optimal online value function in the large market regime differs from $qF^{-1}(1-q/\lambda t)$, which is the first order term for the optimal online value function in the fluid regime of Gallego and Van Ryzin (1994).

Next note that the remainder term in (40) is $o(a(\lambda t))$, where $a(\lambda t) = u(b(\lambda t))$ with u being the Mills ratio of the item valuation distribution F and $b(\lambda t) = F^{-1}(1-1/\lambda t)$. The implications of this for the approximation (40) are as follows. If the hazard rate function of F converges to a positive constant as $x \to x_U$, then the function $a(\lambda t)$ in the remainder term in (40) converges to the reciprocal of this constant as $\lambda t \to \infty$, and so the remainder term itself vanishes. More generally, the function $a(\lambda t)$ in (40) is decreasing if the hazard rate function of F is increasing, in which case the remainder term vanishes too. If the hazard rate function of F is decreasing, then $a(\lambda t)$ is increasing as $\lambda t \to \infty$. In the particular case of which the support of F is bounded, then under the assumptions of Theorem 2 it is guaranteed (Resnick 2013) that the hazard rate function diverges to ∞ as $x \to x_U$ and so the remainder term in (40) vanishes as $\lambda t \to \infty$. In all cases, $a(\lambda t)$ is a slowly varying function that is $o(b(\lambda t))$ (Resnick 2013).

Finally, by (41) if F is a von-Mises function in the Gumbel domain of attraction, then up to $o(a(\lambda t))$ the optimal pricing policy in the large market regime follows the same run-out rate form as in the fluid regime of Gallego and Van Ryzin (1994). By (42), the optimal purchasing probabilities are up to $o(1/\lambda t)$ of a run-out rate form as well. In Section 9, we test the accuracy of our approximations by running several numerical experiments.

8.3. Examples

We now present two examples applying the results of Section 8.2 to commonly used customer item valuation distributions in the Gumbel domain of attraction.

Example 3 (Finite Right Endpoint). For our first example of this section we consider the case in which customers value the items being sold according to the distribution from Example 1 of Section 7. Specifically, suppose that $x_U < \infty$ and

$$1 - F(x) = \beta \exp\left(-\frac{\alpha}{x_U - x}\right) \text{ for } 0 \le x < x_U,$$

where $\alpha > 0$ and and $\beta = \exp(\alpha/x_U)$. Note that in this case the item valuation distribution is bounded. Solving the first order condition in (37) one obtains that the optimal monopoly price is given by

$$p^\star_\alpha = x_U - \frac{\alpha}{2}\left(\sqrt{1 + 4x_U/\alpha} - 1\right). \tag{43}$$

It is also straightforward to show that

$$F^{-1}(p) = x_U - \frac{\alpha}{\ln(\beta/(1-p))} \quad \text{for} \quad 0 \le p < 1. \tag{44}$$

Substituting into Table 6, we obtain the following approximations to $J^{\star}(q,\lambda t)$ corresponding to each of the 3 asymptotic regimes. We also note that in the large market regime the error term in the approximation to the optimal online value function is $o((\ln\lambda t)^{-2})$.

Small Market$(q,\lambda t)$	Fluid$(q,\lambda t)$	Large Market$(q,\lambda t)$
$\lambda t p_{\alpha}^{\star}(1-F(p_{\alpha}^{\star}))$	$qx_U - q\alpha\ln^{-1}\left(c\cdot\frac{\lambda t}{q}\right)$	$qx_U - q\alpha\ln^{-1}\left(c\cdot\left(\frac{(q/e)^q}{q!}\right)^{1/q}\cdot\frac{\lambda t}{q}\right)$

Table 7 Approximations to the optimal online value function assuming customer item valuation distributions in the Gumbel domain of attraction following the distribution (23).

Example 4 (The Weibull Distribution). We next consider the customer item valuation distribution from Example 2 of Section 7. Specifically, suppose that customers value the items being sold according to a Weibull distribution with scale parameter $\mu>0$ and shape parameter $k>0$. The CDF of the item valuation distribution in this case is given by $F(x)=1-\exp(-(x/\mu)^k)$ for $x\geq 0$. Moreover,

$$F^{-1}(p)=\mu\ln^{1/k}\left(\frac{1}{1-p}\right)\quad\text{for}\quad 0\leq p<1. \tag{45}$$

Solving the first order condition in (37), we obtain that the optimal monopoly price is given by $p_k^{\star}=\mu\,(1/k)^{1/k}$. Substituting into Table 6, we then have the following approximations to $J^{\star}(q,\lambda t)$ corresponding to each of the 3 asymptotic regimes.

Small Market$(q,\lambda t)$	Fluid$(q,\lambda t)$	Large Market$(q,\lambda t)$
$\lambda t\mu(ke)^{-1/k}$	$\mu q\ln^{1/k}\left(\frac{\lambda t}{q}\right)$	$\mu q\ln^{1/k}\left(\left(\frac{(q/e)^q}{q!}\right)^{1/q}\cdot\frac{\lambda t}{q}\right)$

Table 8 Approximations to the optimal online value function assuming a Weibull customer item valuation distribution with scale parameter μ and shape parameter k.

The error term for the optimal value function approximation in the large market regime in this example is $o(\ln^{(1-k)/k}(\lambda t))$. The case of a shape parameter $k=1$ corresponds to an exponential distribution with a mean of μ and the function inside the o notation is constant. If $k<1$, the function inside the o notation is decreasing, and if $k>1$ it is increasing. This makes the Weibull distribution a good choice to test our approximations numerically, which we do in Section 9.

8.4. Regret Analysis

We now study the regret of the optimal online value function $J^\star(q,\lambda t)$ relative to its upper bound given by the optimal offline value function $J^\star_{\text{OFF}}(q,\lambda t)$. It turns out that the asymptotics of the regret depends on which asymptotic regime we consider. As in Section 8.2, our fluid regime results in this section assume that the customer item valuation distribution satisfies the assumptions in Section 3 and the additional assumption that the function $r(p)=p(1-F(p))$ is unimodal on $\mathbb{R}_+$. Our large market regime results assume that F lies in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17). We now proceed as follows.

First note that for each admissible pricing policy $p\in\mathcal{V}$, customer arrival rate $\lambda>0$, initial inventory level $q\in\mathbb{N}$, and selling horizon $t>0$, we have the inequality

$$J_p(\lambda;q,t) \le J^\star_{\text{OFF}}(q,\lambda t). \tag{46}$$

Therefore for each admissible pricing policy $p\in\mathcal{V}$, we may define its regret relative to the optimal offline policy by setting

$$\text{Regret}_p(\lambda;q,t) = J^\star_{\text{OFF}}(q,\lambda t) - J_p(\lambda;q,t) \;\ge\; 0. \tag{47}$$

It follows that maximizing the value function in (3) is equivalent to minimizing the regret in (47). The minimal regret may thus be represented as a function of the initial inventory level and market size. Specifically,

$$\text{Regret}^\star(q,\lambda t) = J^\star_{\text{OFF}}(q,\lambda t) - J^\star(q,\lambda t).$$

Small Market$(q,\lambda t)$	Fluid$(q,\lambda t)$	Large Market$(q,\lambda t)$
$\lambda t(E[X]-p^\star(1-F(p^\star))$	$qE[X-p(c)\|X>p(c)]$	$\ln\left(\frac{e^q q!}{e^{C_q}}\right)a(\lambda t)$

Table 9 Approximations to the minimal regret in each of the 3 asymptotic regimes.

Table 9 provides our approximations to the minimal regret in each of the 3 asymptotic regimes. The approximations are derived as follows.

- Small market approximation: Recall from Section 8.2 that in the small market regime for a fixed market size $\lambda t>0$, the optimal value function has the asymptotic

$$J^\star(q,\lambda t) = \lambda t p^\star(1-F(p^\star)) + O\left((\lambda t)^q/\Gamma(q+1)\right) \text{ as } q\to\infty, \tag{48}$$

where $p^\star$ is the optimal monopoly price (37). Next, it follows Proposition A4 in the appendix that the optimal offline value function satisfies

$$J^\star_{\mathrm{OFF}}(q,\lambda t) = \lambda t E[X] + O((\lambda t)^q/\Gamma(q+1)) \text{ as } q \to \infty. \tag{49}$$

The minimal regret in the small market regime is then given by

$$\mathrm{Regret}^\star(q,\lambda t) = \lambda t(E[X] - p^\star(1-F(p^\star)) + O((\lambda t)^q/\Gamma(q+1)) \text{ as } q \to \infty. \tag{50}$$

- Fluid regime: Let $c > 0$ and set $q = c\lambda t$. Next, recall from Section 8.2 that if $c < 1 - F(p^\star)$, then the optimal value function has the asymptotic $J^\star(q,\lambda t) = q \cdot F^{-1}(1 - q/\lambda t) + o(\lambda t)$ as $\lambda t \to \infty$. Also recall by Proposition A5 in the appendix that

$$J^\star_{\mathrm{OFF}}(q,\lambda t) = \lambda t \int_0^{q/\lambda t} F^{-1}(1-s)ds + o(\lambda t) \text{ as } \lambda t \to \infty. \tag{51}$$

Integrating-by-parts and using the change-of-variables $v = F^{-1}(1-s)$, it follows that

$$\lambda t \int_0^{q/\lambda t} F^{-1}(1-s)ds = qF^{-1}(1-q/\lambda t) + \lambda t \int_{F^{-1}(1-q/\lambda t)}^{\infty} (1-F(v))dv. \tag{52}$$

Moreover,

$$\int_{F^{-1}(1-q/\lambda t)}^{\infty} (1-F(v))dv = \frac{q}{\lambda t} E[(X - F^{-1}(1-q/\lambda t))|X > F^{-1}(1-q/\lambda t)]. \tag{53}$$

Recalling that $c = q/\lambda t$ and letting $p(c) = F^{-1}(1-c)$, the above then yields that

$$\mathrm{Regret}^\star(q,\lambda t) = qE[X - p(c)|X > p(c)] + o(\lambda t) \text{ as } \lambda t \to \infty. \tag{54}$$

- Large market regime: Recall from Section 8.2 that if F is in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17), then for each initial inventory level $q \in \mathbb{N}_+$ the optimal value function satisfies

$$J^\star(q,\lambda t) = qF^{-1}\left(1 - \left(\frac{q!}{(q/e)^q}\right)^{1/q} \frac{q}{\lambda t}\right) + o(a(\lambda t)) \text{ as } \lambda t \to \infty. \tag{55}$$

Next, it follows by Proposition A6 in the appendix that the large market asymptotic of the optimal offline value function is given by

$$J^\star_{\mathrm{OFF}}(q,\lambda t) = qF^{-1}\left(1 - \frac{e^{C_q/q}}{\lambda t}\right) + o(a(\lambda t)) \text{ as } \lambda t \to \infty, \tag{56}$$

where

$$C_q = \int_0^\infty \ln(v) Q(q,v)dv, \tag{57}$$

and $Q(\cdot,\cdot)$ is the regularized upper incomplete gamma function. It then follows by (A83) in the appendix (see also Proposition 0.10 of Resnick (2013)) that the minimal regret is of the form

$$\text{Regret}^{\star}(q,\lambda t) = c(q)a(\lambda t) + o(a(\lambda t)) \text{ as } \lambda t \to \infty, \tag{58}$$

where

$$c(q) = \ln\left(\frac{e^q q!}{e^{C_q}}\right).$$

We now make some observations regarding the minimal regret approximations in Table 9. First, note that the dominant term in the minimal regret for the small market regime is linear in the market size λt and constant with respect to the initial inventory level q. On the other hand, the first-order term for the minimal regret in the fluid regime is the product of the market size λt and a function of c=$q/\lambda t$. This can be seen by rewriting the approximation to the minimal regret as $\lambda t \cdot c \cdot E[X - p(c)|X > p(c)]$. Finally, the first order term for the minimal regret in the large market regime is the product of a function of the market size λt and a function of the initial inventory level q. We therefore see that for each regime the dominant term of the minimal regret takes a product form.

We complete this section by analyzing the minimal regret for the large market regime in more detail. First recall by Section 8.1 that the minimal static pricing regret in the large market regime is equal to ∞ on the scale of $a(\lambda t)$ as $\lambda t \to \infty$. This contrasts with the result (58) above which shows that the minimal dynamic pricing regret in the large market regime is on the scale of $o(a(\lambda t))$ as $\lambda t \to \infty$. Dynamic pricing therefore provides a significant improvement over static pricing in the large market regime.

Next, recall that $a(\lambda t) = u(b(\lambda t))$ where u is the reciprocal of the hazard rate function of the customer item valuation distribution F. It then follows from (58) that for a fixed initial inventory level $q \in \mathbb{N}_+$ and as the market size $\lambda t \to \infty$, the first order term of the minimal regret is decreasing if the hazard rate function of the customer item valuation distribution is increasing. Conversely, the first order term in the minimal regret is increasing if the hazard rate function of the customer item valuation distribution is decreasing. In the case when the hazard rate function of F converges to a constant, the minimal regret converges to the reciprocal of that constant multiplied by $c(q)$.

Finally, regarding the multiplier $c(q)$, using Lemma A3 and some elementary asymptotics of the harmonic numbers, it may be shown that

$$C_q = q\ln q - q + \frac{1}{2} + o(1) \text{ as } q \to \infty. \tag{59}$$

By (59) and Stirling's approximation (Olver et al. 2010), it then follows that

$$c(q) = q + \ln\sqrt{2\pi q} - \frac{1}{2} + o(1) \text{ as } q \to \infty. \tag{60}$$

8.5. Asymptotic Optimalliy

The results in Section 8.2 provide asymptotics for the optimal value function, pricing policy, and purchasing probability policy in the large market regime. However, for most customer item valuation distributions, the optimal pricing policy is still unknown. In this section, we construct an easy-to-implement pricing policy whose value function achieves the same asymptotics in the large market regime as in Theorem 2 for the optimal value function. Throughout this section, we assume that F lies in the Gumbel domain of attraction and satisfies the von-Mises conditions (16)-(17).

Recall from Section 8.4 that for each admissible pricing policy $p \in \mathcal{V}$, its regret relative to the optimal offline value function is defined for each initial inventory level $q \in \mathbb{N}_+$ and selling horizon of length $t > 0$ by

$$\text{Regret}_p(\lambda; q, t) = J^\star_{\text{OFF}}(q, \lambda t) - J_p(\lambda; q, t) \;\geq\; 0. \tag{61}$$

The following is then our definition of an asymptotically optimal pricing policy in the large market regime.

Definition 2. An admissible pricing policy $p \in \mathcal{V}$ is said to be asymptotically optimal in the large market regime if for each $q \in \mathbb{N}_+$,

$$\lim_{\lambda t \to \infty} \frac{\text{Regret}_p(\lambda; q, t)}{a(\lambda t)} = c(q), \tag{62}$$

where $c(q)$ is given by (60) and the limit above holds for any sequence of (λ, t) such that $\lambda t \to \infty$.

The results of Section 8.4 imply that an admissible pricing policy $p \in \mathcal{V}$ is asymptotically optimal in the large market regime if on a scale of $a(\lambda t)$, it achieves the same regret with respect to the optimal offline value function as does the optimal pricing policy $p^\star$. Using the relationship (48) between the minimal regret and the optimal value function, another way to state this is to say that $p \in \mathcal{V}$ is asymptotically optimal if

$$\lim_{\lambda t \to \infty} \frac{J^\star(q, \lambda t) - J_p(\lambda; q, t)}{a(\lambda t)} = 0, \tag{63}$$

for any sequence of (λ, t) such that $\lambda t \to \infty$.

Now consider the pricing policy $p \in \mathcal{V}$ such that for each $\lambda > 0$ and $q \in \mathbb{N}_+$ and $t > 0$,

$$p(\lambda; q, t) = F^{-1}\left(1 - \frac{q}{\lambda t}\right) \text{ for } \lambda t > q, \tag{64}$$

and $p(\lambda; q, t) = p^\star$ otherwise, where $p^\star$ is the optimal monopoly price given by (37). The pricing policy (64) is a dynamic run-out rate policy that is obtained by continuously resolving the fluid approximation of Gallego and Van Ryzin (1994) with limited inventory. The following is our main result on its asymptotic performance.

Theorem 3. *The policy $p \in \mathcal{V}$ given by (64) is asymptotically optimal in the large market regime.*

In Section 9, we numerically evaluate the performance of the dynamic run-out rate policy (64).

9. Numerical Experiments

We now complete the paper by running several numerical experiments. First, in Section 9.1 we test the accuracy of our approximations to the optimal online value function from Section 8.2. Next, in Section 9.2 we use the approximations from Section 8.2 to numerically determine the boundaries between the small market, fluid, and large market regimes in the online setting. In Section 9.3, we perform a similar procedure to determine the boundaries between the small market, fluid, and large market regimes in the offline setting. Finally, in Section 9.4 we evaluate the performance of the dynamic run-out rate policy from Section 8.5.

We assume throughout this section that the customer item valuations follow a Weibull distribution with scale parameter $\mu > 0$ and shape parameter $k > 0$. That is,

$$F(x) = 1 - \exp(-(x/\mu)^k) \text{ for } x \geq 0.$$

Recall that the Weibull distribution exhibits a decreasing hazard rate if $k < 1$ and an increasing hazard rate if $k > 1$. This makes the choice of Weibull items valuations a flexible one.

9.1. Accuracy of Optimal Value Function Approximation

In this section, we evaluate the accuracy of the small market, fluid, and large market approximations to the optimal online value function found in Table 6. Our approach is to compare each approximation to the true value of the optimal value function obtained by numerically solving the HJB equations (3). The results are presented in Figure 4 where the initial inventory level q is kept fixed at 10 and the market size λt increases from 0 to 250. We consider 3 cases corresponding to item valuations distributed according to a Weibull distribution with shape parameters $k = 1/2, 1$ and 2. In each case, the mean of the distribution is kept fixed at 1 by varying the scale parameter appropriately.

Note first that as expected for each value of the shape parameter the small market approximation performs best for small values of λt, the fluid approximation is the most accurate for moderate values of λt and the large market approximation performs best when λt is large. It is also apparent that the behavior of the hazard rate function of the item valuation distribution plays a significant role in our results in Figure 4. This corroborates our results from Section 9.4. Specifically, when the hazard rate function is decreasing (corresponding to the case of $k = 0.5$), the large market approximation (40) provides a lower bound to the optimal value function with the bound tightening as λt increases. On the other hand, when the hazard rate function is increasing (corresponding to the case of $k = 2$), the large market approximation (40) provides a upper bound to the optimal value function. The hazard rate function being constant (corresponding to $k = 1$) coincides with the norming function a being constant, and therefore, the $o(a(\lambda t))$ term in (40) vanishes. Moreover, judging by Figure 4 it appears to converge to zero from below.

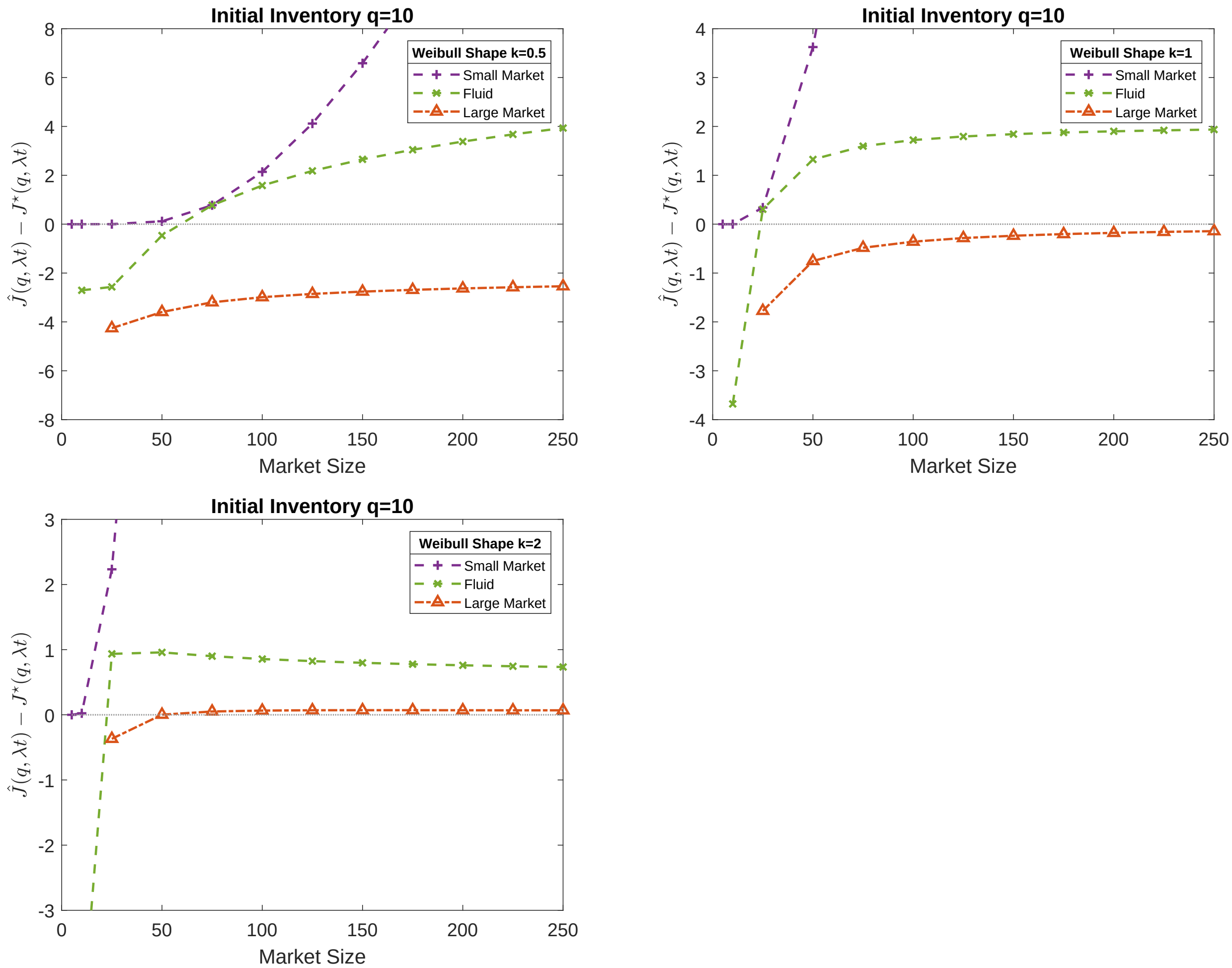

Figure 4 **Accuracy of the optimal online value function approximations for item valuations distributed according to a Weibull distribution**

The fluid approximation to the optimal online value function found in Table 6 does not appear in Figure 4 to be as tight as the large market approximation. It actually turns out that when the item valuation distribution has a decreasing or constant hazard rate function ($k = 0.5$ or 1), the fluid approximation becomes looser as the market size λt increases. With an increasing hazard rate function ($k = 2$), the fluid approximation becomes tighter as λt grows large but at a slower rate than the large market approximation. As for the small market approximation, while it gives a tight approximation for small market sizes, it quickly diverges as the market size λt increases.

9.2. Online Regime Boundaries

We next study the boundaries between the 3 regimes in the online problem. For each pair $(q, \lambda t)$ of initial inventory levels and market sizes, we compare our optimal online value function approximations in Table 8 to the value of the optimal value function itself. Unfortunately, a closed-form expression for the optimal value function does not exist in the case of Weibull item valuation distributions. We therefore numerically solve the HJB equations (4) to determine the optimal value function in this case.

Our results are presented in Figure 5 where we plot as a function of initial inventory level q and market size λt, the regions where each of the approximations of Table 8 is closest in absolute value to the optimal value function. The plots correspond to customer item valuation distributions having shape parameters $k = 1/2$ and 2. In both cases, the scale parameter of the item valuation distribution has been adjusted so that its mean remains equal to 1. Recall that the case of a shape parameter $k = 1$ corresponds to the exponential distribution studied in Section 5 and its plot is given in Figure 3.

The plots in Figure 5 for shape parameters $k = 1/2$ and 2 look similar to those in Figure 3 for the exponential case of $k = 1$. Each plot has 3 regions corresponding to where each asymptotic regime is optimal. The boundaries between the regions in each plot are approximately linear, corresponding to specific initial inventory-to-market size ratios. Moreover, the size of the fluid regime remains relatively consistent across different shape parameters k. However, higher values of k result in a delayed transition between regions based on market size, which is indicated by the larger area of the small market regime.

The lack of a closed-form expression for the optimal value function makes it challenging to derive expressions for the boundaries between the small market, fluid, and large market regimes in the case of item valuations with a Weibull distribution. The following asymptotics that are plotted in Figure 5 do however appear to perform reasonably well. For the small market regime-fluid regime boundary, we have

$$\sigma^{-1}(\lambda t) = \frac{\lambda t}{e^{1/k}} + \gamma^{\star}\sqrt{\frac{\lambda t}{e^{1/k}}} + o\left((\lambda t)^{1/2}\right) \text{ as } \lambda t \to \infty. \tag{65}$$

Next, for the fluid regime-large market regime boundary, we have

$$\tau(q) = qe^{1/k} + k^{(1/2-1/k)}e^{1/k} \cdot (2\pi)^{k/4} q^{3/4} + o(q^{3/4}) \text{ as } q \to \infty. \tag{66}$$

9.3. Offline Regime Boundaries

We next study the boundaries between the 3 regimes in the offline problem. Again, we consider Weibull item valuation distributions. Our approach is similar to that in Section 9.2. For each pair $(q, \lambda t)$ of initial inventory levels and market sizes we find the best (in terms of absolute value) offline approximation in Table 5 to the true optimal offline value function given in Section 7. Note that both the approximations and the optimal offline value function are readily computable using the formulas provided in Section 7.

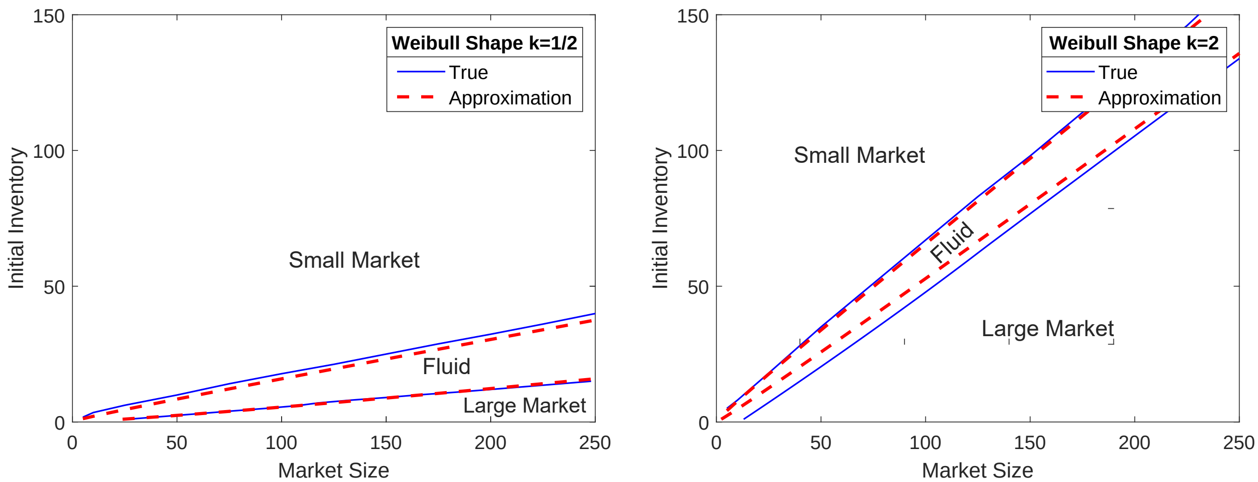

Figure 5 **Online problem boundaries between the 3 asymptotic regimes in the case of Weibull item valuations.**

As in Section 9.1, we consider item valuations with a Weibull distribution having shape parameters $k = 1/2, 1$ and 2. The scale parameter of each item valuation distribution is also adjusted so that its mean remains equal to 1. Figure 6 provides our results and reveals some interesting insights into the offline problem approximations. In particular, note that for the case of item valuations that follow an exponential distribtion ($k = 1$), the fluid approximation is dominated everywhere by either the small or large market approximation. On the other hand, for $k \neq 1$, we observe that the large market regions are small relative to the small market and fluid regions.

9.4. Performance of Asymptotically Optimal Policy

We complete the numerics section by comparing the value function J_p of the asymptotically optimal dynamic run-out rate pricing policy given by (64) of Section 8.5 against the optimal value function $J^\star$. Both of these value functions may be approximated by numerically solving the HJB equations (3). The two value functions are plotted in Figure 7 assuming Weibull item valuation distributions with a mean of 1 and shape parameters $k = 1/2, 1$ and 2. The initial inventory level is held fixed at $q = 10$ units.

Recall by (63) of Section 8.5 that for a fixed initial inventory level $q \in \mathbb{N}_+$, the difference between the performance of any asymptotically optimal pricing policy and the optimal pricing policy is $o(a(\lambda t))$ as $\lambda t \to \infty$, where a is the norming function (18). Moreover, a may be expressed as a time change of the Mills ratio (or reciprocal hazard rate function) of the item valuation distribution F. The findings in Figure 7 bear this asymptotic out. For the most part, the dynamic run-out rate pricing policy achieves almost the same performance as the optimal pricing policy. However, if one inspects closely the plot corresponding to item valuations following a Weibull distribution with shape parameter $k = 1/2$, there is a gap between the expected revenue of the dynamic run-out

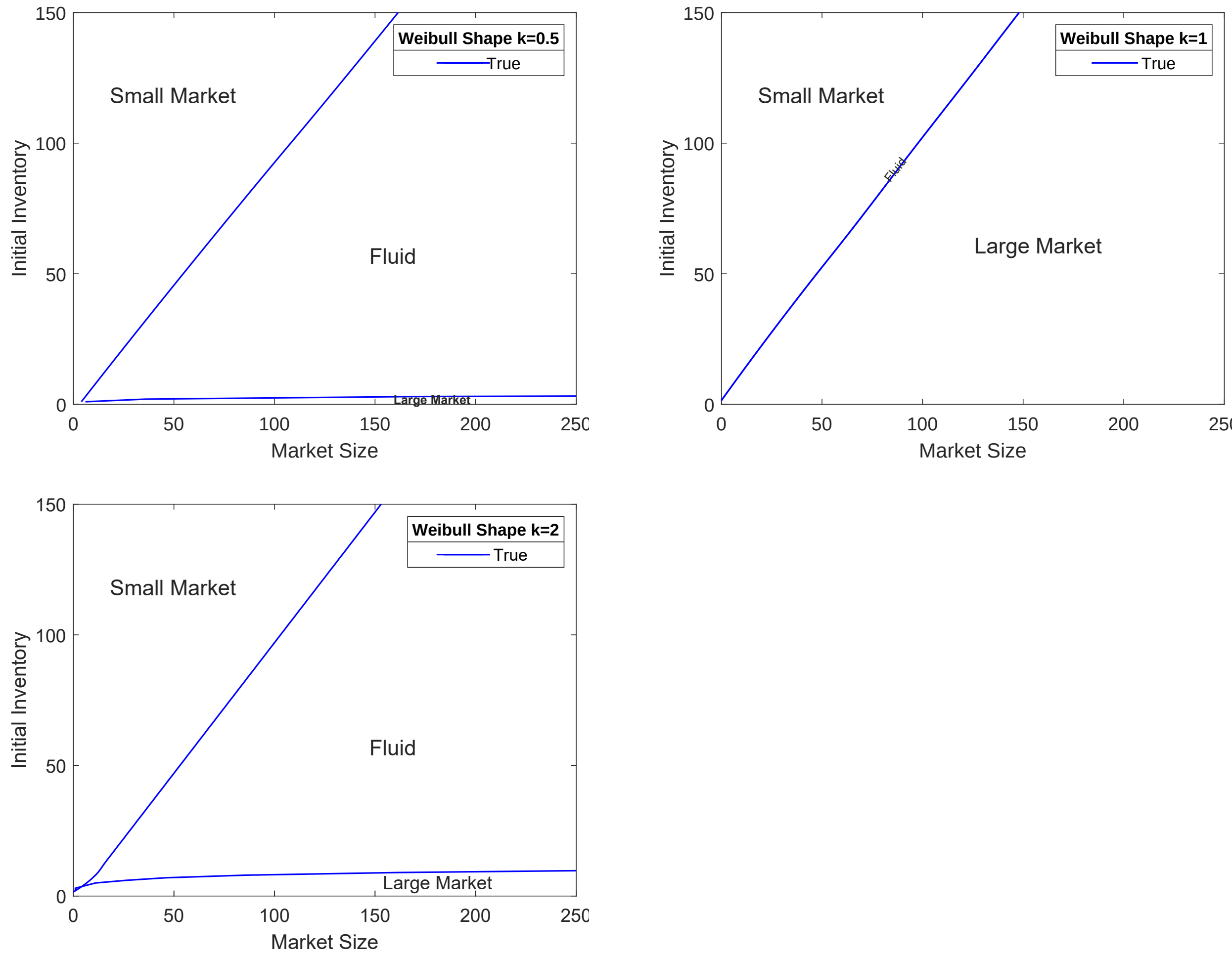

Figure 6 Offline problem boundaries between the 3 asymptotic regimes in the case of Weibull item valuations.

rate pricing policy and that of the optimal pricing policy. This is not surprising since $k = 1/2$ corresponds to a decreasing hazard rate, which implies that the function $a(\lambda t)$ does not vanish.

10. Conclusion

This paper develops a framework for studying the single-item dynamic pricing problem based on three asymptotic regimes characterized by their initial inventory level-to-market size ratio. We provide approximations to the both the offline and online optimal value function in each of these regimes. We also theoretically and numerically identify the boundaries between the small market, fluid and large market regimes. The large market regime is new and asymptotically displays behaviors deviating from the classical fluid regime. In particular, the first and second order approximations to the offline and online optimal value functions and the minimal regret deviate from those of the fluid regime. Despite these differences, we are still able to establish the robustness of a dynamic run-out rate pricing policy in both the fluid and large market regimes.

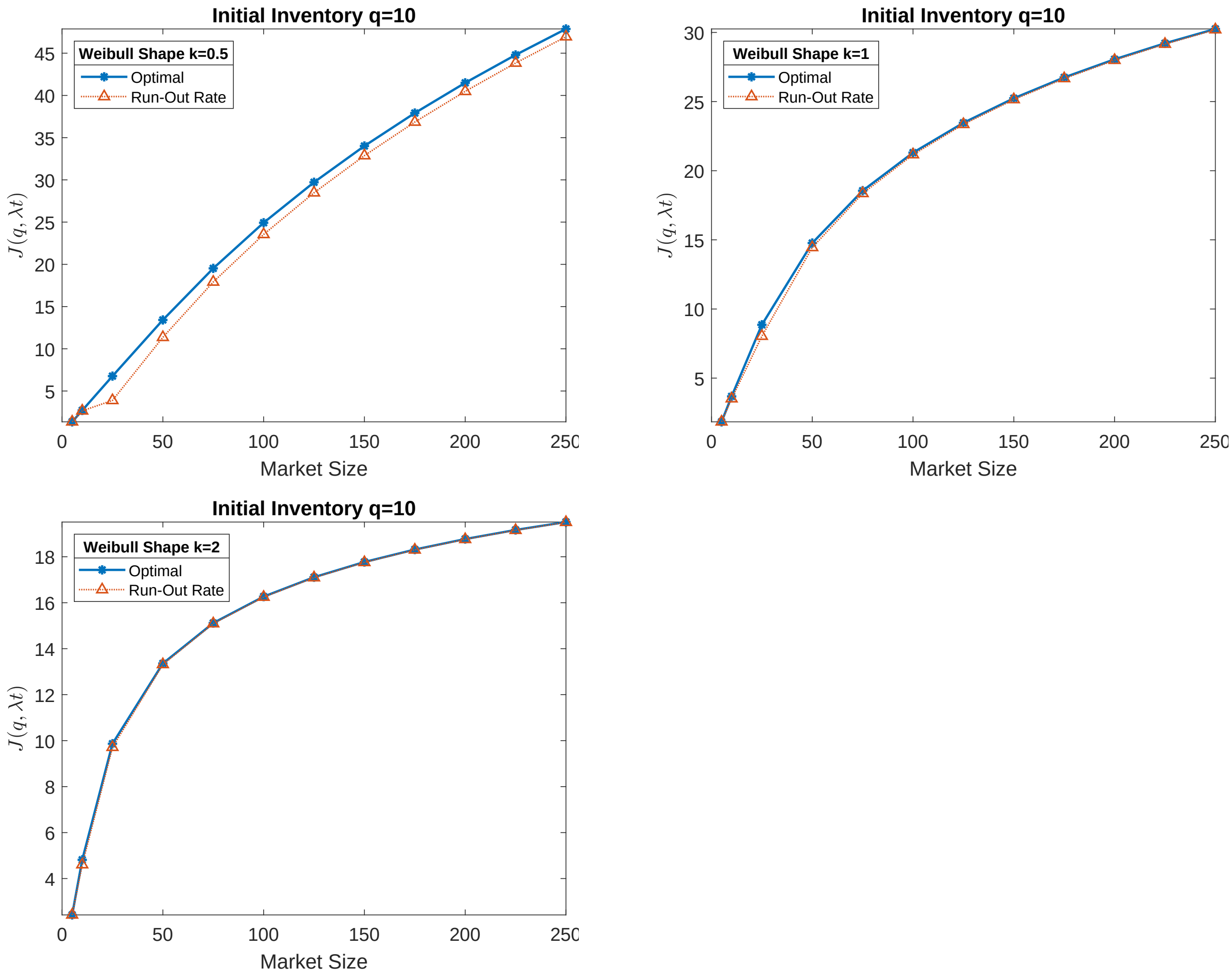

Figure 7 **Value functions of the optimal pricing policy and the asymptotically optimal dynamic run-out rate pricing policy for item valuations distributed according to a Weibull distribution.**

There exist several directions for future research. One natural extension is to study the single-item dynamic pricing problem in the large market regime with customer item valuation distributions lying in the remaining two extreme value domains of attraction, i.e. the Weibull and Fréchet domains. A second interesting direction for future research is to incorporate multi-item settings.

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Appendix. Proofs

A. Proofs for Exponentially Distributed Item Valuations

A1.1. Preliminary Results

For each initial inventory level $q > 0$ and market size $\lambda t > 0$, let $\text{Small}(q, \lambda t)$, $\text{Fluid}(q, \lambda t)$ and $\text{Large}(q, \lambda t)$ denote the approximations to the optimal value function in Table 1, where item valuations follows an exponential distribution with a mean of 1. We then have the following result.

Lemma A1. *For each $q > 0$ and $\lambda t > 0$,*

$$\text{Large}(q, \lambda t) \; < \; J^\star(q, \lambda t) \; < \; \text{Small}(q, \lambda t) \tag{A1}$$

and

$$\text{Large}(q, \lambda t) < \text{Fluid}(q, \lambda t) \; \leq \; \text{Small}(q, \lambda t). \tag{A2}$$

Proof of Lemma A1 It suffices to prove the result for the case of $\lambda = 1$. Let $q > 0$. We first prove (A1) for $t > 0$. Using the fact that $\Gamma(q+1, x) < \Gamma(q+1)$ for $x > 0$, it follows by (6) that $J^\star(q, t) < \text{Small}(q, t)$ for $t > 0$. Next, note that after some algebra

$$J^\star(q, t) - \text{Large}(q, t) \; = \; \ln\left(\left(\frac{t}{e}\right)^{-q} e^{t/e}\Gamma(q+1, t/e)\right). \tag{A3}$$

Then, using the facts that $\Gamma(q+1, x) = s\Gamma(q, x) + x^q e^{-x}$ and $\Gamma(q, x) > 0$ for $x \geq 0$, it follows that $J^\star(q, t) > \text{Large}(q, t)$ for $t > 0$. Hence, (A1) is proven.

Next, we prove (A2). By Table 1,

$$\text{Fluid}(q, t) - \text{Large}(q, t) \; = \; \ln\left(\frac{\Gamma(q+1)}{(q/e)^q}\right). \tag{A4}$$

However, $\Gamma(q+1) > (q/e)^q$ (Jameson 2015) and hence $\text{Large}(q, t) < \text{Fluid}(q, t)$. Next, by Table 1, after some algebra,

$$\text{Small}(q, t) - \text{Fluid}(q, t) \; = \; \ln\left(\left(\frac{t}{e}\right)^{-q} e^{t/e})\right) + q\ln q - q. \tag{A5}$$

Moreover, the function $x^{-q}e^x$ is strictly convex on $\mathbb{R}_+$, obtaining its minimum at $x = q$. Letting $t/e = q$ in (A5) and using the fact that ln is increasing, we then obtain that $\text{Fluid}(q, t) \leq \text{Small}(q, t)$. $\square$

It turns out that in the case of item valuations with an exponential distribution, the fluid approximation may be either less or greater than $J^\star(q, \lambda t)$. In particular, we have the following result.

Lemma A2. *For each $q > 0$, there exists a $u(q) < eq$ such that*

$$\text{Fluid}(q, \lambda t) \; < \; J^\star(q, \lambda t) \text{ for } \lambda t < u(q) \;\; \text{and} \;\; \text{Fluid}(q, \lambda t) \; > \; J^\star(q, \lambda t) \text{ for } \lambda t > u(q). \tag{A6}$$

Proof of Lemma A2 It suffices to prove the result for the case of $\lambda = 1$. We first prove (A6). Let $q > 0$. By (6) and Table 1, it follows that

$$\text{Fluid}(q,t) - J^\star(q,t) = \ln\left(\frac{q!}{(q/e)^q}\right) - \ln\left(\left(\frac{t}{e}\right)^{-q} e^{t/e}\Gamma(q+1, t/e)\right) \text{ for } t > 0. \tag{A7}$$

Taking the derivative with respect to t/e of the expression inside the ln above and making using of the identities

$$\frac{\partial\Gamma(q+1,x)}{\partial x} = -x^q e^{-x} \quad \text{and} \quad \Gamma(q+1,x) = q\Gamma(q,x) + x^q e^{-x} \text{ for } x > 0, \tag{A8}$$

we obtain

$$q\left(\frac{t}{e}\right)^{-q} e^{t/e}\Gamma(q,t/e)\cdot\left(1 - \frac{q}{t/e}\right) - \frac{q}{t/e}, \tag{A9}$$

which is negative whenever

$$\left(\frac{t}{e}\right)^{-(q-1)} e^{t/e}\Gamma(q,\ t/e)\cdot\left(1 - \frac{q}{t/e}\right) < 1. \tag{A10}$$

The above inequality clearly holds for $0 < t \le eq$. By the inequalities of Natalini and Palumbo (2000) for the incomplete gamma function, it also holds for $t > eq$ so long as $q \neq 1$. Finally, (A10) trivially holds for all $t > 0$ when $q = 1$. Thus, $\text{Fluid}(q,\cdot) - J^\star(q,\cdot)$ is continuous and strictly increasing for $t > 0$.

Next, by (6) and Table 1 it follows that

$$\text{Fluid}(q,t) - J^\star(q,t) \to -\infty \text{ as } t \to 0. \tag{A11}$$

In order to show that (A6) holds, it is now sufficient to show that $\text{Fluid}(q,t) > J^\star(q,t)$ for $t = eq$. Substituting into (A7), this is equivalent to showing that

$$e^q > e^q \cdot \frac{\Gamma(q+1,q)}{\Gamma(q+1)}, \tag{A12}$$

which is true since $\Gamma(q+1,x)/\Gamma(q+1) < 1$ for $x > 0$. □

A1.2. Regions Proof

Proof of Proposition 1 It suffices to prove the result for the case of $\lambda = 1$. Let $q > 0$ and recall by Lemmas A1 and A2 that for $0 < t \le u(q)$ we have the ordering

$$\text{Large}(q,t) < \text{Fluid}(q,t) \le J^\star(q,t) < \text{Small}(q,t). \tag{A13}$$

Thus, for $0 < t \le u(q)$ either $\text{Fluid}(q,t)$ or $\text{Small}(q,t)$ is optimal. Next, note by Table 1 and (6) that for $t \ge 0$,

$$\text{Small}(q,t) - J^\star(q,t) = \ln\left(\frac{\Gamma(q+1)}{\Gamma(q+1,t/e)}\right), \tag{A14}$$

which is continuous and strictly increasing for $t \ge 0$ and equal to 0 at $t = 0$. On the other hand, as in the proof of Lemma A2, $J^\star(q,t) - \text{Fluid}(q,t)$ is continuous and strictly decreasing for $t > 0$ with

$$\lim_{t\downarrow 0} J^\star(q,t) - \text{Fluid}(q,t) = \infty. \tag{A15}$$

Thus, since $J^\star(q,u(q)) = \text{Fluid}(q,u(q))$, there exists some $\sigma(q) < u(q)$ such that $\text{Approx}^\star(q,t) = \text{S}$ for $0 < t < \sigma(q)$ and $\text{Approx}^\star(q,t) = \text{F}$ for $\sigma(q) < t \le u(q)$. Moreover, since $u(q) < eq$ by Lemma A2, this implies that $\sigma(q) < eq$ as well.

Next, consider the case of $t > u(q)$. By Lemmas A1 and A2, for $t > u(q)$ we have the ordering

$$\text{Large}(q,t) \;<\; J^\star(q,t) \;<\; \text{Fluid}(q,t) \;\le\; \text{Small}(q,t). \tag{A16}$$

Hence, for $t > u(q)$ either $\text{Large}(q,t)$ or $\text{Fluid}(q,t)$ is optimal. Now recall from the proof of Lemma A2 that $\text{Fluid}(q,t) - J^\star(q,t)$ is continuous and strictly increasing for $t > 0$ with a value of 0 at $t = u(q)$. Moreover, using the asymptotic $x^{-q}e^x\Gamma(q+1,x) \to 1$ as $x \to \infty$, it follows from (A7) above that

$$\text{Fluid}(q,t) - J^\star(q,t) \;\to\; \ln(q!/(q/e)^q) \text{ as } t \to \infty. \tag{A17}$$

Next, it follows by (A4) that $J^\star(q,t) - \text{Large}(q,t)$ is continuous and strictly decreasing for $t > 0$ with a limiting value of 0 at $t = \infty$. Thus, there exists some $\tau(q) > u(q)$ such that $\text{Approx}^\star(q,t) = \text{F}$ for $u(q) < t < \tau(q)$ and $\text{Approx}^\star(q,t) = \text{L}$ for $t > \tau(q)$. Finally, in order to show that $\tau(q) > eq$ it suffices to show that

$$\text{Fluid}(q,eq) - J^\star(q,eq) \;<\; J^\star(q,eq) - \text{Large}(q,eq). \tag{A18}$$

By (6) and Table 1, along with the fact that ln is strictly increasing, the above is equivalent to

$$\frac{\Gamma(q+1,q)}{\Gamma(q+1)} > \left(\frac{(q/e)^q}{\Gamma(q+1)}\right)^{1/2}. \tag{A19}$$

Now note that $\Gamma(q,\cdot)/\Gamma(q)$ is the tail CDF of a gamma distribution with shape parameter q and scale parameter 1. It therefore follows by by Theorem 1 of Chen and Rubin (1986) that

$$\frac{\Gamma(q+1,q)}{\Gamma(q+1)} > \frac{1}{2} \text{ for } q > 0. \tag{A20}$$

On the other hand, by (Jameson 2015) we have the upper bound

$$\left(\frac{(q/e)^q}{\Gamma(q+1)}\right)^{1/2} < (2\pi q)^{-1/4} \text{ for } q > 1. \tag{A21}$$

It then follows that (A19) holds for $q > 8/\pi \approx 2.55$, as desired. □

A1.3. Monotonicty Proof

Proof of Proposition 2 It suffices to prove the result for the case of $\lambda = 1$. We first show that $\sigma(q)$ is strictly increasing for $q > 0$. Let $q > 0$ and recall by the proof of Proposition 1 that for $0 < t \le \sigma(q)$ we have the ordering

$$\text{Large}(q,t) < \text{Fluid}(q,t) \;\le\; J^\star(q,t) \;<\; \text{Small}(q,t). \tag{A22}$$

It threfore suffices to show that for each $q_2 > q_1 > 0$ and $0 < t \le \sigma(q_1)$,

$$\text{Small}(q_2,t) - J^\star(q_2,t) < J^\star(q_2,t) - \text{Fluid}(q_2,t). \tag{A23}$$

We recall first by Tricomi (1950b) (see also Gautschi (1998)) that for fixed $t > 0$, $\Gamma(q,t)/\Gamma(q)$ is strictly increasing in $q > 0$. It then follows by (A14) that

$$\text{Small}(q_2,t) - J^\star(q_2,t) < \text{Small}(q_1,t) - J^\star(q_1,t) \text{ for } t > 0. \tag{A24}$$

Hence, in order to prove (A23) it remains to show that

$$J^\star(q_2,t) - \text{Fluid}(q_2,t) \;>\; J^\star(q_1,t) - \text{Fluid}(q_1,t) \text{ for } 0 < t \le \sigma(q_1). \tag{A25}$$

First note by (6) and Table 1 that for each $q > 0$ and $t > 0$,

$$J^\star(q,t) - \text{Fluid}(q,t) \;=\; \ln\left(e^{t/e}\frac{\Gamma(q+1,t/e)}{\Gamma(q+1)}\cdot\left(\frac{t}{q}\right)^{-q}\right). \tag{A26}$$

Then, recall again by Tricomi (1950b) that for fixed $t > 0$, $\Gamma(q,t)/\Gamma(q)$ is strictly increasing in $q > 0$. Next, it is straightforward to verify that for fixed $t > 0$, $(t/q)^{-q}$ is strictly increasing in q for $q > t/e$. Thus, since $\sigma(q_1) \le eq_1$ by Proposition 1, it follows that for $0 \le t \le \sigma(q_1)$, $(t/q)^{-q}$ is increasing in $q > q_1$. Using (A26) and the monotonicity of ln, the inequality (A25) is now proven. Thus, $\sigma(q)$ is strictly increasing for $q > 0$.

We now show that $\sigma(q) \to 0$ as $q \to 0$ and $\sigma(q) \to \infty$ as $q \to \infty$. The fact that $\sigma(q) \to 0$ as $q \to 0$ follows since by Proposition 1, $\sigma(q) < eq$ for $q > 0$. We next show that $\sigma(q) \to \infty$ as $q \to \infty$. Suppose not. Then, since σ is strictly increasing, there exists some $c \in \mathbb{R}_+$ such that $\sigma(q) \to c$ as $q \to \infty$. This implies that the small market approximation is never the best for $t > c$. However, this is not possible since $J^\star(q,t) \to \text{Small}(q,t) = t/e$ as $q \to \infty$ and $\text{Fluid}(q,t), \text{Large}(q,t) \to -\infty$ as $q \to \infty$. Thus, it must be the case that $\sigma(q) \to \infty$ as $q \to \infty$.

Next, we show that $\tau(q)$ is strictly increasing for q sufficiently large. By definition, for $q > 0$,

$$\text{Fluid}(q,\tau(q)) - J^\star(q,\tau(q)) \;=\; J^\star(q,\tau(q)) - \text{Large}(q,\tau(q)). \tag{A27}$$

In order to prove the result, it then suffices by (10) of Proposition 1 to show that for q large,

$$\frac{\partial}{\partial q}(\text{Fluid}(q,\tau(q)) - J^\star(q,\tau(q))) \;<\; \frac{\partial}{\partial q}(J^\star(q,\tau(q)) - \text{Large}(q,\tau(q))), \tag{A28}$$

where the partial derivatives are taken with respect to the first argument of each function, holding $\tau(q)$ fixed. Performing the necessary calculations and using the identity

$$\left(\frac{\tau(q)}{q}\right)\cdot\left(\frac{\tau(q)}{e}\right) \;=\; e^{2\tau(q)/eq}\left(\frac{(\Gamma(q+1,\tau(q)/e))^2}{\Gamma(q+1)}\right)^{1/q}, \tag{A29}$$

it then follows that to prove the result it suffices to prove that for q sufficiently large,

$$\begin{aligned}&\Psi(q+1) - 1 + 2\cdot\frac{\tau(q)}{qe} + \frac{2}{q}\ln\Gamma(q+1,\tau(q)/e) - \frac{1}{q}\ln\Gamma(q+1)\\ &< \left(1+\frac{1}{\Gamma(q+1,\tau(q)/e)}\right)\frac{\partial\Gamma(q+1,\tau(q)/e)}{\partial q}.\end{aligned} \tag{A30}$$

Now recall that for $q > 0$, $\Psi(q) < \ln q$ (see Alzer (1997)) and clearly $\Gamma(q,x) < \Gamma(q)$ for $x > 0$. Also, recall by Anderson and Qiu (1997) that $\ln\Gamma(q) < q\ln q$ for $q > 1$. Moreover, using the fact by Proposition 1 that $\tau(q)/e > q$ for $q > 8/\pi$, it is straightforward to show that for $q > 8/\pi$,

$$\frac{\partial\Gamma(q+1,\lambda\tau(q)/e)}{\partial q} \;>\; \ln q\cdot\Gamma(q+1,\tau(q)/e). \tag{A31}$$

Piecing the above facts together, in order to prove that $\tau(q)$ is eventually strictly increasing it suffices from (A30) to show that for q sufficiently large,

$$\Gamma(q+1,\tau(q)/e) \;>\; 3 + \frac{2}{\ln q}\cdot\frac{\tau(q)}{qe}. \tag{A32}$$

First note that (A29) may be rewritten as

$$\left(\frac{\tau(q)}{e}\right)^{-q} e^{\tau(q)/e}\Gamma(q+1,\tau(q)/e) = \left(\frac{\Gamma(q+1)}{(q/e)^q}\right)^{1/2}. \tag{A33}$$

Next, by (Jameson 2015) for $q > 1$,

$$\left(\frac{\Gamma(q+1)}{(q/e)^q}\right)^{1/2} > (2\pi q)^{1/4}. \tag{A34}$$

It then follows as in Example 3.2.2 of Natalini and Palumbo (2000) that

$$\frac{\tau(q)}{qe} < C_q = \left(1 + \frac{1}{(2\pi q)^{1/4} - 1}\right) \to 1 \text{ as } q \to \infty. \tag{A35}$$

In order to prove that (A32) holds for q sufficiently large, it therefore suffices to show that $\Gamma(q+1,\tau(q)/e) \to \infty$ as $q \to \infty$. However, by the monotonicity of $\Gamma(q+1,\cdot)$ and the inequality of Example 3.2.1 of Natalini and Palumbo (2000),

$$\Gamma(q+1,\tau(q)/e) > \Gamma(q+1, qC_q) > (qC_q)^q e^{-qC_q} \to \infty \text{ as } q \to \infty. \tag{A36}$$

Thus, $\tau(q)$ is strictly increasing for q sufficiently large. The fact that $\tau(q) \to \infty$ as $q \to \infty$ follows since by Proposition 1, $\tau(q) > eq$ for $q > 0$.

In remains to prove the continuity properties of $\sigma(q)$ and $\tau(q)$. The fact that $\sigma(q)$ is continuous follows from the continuity of $J^\star(q,t)$, $\text{Fluid}(q,t)$ and $\text{Large}(q,t)$ with respect to both q and t, together with (10) of Proposition 1 and the monotonicity properties of $\text{Small}(q,\cdot) - J^\star(q,\cdot)$ and $J^\star(q,\cdot) - \text{Fluid}(q,\cdot)$. The details are omitted. The fact that $\tau(q)$ is continuous follows in a similar manner. □

A1.4. σ^{-1} Asymptotics Proof

In order to prove Proposition 4, it suffices to prove the following result.

PROPOSITION A1. *The function σ has the asymptotic expansion*

$$\sigma(q) = qe - \gamma^\star e\sqrt{2q} + o(q^{1/2}) \text{ as } q \to \infty, \tag{A37}$$

where $\gamma^\star$ is given by (14).

Proof of Proposition A1 First note by Proposition 1 and the continuity of $\text{Small}(q,\lambda t)$, $\text{Fluid}(q,\lambda t)$ and $J^\star(q,\lambda t)$ that for each $q > 0$, $\sigma(q)$ is the unique solution to

$$\text{Small}(q,\sigma(q)) - J^\star(q,\sigma(q)) = J^\star(q,\sigma(q)) - \text{Fluid}(q,\sigma(q)). \tag{A38}$$

By (7) and Table 1, the above then implies that

$$\frac{\Gamma(q+1)}{\Gamma(q+1,\sigma(q)/e)} = e^{\sigma(q)/e}\frac{\Gamma(q+1,\sigma(q)/e)}{\Gamma(q+1)} \cdot \left(\frac{\sigma(q)}{q}\right)^{-q}. \tag{A39}$$

Next, recall by Tricomi (1950a) that for each $y \in \mathbb{R}$,

$$\frac{\Gamma(q+1, q+(2q)^{1/2}y)}{\Gamma(q+1)} = \frac{1}{2}\text{erfc}(y) + +o(1/\sqrt{q}) \text{ as } q \to \infty, \tag{A40}$$

where the convergence is uniform for bounded y. Now suppose that

$$\lim_{q\to\infty} \frac{\sigma(q)/e - q}{(2q)^{1/2}} = c \in \mathbb{R}. \tag{A41}$$

Then, it is straightforward to show that

$$\lim_{q\to\infty} \exp(\sigma(q)/e)\left(\frac{\sigma(q)}{q}\right)^{-q} = e^{c^2}. \tag{A42}$$

Taking limits as $q\to\infty$ on both sides of (A39), we then obtain that

$$\left(\frac{1}{2}\mathrm{erfc}(c)\right)^{-1} = e^{c^2}\left(\frac{1}{2}\mathrm{erfc}(c)\right). \tag{A43}$$

To complete the proof it suffices to show that the lefthand side of (A43) is strictly increasing in c, the righthand side of (A43) is strictly decreasing in c, and there exists a unique $c=\gamma^\star$ satisfying (A43). Then, since by (A10) in the proof of Lemma A2, the lefthand side of (A39) is strictly increasing in $\sigma(q)/e$ and the righthand side of (A39) is strictly decreasing in $\sigma(q)/e$, it must be the case that (A41) holds with $c=\gamma^\star$, which after some algebra yields (A37).

We proceed as follows. By (12) and since $\mathrm{erfc}=1-\mathrm{erf}$, it follows that the lefthand side of (A43) is strictly increasing in c with

$$\lim_{c\to-\infty}\left(\frac{1}{2}\mathrm{erfc}(c)\right)^{-1} = 1 \quad \text{and} \quad \lim_{c\to\infty}\left(\frac{1}{2}\mathrm{erfc}(c)\right)^{-1} = \infty. \tag{A44}$$

Regarding the righthand side of (A43), first note that

$$\lim_{c\to-\infty} e^{c^2}\left(\frac{1}{2}\mathrm{erfc}(c)\right) = \infty \quad \text{and} \quad \lim_{c\to\infty} e^{c^2}\left(\frac{1}{2}\mathrm{erfc}(c)\right) = 0, \tag{A45}$$

where the second limit follows by the inequality $\mathrm{erfc}(c) < e^{-c^2}/c\sqrt{\pi}$ for $c>0$ from (Zhang et al. 2020). To see that the righthand side of (A43) is strictly decreasing in c, note that taking its derivative with respect to c it suffices to show that

$$ce^{c^2}\mathrm{erfc}(c) - \frac{1}{\sqrt{\pi}} < 0. \tag{A46}$$

(A46) is clearly true for $c\le 0$. On the other hand, if $c>0$, then (A46) follows again by the inequality from (Zhang et al. 2020). The preceding now implies that there exists a unique $c=\gamma^\star$ satisfying (A43). ☐

A1.5. τ^{-1} Asymptotics Proof

It order to prove Proposition 5, it suffices to prove the following result.

Proposition A2. *The function τ has the asymptotic expansion*

$$\tau(q) = eq + \frac{e}{(2\pi)^{1/4}}\cdot q^{3/4} + o(q^{3/4}) \text{ as } q\to\infty. \tag{A47}$$

Proof of Proposition A2 Note first by Proposition 1 and the continuity of $\mathrm{Fluid}(q,\lambda t)$, $\mathrm{Large}(q,\lambda t)$ and $J^\star(q,\lambda t)$ that for each $q>0$, $\tau(q)$ is the unique solution to

$$\mathrm{Fluid}(q,\tau(q)) - J^\star(q,\tau(q)) = J^\star(q,\tau(q)) - \mathrm{Large}(q,\tau(q)). \tag{A48}$$

By (7) and Table 1, the above then implies that

$$e^{-\tau(q)/e}\frac{\Gamma(q+1)}{\Gamma(q+1,\tau(q)/e)}\cdot\left(\frac{\tau(q)}{q}\right)^{q} = e^{\tau(q)/e}\frac{\Gamma(q+1,\tau(q)/e)}{\Gamma(q+1)}\cdot\left(\frac{\tau(q)}{q}\right)^{-q}\left(\frac{\Gamma(q+1)}{(q/e)^q}\right). \tag{A49}$$

Next, recall by (Tricomi 1950a) that for each $\gamma > 1$ such that $\gamma - 1 = O(q^{1/4})$,

$$\Gamma(q, \gamma q) = \frac{(\gamma q)^q e^{-\gamma q}}{q(\gamma - 1)} \left(1 + O\left(\frac{1}{q}\right)\right) \text{ as } q \to \infty.$$

Now suppose that

$$\lim_{q\to\infty} \frac{\tau(q)/e - q}{q^{3/4}} = c \in \mathbb{R}, \tag{A50}$$

and recall by Stirling's approximation that $\Gamma(q+1)/(q/e)^q \sim \sqrt{2\pi q}$. Then, dividing both sides of (A49) by $q^{1/4}$ and taking limits as $q \to \infty$, we obtain that

$$c\sqrt{2\pi} = \frac{1}{c}. \tag{A51}$$

The lefthand side of (A51) is strictly increasing in c and the righthand side is strictly decreasing in c. Moreover, equality is uniquely satisfied at $c^\star = (2\pi)^{-1/4}$. Then, since by (A10) in the proof of Lemma A2, the lefthand side of (A49) is strictly increasing in $\tau(q)/e$ and the righthand side of (A49) is strictly decreasing in $\tau(q)/e$, it must be the case that (A50) holds with $c = c^\star$, which after some algebra yields (A47). □

B. Offline Problem

A2.1. General Expression for the Optimal Offline Value Function

Let F be an arbitrary item valuation distribution and denote by F^{-1} its left-continuous inverse. We then have the following result.

PROPOSITION A3. *If F has a finite mean, then for each $q \in \mathbb{N}_+$ and $\lambda t > 0$,*

$$J^\star_{\text{OFF}}(q, \lambda t) = \frac{1}{\Gamma(q)} \int_0^{\lambda t} F^{-1}\left(1 - \frac{v}{\lambda t}\right) \cdot \Gamma(q, v) \cdot dv. \tag{A52}$$

Proof of Proposition A3 It suffices to prove the result for the case of $\lambda = 1$. Now recall from Section 7 the definition of $X_{m,n}$ as the mth largest value in the sequence $\{X_1, X_2, ..., X_n\}$ and let $X_{m,n} = 0$ if $m > n$. Then, in order to prove the result it suffices by (19) to show that for each $q \in \mathbb{N}_+$ and $t > 0$,

$$\sum_{m=1}^{q} E[X_{m,N_t}] = \frac{1}{\Gamma(q)} \int_0^{t} F^{-1}\left(1 - \frac{v}{t}\right) \cdot \Gamma(q, v) \cdot dv. \tag{A53}$$

We proceed as follows.

For each $m \in \mathbb{N}_+$ and $t > 0$, it may be shown that the CDF of X_{m,N_t} is given by

$$P\left(X_{m,N_t} \le x\right) = Q(m, \Lambda_t(x)), \tag{A54}$$

where $\Lambda_t(x) = t(1 - F(x))$ and $Q(\cdot,\cdot) = \Gamma(\cdot,\cdot)/\Gamma(\cdot)$ is the regularized upper incomplete gamma function (Olver et al. 2010). Thus,

$$E[X_{m,N_t}] = \int_0^\infty (1 - Q(m, \Lambda_t(x)))dx, \tag{A55}$$

and integrating-by-parts we obtain

$$E[X_{m,N_t}] = x(1 - Q(m, \Lambda_t(x)))|_0^\infty + \int_0^\infty x \cdot dQ(m, \Lambda_t(x)). \tag{A56}$$

Using the asymptotic $x^{-m}(1-Q(m,x)) \to 1/m$ as $x \to 0$, it follows that the first term on the righthand side above is equal to zero so long as F has a finite first moment. It follows that

$$E[X_{m,N_t}] = \int_0^\infty x \cdot dQ(m, \Lambda_t(x)). \tag{A57}$$

However,

$$dQ(m,\Lambda_t(x)) = \frac{t}{\Gamma(m)} \cdot \Lambda_t^{m-1}(x)\exp(-\Lambda_t(x))dF(x), \tag{A58}$$

and so

$$E[X_{m,N_t}] = t\int_0^\infty x \cdot \frac{\Lambda_t^{m-1}(x)}{\Gamma(m)} \exp(-\Lambda_t(x)) \cdot dF(x). \tag{A59}$$

Now recall that for each $q \in \mathbb{N}_+$ and $s > 0$,

$$\sum_{m=1}^{q} \frac{s^{m-1}}{\Gamma(m)} = e^s \frac{\Gamma(q,s)}{\Gamma(q)}. \tag{A60}$$

Thus,

$$\sum_{m=1}^{q} E[X_{m,N_t}] = t\int_0^\infty x \cdot \sum_{m=1}^{q} \frac{\Lambda_t^{m-1}(x)}{\Gamma(m)} \cdot \exp(-\Lambda_t(x)) \cdot dF(x) \tag{A61}$$

$$= \frac{t}{\Gamma(q)} \int_0^\infty x \cdot \Gamma(q, \Lambda_t(x)) \cdot dF(x). \tag{A62}$$

Making the substitution $v = t(1-F(x))$ in the final integral above, we then obtain (A53). □

A2.2. Monopoly Approximation of Optimal Offline Value Function

Proposition A4. *If F has a finite mean, then for each $\lambda t > 0$,*

$$J^\star_{\mathrm{OFF}}(q, \lambda t) = \lambda t E[X] + O((\lambda t)^q / \Gamma(q+1)) \text{ as } q \to \infty. \tag{A63}$$

Proof of Proposition A5 It suffices to prove the result for case of $\lambda = 1$. By Proposition A3 it follows after some algebra that

$$J^\star_{\mathrm{OFF}}(q,\lambda t) - \lambda t E[X] = \lambda t \int_0^1 F^{-1}(1-u) P(q, \lambda t u) du, \tag{A64}$$

where $P(q,\cdot) = \gamma(q,\cdot)/\Gamma(q)$ denotes the regularized lower incomplete gamma function. Hence, in order to complete the proof it suffices to show that

$$\lambda t \int_0^1 F^{-1}(1-u) P(q,\lambda t u) du = O((\lambda t)^q/\Gamma(q)) \text{ as } q \to \infty. \tag{A65}$$

First note that since $P(q,\cdot)$ is an increasing function, we have the inequality

$$\lambda t \int_0^1 F^{-1}(1-u) P(q, \lambda t u) du \le \lambda t E[X] P(q, \lambda t). \tag{A66}$$

Then, recall by (Olver et al. 2010) the asymptotic $P(a,z) \sim z^a e^{-z}/\Gamma(1+a)$ as $a \to \infty$. □

A2.3. Fluid Approximation of Optimal Offline Value Function

PROPOSITION A5. *If F has a finite mean, then for each $q = c\lambda t$ where $0 < c < 1$,*

$$J^{\star}_{\mathrm{OFF}}(q,\lambda t) = \lambda t \int_0^{q/\lambda t} F^{-1}(1-s)ds + o(\lambda t) \text{ as } \lambda t \to \infty. \tag{A67}$$

Proof of Proposition A5 By Proposition A3 it follows after some algebra that

$$J^{\star}_{\mathrm{OFF}}(q,\lambda t) - \lambda t \int_0^{q/\lambda t} F^{-1}(1-s)ds = \int_q^{\lambda t} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv - \int_0^{q} F^{-1}\left(1-\frac{v}{\lambda t}\right) P(q,v)dv, \tag{A68}$$

where $P(q,v) = 1 - Q(q,v)$ is the regularized lower incomplete gamma function (Olver et al. 2010) . In order to complete the proof, it now suffices to show that each integral on the righthand side of (A68) is $o(\lambda t)$.

We begin with the first integral on the righthand side of (A68). Letting $y > 0$, we have the decomposition

$$\int_q^{\lambda t} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv = \int_q^{q+yq^{1/2}} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv + \int_{q+yq^{1/2}}^{\lambda t} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv. \tag{A69}$$

Now consider the first integral on the righthand side of (A69). Recalling that $q = c\lambda t$ and since F^{-1} is increasing and $Q(\cdot,\cdot)$ is decreasing, it follows that

$$\int_q^{q+yq^{1/2}} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv < yF^{-1}(1-c)Q(q,q)\sqrt{c\lambda t}. \tag{A70}$$

Thus, since $Q(q,q) \to 1/2$ as $q \to \infty$ (see (3.1) of Nemes and Olde Daalhuis (2019)), it follows that

$$\frac{1}{\lambda t}\int_q^{q+yq^{1/2}} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv \to 0 \text{ as } \lambda t \to \infty. \tag{A71}$$

Next, consider the second integral on the righthand side of (A69). First note that

$$\int_{q+yq^{1/2}}^{\lambda t} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv < Q(q,q+yq^{1/2}) \int_0^{\lambda t} F^{-1}\left(1-\frac{v}{\lambda t}\right) = \lambda t E[X] Q(q,q+yq^{1/2}). \tag{A72}$$

Next, by (3.1) of Nemes and Olde Daalhuis (2019),

$$Q(q,q+q^{1/2}y) \to \frac{1}{2}\mathrm{erfc}(2^{-\frac{1}{2}}y) \text{ as } q \to \infty. \tag{A73}$$

Thus,

$$\limsup_{\lambda t\to\infty} \frac{1}{\lambda t}\int_{q+yq^{1/2}}^{\lambda t} F^{-1}\left(1-\frac{v}{\lambda t}\right) Q(q,v)dv < \frac{E[X]}{2}\mathrm{erfc}(2^{-\frac{1}{2}}y) \to 0 \text{ as } y \to \infty. \tag{A74}$$

Combining (A69), (A71) and (A74) it now follows that the first integral on the righthand side of (A68) is $o(\lambda t)$. The fact that the second integral on the righthand side of (A68) is $o(\lambda t)$ follows in a similar manner. The details are omitted for the sake of brevity. □

A2.4. Large Market Approximation of Optimal Offline Value Function

PROPOSITION A6. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions* (16)-(17)*, then for each $q \in \mathbb{R}_+$ as $\lambda t \to \infty$,*

$$J^{\star}_{\mathrm{OFF}}(q, \lambda t) = qF^{-1}\left(1 - \frac{e^{C_q/q}}{\lambda t}\right) + o(a(\lambda t)), \tag{A75}$$

where

$$C_q = \int_0^{\infty} \ln(v) Q(q, v) dv. \tag{A76}$$

Proof of Proposition A6 It suffices to prove the result for the case of $\lambda = 1$. Recalling by (18) the definition of the norming function b, it follows from (20) that for each $q \in \mathbb{R}_+$ and $t \geq 0$,

$$J^{\star}_{\mathrm{OFF}}(q, t) = \int_0^t b\left(\frac{t}{v}\right) \cdot Q(q, v) \cdot dv. \tag{A77}$$

Next, integrating-by-parts it follows that

$$\int_0^t Q(q, v) \cdot dv = q + tQ(q, t) - qQ(q + 1, t), \tag{A78}$$

and so by (A77) we obtain that

$$J^{\star}_{\mathrm{OFF}}(q, t) = qb(t) + \int_0^t \left(b\left(\frac{t}{v}\right) - b(t)\right) \cdot Q(q, v) \cdot dv + b(t)(tQ(q, t) - qQ(q + 1, t)). \tag{A79}$$

Now letting $0 < \varepsilon < 1$, it follows after some algebra that we may write

$$J^{\star}_{\mathrm{OFF}}(q, t) = qb(t) - a(t) \int_0^{\infty} \ln(v) Q(q, v) dv \tag{A80}$$

$$+ a(t) \int_{t^{\varepsilon}}^{\infty} \ln(v) Q(q, v) dv + \int_0^{t^{\varepsilon}} \beta(t, 1/v) Q(q, v) dv \tag{A81}$$

$$+ \int_{t^{\varepsilon}}^t \left(b\left(\frac{t}{v}\right) - b(t)\right) \cdot Q(q, v) \cdot dv + b(t)(tQ(q, t) - qQ(q + 1, t)), \tag{A82}$$

where $\beta(t, 1/v) = b(t/v) - b(t) - \ln(1/v)a(t)$. We now study the asymptotics as $t \to \infty$ of each of the terms on the righthand side above.

Regarding the expression on the righthand side of (A80), recall by Proposition 0.10 of Resnick (2013) that

$$\lim_{t \to \infty} \frac{b(ct) - b(t)}{a(t)} = \ln c \text{ for } c \in \mathbb{R}. \tag{A83}$$

It then follows by (A76) that

$$qb(t) - a(t) \int_0^{\infty} \ln(v) Q(q, v) dv = qF^{-1}\left(1 - \frac{e^{C_q/q}}{\lambda t}\right) + o(a(t)) \text{ as } t \to \infty. \tag{A84}$$

Hence, in order to complete the proof it remains to show that the remaining terms in (A81) and (A82) are $o(a(t))$ as $t \to \infty$.

Consider first the first term on the righthand side of (A81). Using the inequalities of Natalini and Palumbo (2000) for the incomplete gamma function, it is straightforward to show that there exists a constant C such that $\ln(v)Q(q, v) < Ce^{-v/2}$ for v sufficiently large. This then clearly implies that

$$a(t) \int_{t^{\varepsilon}}^{\infty} \ln(v) Q(q, v) dv = o(a(t)) \text{ as } t \to \infty. \tag{A85}$$

We next proceed to the second term on the righthand side of (A81). By (A83), it follows that for each $v > 0$, $\beta(t, 1/v)/a(t) \to 0$ as $t \to \infty$. Thus, in order to show that

$$\frac{1}{a(t)} \int_0^{t^\varepsilon} \beta(t, 1/v) Q(q, v) dv \to 0 \text{ as } t \to \infty, \tag{A86}$$

it suffices by the dominated convergence theorem to bound $1\{v < t^\varepsilon\}\beta(t, 1/v)Q(q, v)/a(t)$ by a function integrable in v for large t. We proceed as follows.

Using the von-Mises condition (16), it is straightforward to show that $b'(t) = a(t)/t$ for t sufficiently large. It then follows that for t sufficiently large and $0 < v < t^\varepsilon$,

$$\frac{\beta(t, 1/v)}{a(t)} = \int_1^{1/v} \frac{a(tz)/a(t)}{z} dz + \ln(v). \tag{A87}$$

Now let $0 < \delta < 1$. Then, since a is slowly varying, $a(t/v)/a(t) \le (1/v)^\delta$ for t sufficiently large and $0 < v \le 1$. Hence, for t sufficiently large and $0 < v \le 1$,

$$\int_1^{1/v} \frac{a(tz)/a(t)}{z} dz < \int_1^{1/v} z^{-(1-\delta)} dz = \frac{1}{\delta}(v^{-\delta} - 1). \tag{A88}$$

By (A87) and (A88), it follows that for t sufficiently large and $0 < v \le 1$,

$$\left| \frac{\beta(t, 1/v)}{a(t)} \right| < \frac{1}{\delta}(v^{-\delta} - 1) + \ln(1/v). \tag{A89}$$

(A89) and the fact that $Q(q, v) \to 1$ as $v \to 0$, now provides an integrable bound on $\beta(t, 1/v)Q(q, v)/a(t)$ for t sufficiently large and $0 < v \le 1$. In a similar manner, for t sufficiently large and $1 < v < t^\varepsilon$ one has the bound

$$\left| \frac{\beta(t, 1/v)}{a(t)} \right| < \frac{1}{\delta}(v^{\delta} - 1) + \ln(v). \tag{A90}$$

Since by Natalini and Palumbo (2000) there exists a constant C such that $Q(q, v) < Ce^{-v/2}$ for $v > 0$, (A90) now provides an integrable bound on $1\{v < t^\varepsilon\}\beta(t, 1/v)Q(q, v)/a(t)$ for t sufficiently large and $v > 1$. It now follows that (A86) holds.

Next, regarding the first term in (A82), since $\varepsilon < 1$ and b is non-decreasing and $Q(\cdot, \cdot)$ is strictly decreasing, it follows that

$$\int_{t^\varepsilon}^{t} \left(b\left(\frac{t}{v}\right) - b(t) \right) \cdot Q(q, v) \cdot dv < 2 \cdot t \cdot b(t) \cdot Q(q, t^\varepsilon). \tag{A91}$$

Now note by Exercise 0.4.3.1 of Resnick (2013) that b is slowly varying. Thus, there exists Resnick (2013) a constant κ such that $b(t) < \kappa t^\varepsilon$ for $t > 1$. Moreover, recall by Natalini and Palumbo (2000) there exists a constant C such that $Q(q, x) < Ce^{-x/2}$ for $x > 0$. It follows there exists a constant η such that for $t > 1$, $2 \cdot t \cdot b(t) \cdot Q(q, t^\varepsilon) < \eta t^\varepsilon e^{-t^\varepsilon/2} = o(a(t))$ as $t \to \infty$, where the final equality holds since a is slowly varying.

In order to complete the proof, it now suffices to show that the second term in (A82) is $o(a(t))$. Since b is slowly varying, there exists a constant κ such that $b(t) < \kappa t^\varepsilon$ for $t > 1$. Moreover, there exists a constant C such that $Q(q, x) < Ce^{-x/2}$ for $x > 0$. It therefore follows that there exists a constant η such that for $t > 1$,

$$b(t)(tQ(q, t) - qQ(q + 1, t)) < \eta t^{1+\varepsilon} e^{-t^\varepsilon/2} = o(a(t)) \text{ as } t \to \infty, \tag{A92}$$

where the final equality holds since a is slowly varying. $\square$

When $q \in \mathbb{N}_+$, the expression above for C_q may be simplified. Let H_{m-1} denote the $(m-1)$st harmonic number with $H_0 = 0$, and set $\gamma \approx 0.57722$ to be the Euler-Mascheroni constant.

Lemma A3. *If $q \in \mathbb{N}_+$, then*

$$C_q = \sum_{m=1}^{q} (H_{m-1} - \gamma). \tag{A93}$$

Proof of Lemma A3 Recall (Olver et al. 2010) first that for $q \in \mathbb{N}_+$ we have the representation

$$Q(q,v) = e^{-v} \sum_{m=1}^{q} \frac{v^{m-1}}{(m-1)!} \quad \text{for } v \in \mathbb{R}_+. \tag{A94}$$

It then follows that

$$C_q = \int_0^\infty \ln(v) Q(q,v) dv = \sum_{m=1}^{q} \frac{1}{(m-1)!} \int_0^\infty e^{-v} v^{m-1} \ln(v) dv. \tag{A95}$$

On the other hand, for $m \in \mathbb{N}_+$ the digamma function Ψ has (Olver et al. 2010) the representation

$$\Psi(m) = \frac{1}{(m-1)!} \int_0^\infty e^{-v} v^{m-1} \ln(t) dt, \tag{A96}$$

and so substituting into (A95) yields

$$C_q = \sum_{m=1}^{q} \Psi(m). \tag{A97}$$

Finally, recall (Olver et al. 2010) that $\Psi(m) = H_{m-1} - \gamma$ for $m \in \mathbb{N}_+$. $\square$

C. Optimal Static Pricing Policy

Proof of Proposition 6 It suffices to prove the result for the case of $\lambda = 1$. Fix $q \in \mathbb{N}_+$ and let $t \geq 0$. Also set $\Lambda_t(p) = t(1 - F(p))$ for $p \in \mathbb{R}_+$. After some algebra, it may then be shown that

$$J^p_{\text{STAT}}(q,t) = p(q - qQ(q, \Lambda_t(p)) + \Lambda_t(p) Q(q-1, \Lambda_t(p))), \tag{A98}$$

where $Q(\cdot,\cdot) = \Gamma(\cdot,\cdot)/\Gamma(\cdot)$ is the regularized upper incomplete gamma function (Olver et al. 2010).

Now consider a sequence of static pricing policies such that $p(q,t)/b(t) \to c$ as $t \to \infty$ where $c < 1$. Then, for t large,

$$\Lambda_t(p(q,t)) = t(1 - F(p(q,t))) = \frac{1 - F(p(q,t))}{1 - F(b(t))} \to \infty \text{ as } t \to \infty, \tag{A99}$$

where the final convergence is a consequence of Proposition 1.1 and Exercise 0.4.3.1 of Resnick (2013). Recalling (Olver et al. 2010) the asymptotic $Q(q,x)/(x^{q-1}e^{-x}) \to 1/\Gamma(q)$ as $x \to \infty$, it now follows that

$$qQ(q, \Lambda_t(p)) + \Lambda_t(p) Q(q-1, \Lambda_t(p)) \to 0 \text{ as } t \to \infty. \tag{A100}$$

This implies by (A98) that $J^p_{\text{STAT}}(q,t)/b(t) \to cq$ as $t \to \infty$. Since by assumption $c < 1$, we then have that

$$\liminf_{t\to\infty} p^\star_{\text{STAT}}(q,t)/b(t) \geq 1 \text{ and } \liminf_{t\to\infty} J^\star_{\text{STAT}}(q,t)/b(t) \geq q. \tag{A101}$$

In order to complete the proof, it suffices to show that

$$\limsup_{t\to\infty} p^\star_{\text{STAT}}(q,t)/b(t) \leq 1 \text{ and } \limsup_{t\to\infty} J^\star_{\text{STAT}}(q,t)/b(t) \leq q. \tag{A102}$$

Consider a sequence of static pricing policies such that $p(q,t)/b(t) \to c$ as $t \to \infty$ where $c > 1$. Then, in a similar manner to (A99), it follows that $\Lambda_t(p(q,t)) \to 0$ as $t \to \infty$. Noting that $Q(q,x) \to 1$ as $x \to 0$, we then have by (A98) that $J^p_{\text{STAT}}(q,t)/b(t) \to 0$ as $t \to \infty$. Since by assumption $c > 1$, it now follows by (A101) that (A102) holds. $\square$

Next, we have the following.

Proposition A7. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions* (16)-(17)*, then for each $q\in\mathbb{N}_+$,*

$$\lim_{\lambda t\to\infty}\frac{J^\star_{\text{STAT}}(q,\lambda t)-qb(\lambda t)}{a(\lambda t)} = \lim_{\lambda t\to\infty}\frac{p^\star_{\text{STAT}}(q,\lambda t)-b(\lambda t)}{a(\lambda t)} = -\infty. \tag{A103}$$

Proof of Proposition A7 It suffices to prove the result for the case of $\lambda=1$. Fix $q\in\mathbb{N}_+$ and let $t\geq 0$ and recall from (A98) that for $p\in\mathbb{R}_+$,

$$J^p_{\text{STAT}}(q,t) = p(q-qQ(q,\Lambda_t(p))+\Lambda_t(p)Q(q-1,\Lambda_t(p))), \tag{A104}$$

where $\Lambda_t(p)=t(1-F(p))$. Next, one may calculate that $\partial Q(q,x)/\partial x=-x^{q-1}e^{-x}/\Gamma(q)$ for $x>0$. It is then straightforward to verify that the function $q-qQ(q,x)+xQ(q-1,x)$ is strictly increasing in x. In particular, using the asymptotics of $Q(q,\cdot)$ from the proof of Proposition 6, $q-qQ(q,x)+xQ(q-1,x)\to 0$ as $x\to 0$ and $q-qQ(q,x)+xQ(q-1,x)\to q$ as $x\to\infty$. By (30) of Proposition 6 it thus must be the case that

$$\Lambda_t(p^\star_{\text{STAT}}(q,t)) \to \infty \text{ as } t\to\infty. \tag{A105}$$

Now suppose that

$$\lim_{t\to\infty}\frac{p^\star_{\text{STAT}}(q,t)-b(t)}{a(t)} = c\in\mathbb{R}. \tag{A106}$$

Since F is by assumption in the Gumbel domain of attraction, it then follows by Proposition 1.1 of Resnick (2013) that $\Lambda_t(p^\star_{\text{STAT}}(q,t))\to e^{-c}$ as $t\to\infty$. However, this is not possible by (A105) and so since $\Lambda_t(p)$ is decreasing in p, it must be the case that the second limit in (A106) holds. The first limit in (A106) then follows by the inequality $J^\star_{\text{STAT}}(q,t)\leq qp^\star_{\text{STAT}}(q,t)$. □

D. Euler-Lagrange Equations

Recall from Section 3 that for each admissible pricing policy $p\in\mathcal{V}$, we may define an admissible purchasing probability policy by setting $\pi=1-F(p)$. It then follows in a straightforward manner that the value function for each purchasing probability policy π is given by

$$J_\pi(\lambda;q,t) = E\left[\lambda\int_0^t \pi(\lambda;Q_s,t-s)F^{-1}(1-\pi(\lambda;Q_s,t-s))ds|Q_0=q\right], \tag{A107}$$

where

$$Q_s = Q_0 - N\left(\lambda\int_0^s \pi(\lambda;Q_u,t-u)du\right) \text{ for } 0\leq s\leq t. \tag{A108}$$

Since by assumption F has a positive density on the entirety of its support, the mapping from pricing policies to purchasing probability policies is invertible. It turns out that the set of admissible purchasing probability policies may also be characterized as follows. A $[0,1]$-valued purchasing probability policy $\pi=\{\pi_s, 0\leq s\leq t\}$ on $(\Omega,\mathcal{F},P)$ is said to be an admissible purchasing probability policy if the following 2 conditions are satisfied.

I. For each $0\leq s\leq t$, $\pi_s=\pi(\lambda;Q_s,t-s)$, where Q_s is given by (A108) above.

II. The family of functions $\{\pi(\lambda;q,\cdot),q\in\mathbb{N}\}$ are measurable, with $\pi(\lambda;0,\cdot)=0$.

We denote the set of admissible purchasing probability policies by $\mathcal{W}$.

This section derives a set of optimality conditions for the purchasing probabilities of the firm that differ significantly from the optimality conditions in Section 3. The conditions we obtain follow from a calculus of variations (Gelfand et al. 2000) reformulation of the control problem (3). In particular, they consist of a system of differential equations commonly referred to as the Euler-Lagrange equations in the calculus of variations literature. This system of equations is apparently new for the single-item revenue management problem and simplifies the proofs of our main results.

Now note that the optimization problem (3) may be rewritten in terms of purchasing probabilities as

$$J^\star(\lambda;q,t) = \sup_{\pi\in\mathcal{W}} J_\pi(\lambda;q,t). \tag{A109}$$

We next show how to transform (A109) into a calculus of variations problem. This is at a high level achieved by decomposing expression (A107) for the expected revenue of the firm into the expected revenue obtained by the first sale after time 0, and the expected revenue obtained thereafter. Specifically, for each $\pi\in\mathcal{V}$, let $\tau=\inf\{s\geq 0: Q_s<Q_0\}$ be the time of the first sale after time 0 under the policy π. It then follows by known results (Ross 2014) on the thinning of Poisson processes that for each $\pi\in\mathcal{W}$ and $q\in\mathbb{N}$,

$$P(\tau>s|Q_0=q) = \exp\left(-\lambda\int_0^s \pi(q,t-u)du\right) \text{ for } 0\leq s<t. \tag{A110}$$

Hence, by Fubini's theorem (Lieb and Loss 2001),

$$\begin{aligned} &E\left[\int_0^\tau \pi(Q_s,t-s)F^{-1}(1-\pi(Q_s,t-s))ds|Q_0=q\right] \\ &= \int_0^t \exp\left(-\lambda\int_0^s \pi(q,t-u)du\right)\pi(q,t-s)F^{-1}(1-\pi(q,t-s))ds. \end{aligned} \tag{A111}$$

On the other hand,

$$E\left[\int_\tau^t \pi(Q_s,t-s)F^{-1}(1-\pi(Q_s,t-s))ds|\tau,Q_0=q\right] = J_\pi(q-1,t-\tau)/\lambda,$$

and so by the tower property of conditional expectation (Chung 2001) and (A110),

$$\begin{aligned} &E\left[\int_\tau^t \pi(Q_s,t-s)F^{-1}(1-\pi(Q_s,t-s))ds|Q_0=q\right] \\ &= \int_0^t \exp\left(-\lambda\int_0^s \pi(q,t-u)du\right)\pi(q,t-s)J_\pi(q-1,t-s)ds. \end{aligned} \tag{A112}$$

Placing (A107), (A111) and (A112) together, we arrive at the following result.

PROPOSITION A8. *For each $\pi\in\mathcal{W}$, the family of value functions $\{J_\pi(\lambda;q,\cdot),q\in\mathbb{N}\}$ are the unique solution to*

$$J_\pi(\lambda;q,t) = \int_0^t L_q(\Pi(\lambda;q,s,t),\pi(q,s),s)ds \text{ for } t\geq 0,\ q\in\mathbb{N}_+, \tag{A113}$$

where

$$\Pi(q,s,t) = -\int_s^t \pi(q,u)du \quad\text{for}\quad s\geq 0, \tag{A114}$$

and $L_q:(\mathbb{R}_+;\mathbb{R},[0,1],\mathbb{R}_+)\mapsto\mathbb{R}$ is the Lagrangian given by

$$L_q(\lambda;\Pi,\pi,s) = \lambda\exp(\lambda\Pi)\pi(F^{-1}(1-\pi)+J_\pi(q-1,s)) \text{ for } q\in\mathbb{N}_+,$$

together with $J_\pi(0,\cdot)=0$.

For each $\pi \in \mathcal{W}$, $q \in \mathbb{N}_+$ and $t \geq 0$, the function $\Pi(q,\cdot,t)$ defined in (A114) is absolutely continuous on $\mathbb{R}_+$ with a density given by $\pi(q,\cdot)$. Hence, letting $\mathcal{G}$ denote the set of functions Π that are absolutely continuous on $\mathbb{R}_+$ with a density $\pi \in [0,1]$, it follows by (3), Proposition A8 and the fundamental theorem of Lebesgue calculus (Athreya and Lahiri 2006) that we have the following result.

Theorem A1. *For each $q \in \mathbb{N}_+$ and $t \geq 0$, $J^\star(\lambda; q, t)$ is equal to the optimal value of the optimization problem*

$$\begin{aligned} \sup_{\Pi \in \mathcal{G}} \ & \int_0^t L_q^\star(\lambda; \Pi_s, \pi_s, s) ds \\ \text{subject to} \ & \Pi_t = 0, \end{aligned} \tag{A115}$$

where $L_q^\star : (\mathbb{R}_+; \mathbb{R}, [0,1], \mathbb{R}_+) \mapsto \mathbb{R}$ is the Lagrangian given by

$$L_q^\star(\lambda; \Pi, \pi, s) = \lambda \exp(\lambda \Pi) \pi (F^{-1}(1-\pi) + J^\star(q-1, s)) \text{ for } q \in \mathbb{N}_+. \tag{A116}$$

Moreover, each optimal solution $\{\Pi^\star(\lambda; q, \cdot), q \in \mathbb{N}_+\}$ to (A115) corresponds to an optimal solution $\{\pi^\star(\lambda; q, \cdot), q \in \mathbb{N}\}$ to (3) via $\pi^\star(\lambda; 0, \cdot) = 0$ together with $\pi^\star(\lambda; q, s) = \partial \Pi^\star(\lambda; q, s)/\partial s$ for a.e. $s \in [0,t]$ for $q \in \mathbb{N}_+$, and this relationship holds vice-versa as well.

The optimization problem (A115) is a calculus of variations problem where the left endpoint is variable and the right endpoint is fixed. We next provide a sufficient condition on the item valuation distribution F such that the optimal solution to (A115) is unique for each $q \in \mathbb{N}_+$, and we provide a corresponding system of Euler-Lagrange equations that the family of optimal solutions satisfies. Let $\pi_0^\star$ be the maximizer of $\pi F^{-1}(1-\pi)$ over $\pi \in [0,1]$. We then have the following result whose proof may be found in Section E of the appendix.

Theorem A2. *If F is differentiable on $\mathbb{R}_+$ with a positive density f, and $p = (1-F)^2/f$ is decreasing and absolutely continuous on $[0, x_U)$, then there exists a unique solution $\{\Pi^\star(\lambda; q, \cdot), q \in \mathbb{N}_+\}$ to (A115) and each $\{\pi^\star(\lambda; q, \cdot), q \in \mathbb{N}_+\}$ is absolutely continuous and satisfies the system of equations*

$$\begin{aligned} & \frac{\partial \pi^\star(\lambda; q, s)}{\partial s} \\ & = \frac{\pi^\star(\lambda; q, s)}{p'(F^{-1}(1-\pi^\star(\lambda; q, s)))} \left(\lambda (\pi^\star(\lambda; q, s))^2 - f(F^{-1}(1-\pi^\star(\lambda; q, s))) \cdot \frac{\partial J^\star(\lambda; q-1, s)}{\partial s} \right), \end{aligned} \tag{A117}$$

for each $q \in \mathbb{N}_+$ and a.e. $0 \leq s \leq t$, with initial condition $\pi^\star(\lambda; q, 0) = \pi_0^\star$, which is unique.

Note that $J^\star(q,\cdot)$ may be recovered from $\pi^\star(q,\cdot)$ using the identity (A113), and hence the system of equations (A117) may be solved for recursively starting from $q = 1$. Next, the assumption in Theorem A2 that $(1-F)^2/f$ is decreasing is the same that Bitran and Mondschein (1997) used to show the uniqueness of the optimal solution $\pi^\star$ to (3). This then implies the uniqueness portion of Theorem A2. We now apply Theorem A2 to the case in which the customer item valuations are exponentially distributed.

Example 5 (The Exponential Distribution). Consider the case in which F is exponentially distributed with a mean of $1/\mu > 0$. In this case, $(1-F(x))^2/f(x) = \exp(-\mu x)/\mu$ for $x \in \mathbb{R}_+$, which is clearly decreasing. Moreover, $F^{-1}(\pi) = -\ln(1-\pi)/\mu$ for $\pi \in [0,1)$ and so the Euler-Lagrange equations (A117) simplify to

$$\frac{\partial \pi^\star(\lambda;q,s)}{\partial s} = -\left(\lambda(\pi^\star(\lambda;q,s))^2 - \mu \cdot \frac{\partial J^\star(\lambda;q-1,s)}{\partial s} \cdot \pi^\star(\lambda;q,s)\right), \tag{A118}$$

for each $q \in \mathbb{N}_+$ and a.e. $s \geq 0$. On the other hand, in this case it is well known (Gallego and Van Ryzin 1994) that

$$J^\star(\lambda;q,s) = \frac{1}{\mu}\ln\left(e^{\lambda s/e}\frac{\Gamma(q+1,\lambda s/e)}{\Gamma(q+1)}\right) \text{ and } \pi^\star(\lambda;q,s) = \frac{q}{e}\cdot\left(\frac{\Gamma(q,\lambda s/e)}{\Gamma(q+1,\lambda s/e)}\right),$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function. Then, using the fact that

$$\frac{\partial \Gamma(s,x)}{\partial x} = -x^{s-1}e^{-x} \text{ for } s,x > 0,$$

it is straightforward to verify by substitution that (A118) holds.

E. Proof of Euler-Lagrange Equations

Proof of Theorem A2. In order to ease the burden on the notation, we suppress λ as a parameter in $\Pi^\star, \pi^\star, J^\star$ and $L^\star$. We first establish that for fixed $t \geq 0$, under the assumptions of the theorem there exists a unique optimal solution $\{\Pi^\star(q,\cdot), q \in \mathbb{N}_+\}$ on $[0,t]$ to (A115). By Theorem A1, it suffices to show that up to a.e. equivalence there exists a unique optimal solution $\{\pi^\star(q,\cdot), q \in \mathbb{N}\}$ on $[0,t]$ to (3), with $\pi^\star(0,\cdot) = 0$. We proceed as follows. Using Theorem 1 of Zhao and Zheng (2000) it is straightforward to verify that for each $q \in \mathbb{N}_+$, $\Delta J^\star(q,\cdot) = J^\star(q,\cdot) - J^\star(q-1,\cdot) < x_U$. Thus, since F has a finite mean, it follows from the HJB equation (4) that for a.e. $s \geq 0$, $\pi^\star(q,s)$ must satisfy the first order condition

$$F^{-1}(1-\pi^\star(q,s)) - \Delta J^\star(q,s) = u(F^{-1}(1-\pi^\star(q,s))), \tag{A119}$$

where u is the Mills ratio $(1-F)/f$. Next, since $(1-F)^2/f$ is by assumption absolutely continuous on $[0,x_U)$, and $1-F$ is differentiable on $[0,x_U)$, it follows that u is absolutely continuous on $[0,x_U)$ as well. Then, since by assumption $(1-F)^2/f$ is strictly decreasing on $[0,x_U)$, it must be the case that $u^{'}(x) < 1$ for a.e. $x \in [0,x_U)$. It is then straightforward to show that for each $s \geq 0$, there exists a unique $\pi^\star(q,s)$ satisfying (A119). This proves that $\pi^\star(q,\cdot)$ is up to a.e. equivalence unique on $[0,t]$.

Now let $q \in \mathbb{N}_+$ and let $\pi^\star(q,\cdot)$ be such that (A119) is satisfied for every $s \geq 0$. By the preceding it is clear that $\pi^\star(q,\cdot)$ is unique and we now show that $\pi^\star(q,\cdot)$ is absolutely continuous, decreasing and positive as well. First note that since $u^{'}(x) < 1$ for a.e. $x \in [0,x_U)$, it follows that $x - u(x)$ is strictly increasing on $[0,x_U)$ and so we may define its continuous, strtictly increasing inverse function h^{-1} on $[-u(0), x_U - u(x_U))$. Thus, by (A119) we may write $\pi^\star(q,t) = 1 - F(h^{-1}(\Delta J^\star(q,t)))$ for $t \geq 0$. Moreover, since $x - u(x)$ is absolutely continuous with a positive derivative for a.e. $x \in [0,x_U)$, it follows (Royden and Fitzpatrick 1968) that its inverse h^{-1} is absolutely continuous. Finally, since using the HJB equation (4) and Theorem 1 of Zhao and Zheng (2000) it is straightforward to show that $\Delta J^\star(q,\cdot)$ is differentiable and non-decreasing, it follows from the Banach-Zarecki theorem (Yeh 2000) that $\pi^\star(q,t) = 1 - F(h^{-1}(\Delta J^\star(q,t)))$ is absolutely continuous. Finally, the fact that $\pi^\star(q,\cdot)$ is decreasing follows from Theorem 4 of Zhao and Zheng (2000) and since $\Delta J^\star(q,\cdot) < x_U$, it is straightforward to show that $\pi^\star(q,\cdot) > 0$ too.

We now mostly follow the proof of Theorem 1 of (Gelfand et al. 2000). Let $q\in\mathbb{N}_+$. Since by the discussion above the family of functions $\{\pi^\star(q,\cdot), q\in\mathbb{N}_+\}$ are positive, it follows that in (A115) we may optimize over the slightly smaller set of functions Π that are absolutely continuous on $\mathbb{R}_+$ with a density $\pi\in(0,1]$. Now note by (A116) that for all $(\Pi,\pi,s)\in(\mathbb{R},(0,1],\mathbb{R}_+)$,

$$\frac{\partial L_q^\star}{\partial \Pi} = \lambda L_q^\star \text{ and } \frac{\partial L_q^\star}{\partial \pi} = \exp(\lambda\Pi)\left(F^{-1}(1-\pi)+J^\star(q-1,s)-a(1/\pi)\right), \tag{A120}$$

both of which are continuous on $(\Pi,\pi,s)\in(\mathbb{R},(0,1],\mathbb{R}_+)$, and

$$\frac{\partial^2 L_q^\star}{\partial \Pi^2} = \lambda^2 L_q^\star \text{ and } \frac{\partial^2 L_q^\star}{\partial \Pi\partial\pi} = \frac{\partial^2 L_q^\star}{\partial \pi\partial\Pi} = \lambda\frac{\partial L_q^\star}{\partial \pi}.$$

Moreover, since $a=u\circ b$ is absolutely continuous with $a'(t)=u'(b(t))/(t^2 f(b(t)))$ for a.e. $t\ge 1$, it follows that $\partial L_q^\star/\partial\pi$ is absolutely continuous with respect to π for $(\Pi,\pi,s)\in(\mathbb{R},(0,1],\mathbb{R}_+)$, and after some algebra that for $(\Pi,\pi,s)\in(\mathbb{R},(0,1]\setminus\mathcal{N},\mathbb{R}_+)$, where $\mathcal{N}$ is a set of measure zero,

$$\frac{\partial^2 L_q^\star}{\partial \pi^2} = \frac{\exp(\lambda\Pi)}{f(F^{-1}(1-\pi))}\left(u'(F^{-1}(1-\pi))-1\right). \tag{A121}$$

Moreover, since by assumption f is strictly positive and continuous, the above is bounded on compact subsets of $(\mathbb{R},(0,1]\setminus\mathcal{N},\mathbb{R}_+)$.

Now let h be continuously differentiable on $[0,t]$ and such that $h(t)=0$. It then follows that for $|\varepsilon|$ sufficiently small, $(\Pi^\star(q,\cdot)+\varepsilon h, \pi^\star(q,\cdot)+\varepsilon h',\cdot)$ lies in a compact subset of $(\mathbb{R},(0,1],\mathbb{R}_+)$. Then, by (A120)-(A121) and proceeding as in (Gelfand et al. 2000) it follows that $\Pi^\star(q,\cdot)$ satisfies

$$\left.\frac{\partial L_q^\star}{\partial \pi}\right|_{s=0}=0 \quad \text{and} \quad \frac{\partial L_q^\star}{\partial \Pi}-\frac{d}{ds}\frac{\partial L_q^\star}{\partial \pi} = 0 \text{ for } 0\le s\le t.$$

Using (A119) and the fact that $J^\star(q,0)=\Delta J^\star(q,0)=0$, it is straightforward to verify that the boundary condition above at $s=0$ is satisfied by our assumption that $\pi^\star(q,0)$ maximizes $\pi F^{-1}(1-\pi)$. Next, since $\pi^\star(q,\cdot)$ is absolutely continuous on $\mathbb{R}_+$, it follows after some algebra that for a.e. $s\in[0,t]$,

$$\begin{aligned}&\frac{d}{ds}\frac{\partial L_q^\star}{\partial \pi}-\lambda\pi^\star(q,s)\frac{\partial L_q^\star}{\partial \pi}\\ &=\exp(\lambda\Pi^\star(q,s))\left(-\frac{(\pi^\star(q,s))'}{f(F^{-1}(1-\pi^\star(q,s)))}+\frac{\partial J^\star(q-1,s)}{\partial s}+(\pi^\star(q,s))'\frac{a'(1/\pi^\star(q,s))}{(\pi^\star(q,s))^2}\right).\end{aligned}$$

Then, using the fact that $\partial L_q^\star/\partial\Pi=\lambda L_q^\star$ and setting $s=t$, we obtain (A117). □

F. Proof of Optimal Dynamic Pricing Policies

We next prove Proposition A9 below, which by Proposition 0.10 of Resnick (2013) (see also (A83) above) implies Theorem 2 in a straightforward manner. For each $q\in\mathbb{N}_+$ and $t>0$, define the centered and scaled optimal value function

$$\tilde{J}^\star(q,\lambda t) = \frac{J^\star(q,\lambda t)-qb(\lambda t)}{a(\lambda t)},$$

centered and scaled optimal selling price

$$\tilde{p}^\star(q,\lambda t) = \frac{p^\star(q,\lambda t)-b(\lambda t)}{a(\lambda t)},$$

and scaled optimal purchasing probability

$$\tilde{\pi}^\star(q,\lambda t) = \lambda t\pi^\star(q,\lambda t).$$

We then have the following.

PROPOSITION A9. *If F is in the Gumbel domain of attraction and satisfies the von-Mises conditions* (16)-(17)*, then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t \to \infty} \tilde{J}^\star(q, \lambda t) = -(q + \ln(q!)) \tag{A122}$$

and

$$\lim_{\lambda t \to \infty} \tilde{p}^\star(q, \lambda t) = -\ln(q) \tag{A123}$$

and

$$\lim_{\lambda t \to \infty} \tilde{\pi}^\star(q, \lambda t) = q. \tag{A124}$$

Proof of Proposition A9. It suffices to prove the result for the case of $\lambda = 1$. Next, assume that $x_0 = 0$ in (16) and that $u^{'}(x) < 1$ for a.e. $x \in [0, x_U)$. This then implies that $(1-F)^2/f$ is decreasing for $x \geq 0$ and so Theorem A2 may be applied. In the more general case in which $u^{'}(x) < 1$ for a.e. x sufficiently large (which is guaranteed by (17)), the proof may be modified accordingly.

Now note that for each $q \in \mathbb{N}_+$, (A117) is a Bernoulli ODE and making the substitution $v^\star(q,\cdot) = 1/\pi^\star(q,\cdot)$, we obtain that

$$(v^\star(q,t))^{'} + \frac{h_1(q,t)}{t} v^\star(q,t) = h_2(q,t) \;\text{ for a.e. }\; t \geq 0, \tag{A125}$$

where

$$h_1(q,t) = \frac{1}{1 - u^{'}(b(1/\pi^\star(q,t)))} \cdot \frac{t}{a(1/\pi^\star(q,t))} \cdot \frac{\partial J^\star(q-1,t)}{\partial t} \text{ and } h_2(q,t) = \frac{1}{1 - u^{'}(b(1/\pi^\star(q,t)))}.$$

Moreover, it is straightforward to verify by Theorem I.5.3 of (Hale 1980) that the solution to (A125) is unique. Also note that using the HJB equation (4) together with the structural results in Zhao and Zheng (2000) it is straightforward to show that $\pi^\star(q,t) \downarrow 0$ as $t \to \infty$. Thus, since $b(t) \uparrow x_U$ as $t \to \infty$, and $u^{'}(x) \to 0$ as $x \to x_U$, it follows that $h_2(q,t) \to 1$ as $t \to \infty$.

Now let $q \geq 2$ and suppose that (A122)-(A124) holds for $\{1, ..., q-1\}$. We then show that

$$q - 1 \;\leq\; \liminf_{t \to \infty} \tilde{\pi}^\star(q,t) \leq \limsup_{t \to \infty} \tilde{\pi}^\star(q,t) \;\leq\; e^{1/(q-1)}(q-1). \tag{A126}$$

The lim inf result follows from the fact that by Theorem 3 of Zhao and Zheng (2000), for each $t \geq 0$, $p^\star(q,t) \leq p^\star(q-1,t)$, together with the assumption that (A123) holds for $\{1, ..., q-1\}$. To prove the lim sup result, first note that using Proposition A8 it is straightforward to show that

$$\Delta J^\star(q,t) \geq \int_0^t \exp\left(-\int_s^t \pi^\star(q-1,u)du\right) \pi^\star(q-1,s) \Delta J^\star(q-1,s) ds \;\text{ for }\; t \geq 0.$$

Next, integrating-by-parts the right-hand side above we obtain that

$$\Delta J^\star(q,t) \geq \Delta J^\star(q-1,t) - \int_0^t \exp\left(-\int_s^t \pi^\star(q-1,u)du\right) \frac{\partial \Delta J^\star(q-1,s)}{\partial s} ds. \tag{A127}$$

Now, by (A83) and the assumption that (A122) holds for $\{1, ..., q-1\}$, it follows that

$$\frac{\Delta J^\star(q-1,t) - b(t)}{a(t)} \to -1 - \ln(q-1) \text{ as } t \to \infty. \tag{A128}$$

Also note that since by assumption (A124) holds for $\{1,...,q-1\}$, it follows that $\tilde{\pi}(q-1,t) \to q-1$ as $t \to \infty$, and so by Karamata's representation theorem (Resnick 2013),

$$\exp\left(-\int_0^t \pi^\star(q-1,u)du\right) = L(t)/t^{q-1} \text{ for } t \geq 0,$$

where the function L is slowly varying. Then, the integral on the right-hand side of (A127) may be written as

$$\frac{L(t)}{t^{q-1}} \int_0^t \frac{a(s)}{L(s)} \cdot s^{q-2} \cdot \frac{s}{a(s)} \cdot \frac{\partial \Delta J^\star(q-1,s)}{\partial s} ds \text{ for } t \geq 0.$$

Using the assumption that (A122) and (A123) hold for $\{1,...,q-1\}$, together the HJB equation (4), it is straightforward to verify that

$$\frac{t}{a(t)} \cdot \frac{\partial \Delta J^\star(q-1,t)}{\partial s} \to 1 \text{ as } t \to \infty.$$

Thus, since the function a is slowly varying, it follows by Karamata's theorem that

$$\lim_{t\to\infty} \frac{1}{a(t)} \cdot \frac{L(t)}{t^{q-1}} \int_0^t \frac{a(s)}{L(s)} \cdot s^{q-2} \cdot \frac{s}{a(s)} \cdot \frac{\partial \Delta J^\star(q-1,s)}{\partial s} ds = \frac{1}{q-1}. \tag{A129}$$

Taken together, (A127), (A128) and (A129) together with the fact that by (4), $p^\star(q,t) \geq \Delta J^\star(q,t)$, now imply that

$$\liminf_{t\to\infty} \frac{p^\star(q,t) - b(t)}{a(t)} \geq -\ln(q-1) - \frac{1}{q-1}.$$

Then, by Proposition 0.10 of (Resnick 2013) this implies the lim sup result.

We now proceed by induction on $q \in \mathbb{N}$ in order to prove that (A122)-(A124) hold. We first show that for each $q \in \mathbb{N}$, $h_1(q,t) \to q-1$ as $t \to \infty$. In the base case of $q = 1$ this is clear since $J^\star(0,\cdot) = 0$. Next, suppose that $q \geq 2$. As noted previously, $1 - u^{'}(b(1/\pi^\star(q,t))) \to 1$ as $t \to \infty$. Next, since a is slowly varying it follows by (A126) that

$$\frac{a(t)}{a(1/\pi^\star(q,t))} \to 1 \text{ as } t \to \infty.$$

Moreover, using the induction hypothesis, (A122), (A123), and the HJB equation (4), it is straightforward to verify that

$$\frac{t}{a(t)} \cdot \frac{\partial J^\star(q-1,t)}{\partial s} \to q-1 \text{ as } t \to \infty. \tag{A130}$$

These three facts now imply that $h_1(q,t) \to q-1$ as $t \to \infty$, as desired.

We now prove (A124) for $q \in \mathbb{N}$. First note that since by the preceding $h_1(q,t) \to q-1$ as $t \to \infty$, it follows by Karamata's representation theorem that

$$\exp\left(\int_0^t \frac{h_1(s)}{s} ds\right) = L(t)t^{q-1} \text{ for } t \geq 0,$$

where L is a slowly varying function. It then follows that the unique solution to (A125) is given by

$$v^\star(q,t) = \frac{1}{L(t)t^{q-1}} v^\star(q,0) + \frac{1}{L(t)t^{q-1}} \int_0^t L(s)s^{q-1} h_2(q,s)ds \quad \text{for} \quad t \geq 0.$$

Thus, since $h_2(q,t) \to 1$ as $t \to \infty$, it follows by Karamata's theorem (Resnick 2013) that $qv^\star(q,t)/t \to 1$ as $t \to \infty$, which using the definition $v^\star(q,t) = 1/\pi^\star(q,t)$ proves (A124).

We next show that (A123) holds. Using the definitions of the centering function b and $\tilde{\pi}^\star(q,t)$, it follows that $p^\star(q,t) = F^{-1}(F(p^\star(q,t))) = b(t/\tilde{\pi}^\star(q,t))$. Thus, since b is non-decreasing and by the preceding, $\tilde{\pi}^\star(q,t) \to q$ as $t \to \infty$, it follows by (A83) that (A123) holds.

Finally, we show that (A122) holds. First recall by Proposition A8 that for $t \geq 0$,

$$J^\star(q,t) = \int_0^t \exp\left(-\int_s^t (\tilde{\pi}^\star(q,u)/u)du\right)(\tilde{\pi}^\star(q,s)/s)(b(s/\tilde{\pi}^\star(q,s)) + J^\star(q-1,s))ds. \tag{A131}$$

Next, note that since by assumption b is absolutely continuous and by Theorem A2, $\pi^\star(q,\cdot)$ is absolutely continuous, it follows by the monotonicity (Zhao and Zheng 2000) of $\pi^\star(q,\cdot)$ that $b(t/\tilde{\pi}^\star(q,t))$ is absolutely continuous (Royden and Fitzpatrick 1968), and so by the chain rule

$$\frac{db(t/\tilde{\pi}^\star(q,t))}{dt} = -\frac{\tilde{\pi}^\star(q,t)}{t} \cdot a(t/\tilde{\pi}^\star(q,t)) \cdot \frac{(\pi^\star(q,t))'}{(\pi^\star(q,t))^2} \text{ for a.e. } t \geq 0.$$

Thus, integrating-by-parts in (A131) and after some algebra we arrive at the fact that

$$\begin{aligned}\tilde{J}^\star(q,t) = {} & \frac{b(t/\tilde{\pi}^\star(q,t)) - b(t)}{a(t)} + \tilde{J}(q-1,t) \\ & -\frac{1}{a(t)}\exp\left(-\int_0^t (\tilde{\pi}^\star(q,u)/u)du\right)\int_0^t \exp\left(\int_0^s (\tilde{\pi}^\star(q,u)/u)du\right) \cdot \frac{a(s)}{s} \cdot \beta(s)ds,\end{aligned} \tag{A132}$$

for $t \geq 0$, where

$$\beta(s) = \left(\frac{s}{a(s)}\frac{\partial J^\star(q-1,s)}{\partial s} - \tilde{\pi}^\star(q,s) \cdot \frac{a(s/\tilde{\pi}^\star(q,s))}{a(s)} \cdot \frac{(\pi^\star(q,s))'}{(\pi^\star(q,s))^2}\right) \text{ for a.e. } s \geq 0.$$

Now note if $q \geq 2$, then by the induction hypothesis and (A122), and since b is non-decreasing and $\tilde{\pi}^\star(q,t) \to q$ as $t \to \infty$, it follows by (A83) that

$$\frac{b(t/\tilde{\pi}^\star(q,t)) - b(t)}{a(t)} + \tilde{J}(q-1,t) \to -\ln(q) - (q-1) \text{ as } t \to \infty. \tag{A133}$$

In the case case of $q = 1$, the above also holds since $J(0,\cdot) = 0$. Next, note that since $\tilde{\pi}^\star(q,t) \to q$ as $t \to \infty$, it follows by Karamata's representation theorem that

$$\exp\left(-\int_0^t (\tilde{\pi}^\star(q,u)/u)du\right) = L(t)/t^q \text{ for } t \geq 0,$$

where L is a slowly varying function. Thus, we may write

$$\begin{aligned}& \frac{1}{a(t)}\exp\left(-\int_{t_q}^t (\tilde{\pi}^\star(q,u)/u)du\right)\int_0^t \exp\left(\int_0^s (\tilde{\pi}^\star(q,u)/u)du\right) \cdot \frac{a(s)}{s} \cdot \beta(s)ds \\ = {} & \frac{1}{a(t)} \cdot \frac{L(t)}{t^q}\int_0^t \frac{s^{q-1}}{L(s)}a(s)\beta(s)ds.\end{aligned} \tag{A134}$$

Now note that $v^\star(q,t)/t = 1/\tilde{\pi}^\star(q,t) \to 1/q$ as $t \to \infty$. Hence, taking the limit as $t \to \infty$ in (A125), and recalling that $h_1(q,t) \to q-1$ and $h_2(q,t) \to 1$ as $t \to \infty$, it follows that $(v^\star(q,t))' \to -1/q$ as $t \to \infty$. Thus, since $(v^\star(q,t))' = (\pi^\star(q,t))'/(\pi^\star(q,t))^2$, together with (A130) and the fact that a is slowly varying, it follows that $\beta(t) \to q$ as $t \to \infty$. By Karamata's theorem, it now follows that

$$\frac{1}{a(t)} \cdot \frac{L(t)}{t^q}\int_0^t \frac{s^{q-1}}{L(s)}a(s)\beta(s)ds \to 1 \text{ as } t \to \infty,$$

which by (A132), (A133) and (A134) implies (A122). $\square$

G. Proof of Asymptotically Optimal Policies

Proof of Theorem 3 It suffices to prove (63) for the case of $\lambda = 1$. Let $p \in \mathcal{V}$ be the policy given by (64). It then follows by standard theory that for each $q \in \mathbb{N}_+$,

$$\frac{\partial J_p(1;q,t)}{\partial t} = \frac{q}{t}(F^{-1}(1-q/t) - J_p(1;q,t) + J_p(1;q-1,t)) \text{ for } t > q.$$

The above is a linear, first order ODE and its solution for $t > q$ is given by

$$J_p(1;q,t) - \left(\frac{q}{t}\right)^q J_p(1;q,q) = \frac{q}{t^q}\int_q^t s^{q-1}F^{-1}(1-q/s)ds + \frac{q}{t^q}\int_q^t s^{q-1}J_p(1;q-1,s)ds.$$

Integrating-by-parts the first term on the right-hand side above it follows that for $t > q$,

$$\begin{aligned} J_p(1;q,t) - qb(t) = {} & F^{-1}(1-q/t) - F^{-1}(1-1/t) \\ & -\left(\frac{q}{t}\right)^q (x_L - J_p(1;q,q)) \\ & -\frac{q}{t^q}\int_q^t s^{q-2}\frac{1}{f(F^{-1}(1-q/s))}ds \\ & +\frac{q}{t^q}\int_q^t s^{q-1}(J_p(1;q-1,s) - (q-1)b(s))ds \\ & +(q-1)\left(\frac{q}{t^q}\int_q^t s^{q-1}F^{-1}(1-1/s)ds - F^{-1}(1-1/t)\right). \end{aligned} \tag{A135}$$

We now proceed by induction on $q \in \mathbb{N}$. In the base case of $q = 0$, it is clear that (63) holds. Next, let $q \geq 1$ and suppose that (63) holds for $q-1$. Regarding the first term on the right-hand side above, by Proposition 0.10 in Resnick (2013),

$$\frac{F^{-1}(1-q/t) - F^{-1}(1-1/t)}{a(t)} \to -\ln q \text{ as } t \to \infty,$$

and, regarding the second term, since $q > 0$ and by Proposition 0.12 in Resnick (2013), a is slowly varying,

$$\frac{1}{a(t)}\left(\frac{q}{t}\right)^q (x_L - J_p(1;q,q)) \to 0 \text{ as } t \to \infty.$$

Next, regarding the third term above, note that

$$\int_q^t s^{q-2}\frac{1}{f(F^{-1}(1-q/s))}ds = \frac{1}{q}\int_q^t s^{q-1}a(s/q)ds.$$

Hence, since a is slowly varying, it follows by Karamata's theorem (Resnick (2013)) that

$$\frac{1}{a(t)}\frac{q}{t^q}\int_q^t s^{q-2}\frac{1}{f(F^{-1}(1-q/s))}ds \to \frac{1}{q} \text{ as } t \to \infty. \tag{A136}$$

Next, since a is a slowly varying function, it follows by Karamata's theorem and the induction hypothesis that

$$\frac{1}{a(t)}\frac{q}{t^q}\int_q^t s^{q-1}(J_p(1;q-1,s) - (q-1)b(s))ds \to -\ln\left(\prod_{k=1}^{q-1} k\right) - \sum_{k=1}^{q-1}\frac{k}{k} \text{ as } t \to \infty.$$

Now note that integrating-by-parts

$$\begin{aligned} \frac{1}{a(t)}\left(\frac{q}{t^q}\int_q^t s^{q-1}F^{-1}(1-1/s)ds - F^{-1}(1-1/t)\right) = {} & -\frac{1}{a(t)}\left(\frac{q}{t}\right)^q F^{-1}(1-1/q) \\ & -\frac{1}{a(t)}\frac{1}{t^q}\int_q^t s^{q-2}\frac{1}{f(F^{-1}(1-1/s))}ds. \end{aligned} \tag{A137}$$

Since a is slowly varying and $q > 0$, it follows that

$$\frac{1}{a(t)}\left(\frac{q}{t}\right)^q F^{-1}(1-1/q) \to 0 \text{ as } t \to \infty.$$

Next, regarding the second term in (A137), we have again the convergence (A136) and so it follows that

$$\frac{(q-1)}{a(t)}\left(\frac{q}{t^q}\int_q^t s^{q-1}F^{-1}(1-1/s)ds - F^{-1}(1-1/t)\right) \to -\frac{(q-1)}{q} \text{ as } t \to \infty.$$

Dividing both side of (A135) by $a(t)$, and taking limits as $t \to \infty$, now yields the desired result. □