Dynamic Pricing in the Large Market Regime

Tarek Abdallah
Kellogg School of Management, Northwestern University, tarek.abdallah@kellogg.northwestern.edu

Josh Reed
Leonard N. Stern School of Business, New York University, jreed@stern.nyu.edu

We study the single-item dynamic pricing problem in the large market regime where the market size grows large and the initial inventory level is fixed. Previous results for this regime assumed item valuation distributions in the Gumbel domain of attraction of extreme value theory. In this paper, we show these results differ for item valuation distributions in the Weibull and Frechet domains of attraction. We first study the offline version of the single-item dynamic pricing problem and provide large market regime approximations to its optimal value function. We next study the online version of this problem and provide large market regime asymptotics of its optimal value function, pricing policy and purchasing probability policy. These results are then used to derive asymptotics of the minimal regret of the optimal online value function relative to the optimal offline value function. We show that asymptotically the optimal pricing policy of the firm is not a classical run-out rate pricing policy, as was previously shown to be true for item valuation distributions in the Gumbel domain of attraction. Instead, a family of generalized run-out rate pricing policies asymptotically achieves the same regret in the large market regime as the optimal pricing policy. Such policies depend on the domain of attraction that the item valuation distribution belongs to and they may price lower (Weibull) or higher (Frechet) than the classical run-out rate pricing policy. Finally, we discuss our main results and present the outcomes of numerical experiments testing their accuracy and performance.

1. Introduction

Dynamic pricing is a revenue management strategy that has been widely adopted in industries such as airlines, hospitality, and retail. Within the revenue management literature, the single-item dynamic pricing problem is a classical model capturing the trade-offs between market size, inventory, and pricing decisions. Despite the intuitive appeal of this model, deriving its optimal pricing policy is, in general, a challenging problem. As a result, regime-dependent approximations are often used where the problem parameters are assumed to occupy different regions of their allowable space. The fluid regime of Gallego and Van Ryzin (1994) is the most popular regime and underlies many of the approximations currently in use. This regime is defined by scaling the market size and initial inventory level to ∞ while also holding them in a fixed proportion to one another.

The proportional scaling assumption of the fluid regime is reasonable for many market conditions but less so for others. This is particularly true when the market size greatly exceeds inventory levels. One straightforward example of this is major concerts or sporting events where the market for tickets far exceeds the limited venue capacities. Another example where market size is significantly larger than inventory levels is online marketplaces, in which case small and medium-sized firms have access to large consumer bases despite having relatively small inventory levels.

To address the limitations of the proportional scaling assumption of the fluid regime, Abdallah and Reed (2025) studied the single-item dynamic pricing problem in a regime where the market size is large relative to the initial inventory level. This regime is referred to as the large market regime (see also Correa et al. (2021) and den Boer (2021)) and technically is defined by scaling the market size to ∞ while holding the initial inventory level fixed. It turns out that extreme value theory plays a central role in analyzing the single-item dynamic pricing problem in this regime. Abdallah and Reed (2025) considered customer item valuation distributions lying in the Gumbel domain of attraction of extreme value theory (see Embrechts et al. (2013) and Resnick (2013)). The Gumbel domain of attraction is the largest of the three extreme value theory domains of attraction. It contains many commonly used item valuation distributions, such as the exponential and normal distributions.

One of the main results from Abdallah and Reed (2025) was that the large market regime approximation to the optimal value function of the single-item dynamic pricing problem differs substantially from the classical fluid regime approximation. It was also demonstrated numerically in Abdallah and Reed (2025) that the large market regime approximation performs better than the fluid regime approximation for medium to low inventory-to-market-size ratios. Despite these differences, Abdallah and Reed (2025) also showed that the classical dynamic run-out rate pricing policy obtained by continuously resolving the fluid problem remains first- and second-order optimal in the large market regime. However, a static run-out rate pricing policy is not even first-order optimal in the large market regime. The optimal static price in the large market regime turns out to be an order of magnitude lower than the static run-out rate price.

Our paper builds upon and extends the work of Abdallah and Reed (2025) by analyzing the large market regime for customer item valuation distributions in the Weibull and Frechet domains of attraction of extreme value theory. Each of these regimes includes relevant distributions different from those in the Gumbel domain of attraction. The Weibull domain of attraction includes distributions such as the Uniform and Beta distributions, which are bounded from above and have heavier tails at the upper limit of their support than the set of bounded distributions in the Gumbel domain of attraction. The Frechet domain of attraction includes distributions such as the Pareto distribution, which decays polynomially and is often referred to as being a heavy-tailed distribution. Such distributions do not possess finite moments of all orders and can even have infinite variance or mean. For a relative positioning of the three extreme value domains of attraction, see Figure 1.

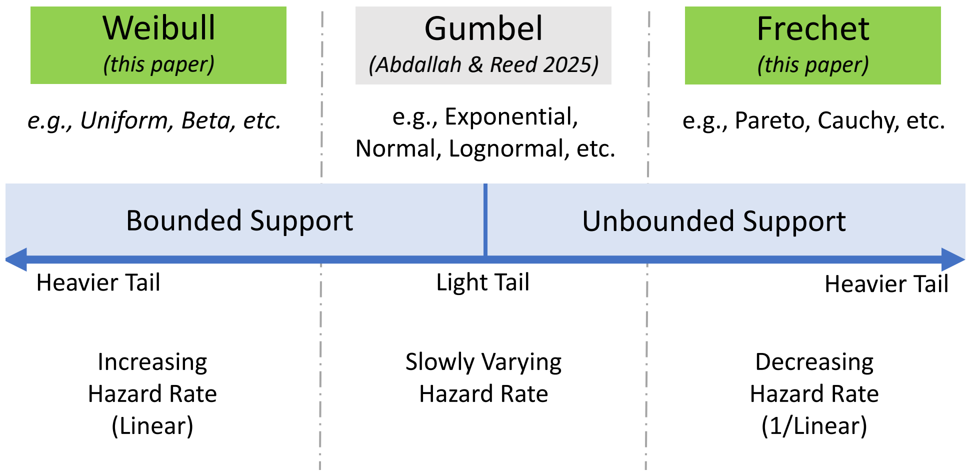


Figure 1 **Relative positioning of the three extreme value theory domains of attraction.**

One of our main results in this paper shows that under the assumption of customer item valuation distributions in the Weibull and Frechet domains of attraction, both the optimal value function and the optimal pricing policy for the single-item dynamic pricing problem in the large market regime behave differently than their counterparts in the fluid regime. In particular, for both of these domains of attraction, the first-order term of the optimal pricing policy in the large market regime is not the same as in the fluid regime. For the Weibull domain of attraction, given a specific inventory level and remaining time in the selling horizon, the optimal policy asymptotically prices lower than the classical dynamic run-out rate price of the fluid regime. The situation is reversed for the Frechet domain of attraction, where given a specific inventory level and remaining time in the selling horizon, the optimal policy asymptotically prices higher than in the fluid regime. Moreover, we show that as time evolves these results imply that for item valuation distributions in the Weibull domain of attraction the optimal pricing policy in expectation behaves similarly to a markup pricing policy. On the other hand, the optimal pricing policy for item valuation distributions in the Frechet domain of attraction behaves like a markdown pricing policy.

An outline of the remainder of the paper is as follows. In Section 3, we provide a rigorous formulation of the single-item dynamic pricing problem and in Section 4 we recall the definition of the large market regime. Next, in Section 5, we characterize the set of customer item valuation distributions lying in the Weibull and Frechet extreme value domains of attraction. In Section 6, we study the offline version of the single-item dynamic pricing problem described in Section 3. In the offline version of the problem, the firm is clairvoyant. It has knowledge of how many customers will arrive throughout the selling horizon as well as each of their item valuations. Our main results for the offline problem are large market regime approximations to the optimal offline value function. It turns out that the form of these approximations depends on the domain of attraction that the customer item valuation distribution lies in.

We next proceed in Section 7 to study the online single-item dynamic pricing problem described in Section 3. In Section 7.1, we provide a system of non-linear equations that first appeared in McAfee and te Velde (2008) where the single-item dynamic pricing problem was studied for demand functions with constant elasticity. The solution to this system of equations is used later in our large market regime approximations. Although the equations cannot be solved in closed form, they can be solved numerically, and we also provide asymptotics for their solution, which include first and second-order terms. Next, in Sections 7.2 and 7.3, we provide the large market regime asymptotics of the optimal online value function, pricing policy, and purchasing probability policy assuming item valuation distributions that lie in the Weibull and Fréchet domains of attraction, respectively. In particular, we show that the first-order term of the optimal pricing policy is no longer the classical dynamic run-out rate policy. In Section 7.4, the asymptotics of the optimal offline and online value functions are used to derive large market regime approximations to the minimal regret of the optimal online value function relative to the optimal offline value function. It turns out that the first-order term of the regret can be factored as a product of two functions: one that depends on the initial inventory level and another that depends on the market size. We then proceed in Section 7.5 to use our minimal regret results from Section 7.4 to define the notion of an asymptotically optimal pricing policy in the large market regime. Moreover, we construct a family of easy to implement generalized run-out rate policies whose form depends on the domain of attraction that the item valuation distribution lies in and are shown to be asymptotically optimal in the large market regime.

Section 8 contains a discussion of our main results. In particular, in Section 8.1 we provide an intuitive interpretation of the asymptotically optimal generalized run-out rate policies of Section 7.5. In Section 8.2, we study the role of the Mills ratio of the customer item valuation distribution in determining the minimal regret. In Section 9, we report the outcomes of several numerical experiments. Specifically, in Section 9.1 we study the the accuracy of our large market regime optimal online value function approximations. In Section 9.2, we use our large market regime approximations together with those of the small market and fluid regimes to numerically determine the boundaries between the three regimes with respect to the two-parameter space of market size and initial inventory levels. All proofs may be found in the Appendix.

2. Related Literature

There exists a large body of research on the single-item dynamic pricing problem. The fluid regime first proposed by Gallego and Van Ryzin (1994) is the standard framework used in the literature to devise pricing heuristics for dynamic pricing problems (Gallego and Topaloglu 2019). Typically, asymptotic analysis in the fluid regime uses a law of large numbers approach to derive a limiting deterministic optimization problem. Atar and Reiman (2013) propose a refinement to the fluid regime where both the market size and initial inventory level are scaled together but the dynamic pricing problem is analyzed at a diffusion scale. The asymptotic analysis in Atar and Reiman (2013) is based on a central limit theorem approach where in the limit a simple control problem involving a Brownian bridge is obtained. Atar and Reiman (2013) then devise a policy that is asymptotically optimal on a diffusion scale.

In most cases, closed-form solutions to the single-item dynamic pricing problem cannot be found. One notable exception is when the demand function has constant elasticity. In this case, McAfee and te Velde (2008) obtained an explicit solution. The results of McAfee and te Velde (2008) involve a system of non-linear equations which also appear in our large market regime asymptotic results for the optimal online value function, pricing policy and purchasing probability policy. The connection between our two sets of results stems from the fact that item valuation distributions in the Weibull and Frechet domains of attraction have regularly varying tails.

In addition to our main asymptotic results, we also derive large market regime results on the regret of the optimal online value function relative to the optimal offline value function. There exists several papers studying a similar regret in the fluid regime. Specifically, the regret of the optimal online value function relative to a fluid-derived upper bound. Gallego and Van Ryzin (1994) establish that the regret of the optimal static price policy relative to the fluid-derived upper bound is $O(\sqrt{\lambda t})$, where λt is the market size. Maglaras and Meissner (2006) were the first to show that a fluid resolving heuristic also exhibits $O(\sqrt{\lambda t})$ regret. Maglaras and Meissner (2006) also demonstrate numerically that in practice the resolving heuristic performs better than the optimal static price policy. Jasin (2014) was the first to prove that the resolving heuristic in fact achieves a lower regret than the optimal static price policy. In particular, in a discrete-time setting Jasin (2014) showed that the resolving heuristic achieves an $O(\log \lambda t)$ regret. Noting the practical challenges of frequent price adjustments, Chen et al. (2016) proposed a pricing policy that mimics the resolving heuristic but requires less frequent price adjustments. Wang and Wang (2022) recently showed that the resolving heuristic in fact achieves a constant regret relative to the performance of the optimal pricing policy. Their proof involves showing that the fluid-derived upper bound is actually a loose upper bound relative to the optimal pricing policy. The analysis of Wang and Wang (2022) requires however additional assumptions of strict concavity of the revenue function as well as smoothness and regularity conditions. A general analysis of the regret of the resolving heuristic in the fluid regime is involved and includes conditions that can be hard to interpret or verify in practice. Interested readers are referred to Jiang et al. (2022) and Balseiro et al. (2023) for a more detailed discussion.

Item valuation distributions in the Frechet domain of attraction lack finite moments of all orders and are considered to be heavy-tailed. Such distributions play an important role in finance and economics (see for example Embrechts et al. (2013) and Gabaix (2009)) but have received less attention in the revenue management and broader operations literature, with queuing theory being a notable exception (see Nair et al. (2022)). One paper closely related to ours where heavy-tailed distributions are considered is den Boer (2021). In den Boer (2021), the single-item dynamic pricing problem is studied in the large market regime assuming a single initial unit of inventory. Other papers considering heavy-tailed distributions include Ibragimov and Walden (2010) on static bundle pricing, Wierman and Zwart (2012) on scheduling, and Bimpikis and Markakis (2016) and Das et al. (2021) on inventory management. We also mention the related work of Correa et al. (2021) where extreme value theory is used to study static pricing policies for the k-selection problem in the large market regime.

3. The Model

The single-item dynamic pricing problem is defined as follows. There exists a firm selling a single item over a finite selling horizon of length $t > 0$. The initial inventory level of the item is q units which may take any value in $\mathbb{N} = \{0, 1, 2, ...\}$. Customers arrive to purchase the item according to a Poisson process $N = \{N_s, 0 \le s \le t\}$ with a rate $\lambda > 0$ that is known to the firm. The quantity λt is the expected number of customers that arrive over the course of the selling horizon and is referred to as the market size.

The arrival time of customer n ($n \ge 1$) is denoted by τ_n and we assume that customer n has a non-negative valuation X_n for the item. The sequence $\{X_n, n = 1, 2, ..., N_t\}$ of customer item valuations is independent and identically distributed (i.i.d.) with a common distribution function F, which we assume to have a finite mean. The firm has knowledge of the item valuation distribution F and the fact that the customer item valuations are i.i.d. The firm does not however know the specific items valuations of each customer. We therefore refer to this setting as the online setting. We denote by x_L and x_U the lower and upper limits of the support of F, respectively. For the sake of simplicity, we also assume that F has a positive density f on the entirety of its support. We may then uniquely define the inverse of F on $[0, 1)$, which we denote by F^{-1}. All random variables and quantities are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, P)$.

At each point in time $s \in [0, t]$, the firm sets a price $p_s \in \mathbb{R}_+$ for the item. An arriving customer is willing to purchase the item if their surplus at the time of their arrival is non-negative. That is, customer n is willing to purchase the item if $X_n \geq p_{\tau_n -}$, where $p_{\tau_n -}$ is the price of the item immediately before customer n arrives. The probability that customer n purchases the item is therefore given by $1 - F(p_{\tau_n -})$. We assume that for each customer the marginal value of more than 1 unit of the item is negligible.

The firm may dynamically adjust the price of the item in response to the previous history of customer purchases and its current inventory position. We restrict our attention to the class of admissible pricing policies defined as follows. An $\mathbb{R}_+ \cup p_\infty$-valued pricing policy[1] on $(\Omega, \mathcal{F}, P)$ is admissible if it satisfies the following two conditions.

I. For each $0 \leq s \leq t$, $p_s = p(\lambda; Q_s, t-s)$, where Q_s is the time s inventory level given by (2) below.

II. The family of functions $\{p(\lambda; q, \cdot), q \in \mathbb{N}\}$ are measurable with $p(\lambda; 0, \cdot) = p_\infty$.

Due to the Markovian nature of the system, such policies can be shown to be optimal amongst the much larger class of predictable pricing policies (Brémaud 1981). We denote the set of admissible pricing policies by $\mathcal{V}$.

Consistent with the revenue management literature (Bitran and Caldentey 2003), we assume that all items have zero marginal cost. Moreover, we assume that the firm is risk-neutral and interested in maximizing its expected revenue. Applying the results of Brémaud (1981), it is straightforward to show that for any admissible pricing policy $p \in \mathcal{V}$ the expected revenue of the firm given an initial inventory level $q \in \mathbb{N}$ is equal to

$$J_p(\lambda; q, t) = E\left[\lambda \int_0^t p(\lambda; Q_s, t-s)(1 - F(p(\lambda; Q_s, t-s)))ds | Q_0 = q\right], \tag{1}$$

where

$$Q_s = Q_0 - N\left(\lambda \int_0^s (1 - F(p(\lambda; Q_u, t-u)))du\right) \text{ for } 0 \leq s \leq t, \tag{2}$$

with N being a standard rate 1 Poisson process. Note that Q_s is the time s inventory level of the firm. Taking the supremum of the above over all admissible pricing policies p, we obtain the optimal value function

$$J^\star(\lambda; q, t) = \sup_{p \in \mathcal{V}} J_p(\lambda; q, t). \tag{3}$$

The following set of HJB equations characterizing the optimal solution to (3) can now be obtained by applying C2 and T3 of VII.2 of Brémaud (1981).

[1] We assume the existence of a null price p_∞ such that $1 - F(p_\infty) = 0$.

Theorem 1. *The family of functions $\{J^\star(\lambda;q,\cdot), q\in\mathbb{N}\}$ are the unique solution to the system of equations*

$$\frac{\partial J^\star(\lambda;q,t)}{\partial t} = \lambda \sup_{p\in\mathbb{R}_+}\left\{(1-F(p))\left(p-[J^\star(\lambda;q,t)-J^\star(\lambda;q-1,t)]\right)\right\}, \quad t\geq 0,\ q\in\mathbb{N}_+, \tag{4}$$
$$J^\star(\lambda;q,0)=0,\quad q\in\mathbb{N}_+,$$
$$J^\star(\lambda;0,t)=0,\quad t\geq 0.$$

Moreover, there exists an optimal solution $p^\star$ to (3) such that for $q\in\mathbb{N}_+$,

$$p^\star(\lambda;q,s)=\operatorname*{arg\,max}_{p\in\mathbb{R}_+}\left\{(1-F(p))(p-[J^\star(\lambda;q,s)-J^\star(\lambda;q-1,s)])\right\},\ s\geq 0. \tag{5}$$

The supermum in (4) is finite since by assumption F has a finite mean. The supremum is also achieved by at least one $p^\star\in\mathbb{R}_+$ since by assumption F is continuous. This justifies the use of the arg max operator in (5). We also note that there may exist more than one optimal family of pricing functions $\{p^\star(\lambda;q,\cdot), q\in\mathbb{N}\}$. Our asymptotic results however hold for each of them.

Next, we recall it is common in the literature (Gallego and Van Ryzin 1994, Maglaras and Meissner 2006) to perform a change-of-variables and consider the problem of selecting purchasing probabilities at each point in time instead of prices. It is straightforward to show that this formulation of the problem is equivalent to the one described above. Specifically, given an admissible pricing policy $p\in\mathcal{V}$ note that at each time $s\in[0,t]$, $\pi_s=1-F(p_{s-})$ is the probability that a customer arriving at time s will purchase the item. Conversely, a desired purchasing probability π_s at times s may be achieved by setting the corresponding price to be $F^{-1}(1-\pi_s)$. Finally, for any optimal pricing policy $p^\star$, we denote its optimal purchasing probability policy by $\pi^\star=1-F(p^\star)$.

We now complete this section by noting as in Abdallah and Reed (2025) that using the HJB equation (4), the optimal value function, optimal pricing policy and optimal purchasing probability policy can all be shown to depend only on the initial inventory level $q\in\mathbb{N}_+$ and market size $\lambda t>0$. We henceforth use the notation $J^\star(q,\lambda t)$, $p^\star(q,\lambda t)$ and $\pi^\star(q,\lambda t)$ to refer to each of them, respectively.

4. The Large Market Regime

In this paper, we consider the single-item dynamic pricing problem (3) in the large market regime (Abdallah and Reed 2025, den Boer 2021, Correa et al. 2021). In this regime, the market size $\lambda t\to\infty$ while all other problem parameters are held fixed. In particular, the initial inventory level is held fixed at $q\in\mathbb{N}_+=\{1,2,3,...\}$ units. This is in contrast to the small market regime where the market size is held fixed but the initial inventory level grows large, and the fluid regime where both the market size and initial inventory level grow large in proportion to one another. See Figure

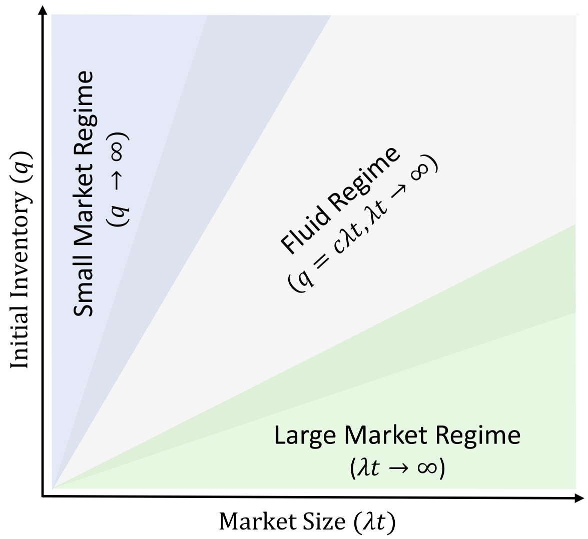


Figure 2 **Relative positioning of the small market, fluid and large market regimes.**

2 below for the relative positioning of the small market, fluid and large market regimes. It turns out that extreme value theory (Embrechts et al. 2013, Resnick 2013) plays an important role in the large market regime and the following central result helps to put our results in context.

For each $n \geq 1$, let $M_n = \max(X_1, X_2, ..., X_n)$ be the maximum item valuation of customers 1 through n. The Fisher-Tippett theorem (Embrechts et al. 2013) then provides the asymptotics of the distribution of M_n as $n \to \infty$. In the following, the notation $\xrightarrow{d}$ is used to denote convergence in distribution.

Theorem 2 **(Fisher-Tippett Theorem)**. *If there exists norming constants $b_n \in \mathbb{R}$ and $a_n > 0$ for $n \geq 1$ and some non-degenerate distribution G such that*

$$\frac{M_n - b_n}{a_n} \xrightarrow{d} G \text{ as } n \to \infty,$$

then G belongs to one of the following three extreme value distributions:

$$\begin{aligned} \text{Type I (Gumbel):} \quad \Lambda(x) &= \quad \exp(-\exp(-x)), x \in \mathbb{R}, \\ \text{Type II (Frechet):} \quad \Phi_\alpha(x) &= \begin{cases} 0, & x \leq 0, \\ \exp(-x^{-\alpha}), & x > 0, \end{cases} \\ \text{Type III (Weibull):} \quad \Psi_\alpha(x) &= \begin{cases} \exp(-(-x)^{\alpha}), & x < 0, \\ 1, & x \geq 0, \end{cases} \end{aligned}$$

where $\alpha > 0$ for either Type II or Type III.

Our main results provide asymptotics for the optimal value function, pricing policy and purchasing probability policy in the large market regime for both the offline and online setting of the single-item dynamic pricing problem assuming customer item valuation distributions in the Weibull and Frechet domains of attraction. We also provide approximations in both the Weibull and Frechet domains of attraction to the regret of the optimal online value function relative to the optimal offline value function. Moreover, we provide asymptotically optimal policies in both of these domains of attraction. Similar results were recently obtained in Abdallah and Reed (2025) for customer item valuation distributions in the Gumbel domain of attraction.

5. The Weibull and Frechet Domains of Attraction

In this section, we provide the specifics of the Weibull and Frechet domains of attraction.

5.1. The Weibull Domain of Attraction

The Weibull domain of attraction consists of item valuation distributions that have an upper bound on their support. That is, $x_U < \infty$. Moreover, their tail behavior at x_U must be regularly varying. There also exists item valuation distributions with an upper bound on their support in the Gumbel domain of attraction and for further details on them see Embrechts et al. (2013) and Resnick (2013).

In order to precisely characterize item valuation distributions in the Weibull domain of attraction, we first require the definition of a slowly varying function (Resnick 2013).

Definition 1 (Slowly Varying Function). A function $L : \mathbb{R}_+ \to (0, \infty)$ is said to be slowly varying (at ∞) if

$$\lim_{t\to\infty} \frac{L(xt)}{L(t)} = 1 \text{ for } x > 0.$$

Any slowly varying function has less than polynomial growth as well as decay. Some examples of common functions that are slowly varying are functions that converge to a constant and $\ln^\beta t$ for any $\beta \in \mathbb{R}$. The following result now characterizes the Weibull domain of attraction (Resnick 2013).

Definition 2 (Weibull Domain of Attraction). The item valuation distribution F is in the Weibull domain of attraction if and only if $x_U < \infty$ and for some $\alpha > 0$,

$$F(x_U - 1/x) = 1 - L(x)x^{-\alpha} \text{ for } x > 0,$$

where $L : \mathbb{R}_+ \to (0, \infty)$ is a slowly varying function.

We refer to $\alpha > 0$ in the above definition as the index of F. It also turns out that if F has a density f in a neighborhood (x_0, x_U) of x_U, then a sufficient condition for F to lie in the Weibull domain of attraction with index $\alpha > 0$ is that

$$\lim_{x\uparrow x_U} (x_U - x)\frac{f(x)}{1 - F(x)} = \alpha. \tag{6}$$

We refer to the above condition as a von-Mises condition and assume it holds for the remainder of the paper when discussing the Weibull domain of attraction. It turns out that this assumption is not very restrictive since every item valuation distribution in the Weibull domain of attraction is tail equivalent to some item valuation distribution satisfying (6) (Embrechts et al. 2013).

The norming constants in the Fisher-Tippett theorem in the case of item valuation distributions in the Weibull domain of attraction are given by

$$b_n \;=\; x_U \quad \text{and} \quad a_n \;=\; x_U - F^{-1}(1-n^{-1}) \text{ for } n \geq 1,$$

and can be naturally extended to $t \geq 1$ by setting

$$b(t) \;=\; x_U \quad \text{and} \quad a(t) \;=\; x_U - F^{-1}(1-t^{-1}). \tag{7}$$

Moreover, it can be shown that there exists a slowly varying function L_1 such that

$$a(t) = t^{-1/\alpha} L_1(t) \text{ for } t \geq 0. \tag{8}$$

5.2. The Frechet Domain of Attraction

We next turn our attention to item valuation distributions in the Frechet domain of attraction. All item valuation distributions in the Frechet domain of attraction have no upper bound on their support. That is, $x_U = \infty$. Moreover, their right tail always exhibits polynomial decay. Such distributions lack finite moments of all orders and in some cases may have infinite variance and possibly even an infinite mean. For these reasons, distributions in the Frechet domain of attraction are sometimes referred to as being heavy-tailed. The following result provides a precise characterization of item valuation distributions in the Frechet domain of attraction (Resnick 2013).

Definition 3 (Frechet Domain of Attraction). The item valuation distribution F is in the Frechet domain of attraction if and only if $x_U = \infty$ and for some $\alpha > 0$,

$$F(x) = 1 - x^{-\alpha} L(x) \text{ for } x > 0, \tag{9}$$

where $L : \mathbb{R}_+ \to (0, \infty)$ is a slowly varying function.

We refer to $\alpha > 0$ in the above definition as the index of F. If $\alpha < 2$, then the second moment of F is infinite and, if $\alpha < 1$, the mean of F is infinite as well. For the remainder of the paper when discussing the Frechet domain of attraction, we assume that $\alpha > 1$ so that the item valuations have a finite mean.

It turns out that if F has a density f in a neighborhood (x_0, ∞) of ∞, then a sufficient condition for F to lie in the Frechet domain of attraction with index $\alpha > 0$ is that

$$\lim_{x \to \infty} \frac{x f(x)}{1 - F(x)} = \alpha. \tag{10}$$

We refer to the above as the von-Mises condition and assume it holds for the remainder of the paper when discussing the Frechet domain of attraction. It is well known that the Frechet domain of attraction consists of distributions that either satisfy the von-Mises condition above or are tail-equivalent to a distribution satisfying it (Embrechts et al. 2013). Hence, this assumption is not too restrictive. A sufficient condition for F to satisfy the von-Mises condition (10) is that F is absolutely continuous in a neighborhood (x_0, ∞) of ∞ with a density f that is ultimately monotone (Resnick 2013).

The norming constants in the Fisher-Tippett theorem in the case of item valuation distributions in the Frechet domain of attraction are given by

$$b_n = 0 \;\text{ and }\; a_n \;=\; F^{-1}(1 - n^{-1}) \text{ for } n \geq 1,$$

and can be naturally extended to $t \geq 1$ by setting

$$b(t) = 0 \;\text{ and }\; a(t) \;=\; F^{-1}(1 - t^{-1}). \tag{11}$$

Moreover, it can be shown that there exists a slowly varying function L_1 such that

$$a(t) = t^{1/\alpha} L_1(t) \text{ for } t \geq 0. \tag{12}$$

6. The Offline Problem

We now study the offline version of the single-item dynamic pricing problem described in Section 3. In the offline version of the problem, the firm is clairvoyant and has full information on the number of customers who will arrive over the selling horizon as well as each of their item valuations. A precise description of the offline problem may be found in Abdallah and Reed (2025). One motivation for studying the offline problem is that its optimal value function is an upper bound on the optimal value function of the online problem. We then in Section 7.4 use this upper bound is characterize the regret of the optimal value function of the online problem relative to that of the offline problem.

The optimal pricing policy of the firm in the offline setting is simple. Suppose that the firm starts out with an initial inventory level of $q \in \mathbb{N}_+$ items. Since the firm has full information, it knows which customers have the top $\min(q, N_t)$ item valuations. Moreover, the firm has knowledge of the item valuations of each of the these customers. It therefore may set the price exactly to the item valuation for each of these $\min(q, N_t)$ customers and equal to the null price p_∞ otherwise.

In Abdallah and Reed (2025), an analytical expression was provided for the value function of the optimal offline policy described above. Specifically, it turns out that the optimal offline value function depends only on the initial inventory level $q \in \mathbb{N}_+$ and market size $\lambda t > 0$, and is given by

$$J^{\star}_{\text{OFF}}(q, \lambda t) = \frac{1}{\Gamma(q)} \int_0^{\lambda t} F^{-1}\left(1 - \frac{v}{\lambda t}\right) \cdot \Gamma(q, v) \cdot dv, \tag{13}$$

where $\Gamma(\cdot)$ and $\Gamma(\cdot,\cdot)$ denote the gamma function and incomplete gamma function (Olver et al. 2010), respectively. The equality above holds for any customer item valuation distribution with a finite mean. In this case, F^{-1} denotes the left-continuous inverse of F. Note also that since $\Gamma(q,v)$ and $\Gamma(q)$ are well-defined for $q>0$, one may use (13) to extend $J^{\star}_{\mathrm{OFF}}(q,\lambda t)$ to all $q>0$.

Approximations to $J^{\star}_{\mathrm{OFF}}$ in the small market and fluid regimes were obtained in Abdallah and Reed (2025) assuming customer item valuation distributions with a finite mean. Also in Abdallah and Reed (2025), approximations to $J^{\star}_{\mathrm{OFF}}$ were obtained in the large market regime assuming customer item valuation distributions in the Gumbel domain of attraction. In the present paper, we complement the results of Abdallah and Reed (2025) by providing approximations to $J^{\star}_{\mathrm{OFF}}$ in the large market regime assuming customer item valuation distribution in the Weibull and Frechet domains of attraction.

In order to provide our approximations to $J^{\star}_{\mathrm{OFF}}$, we first must set up the following notation. For each $q\in\mathbb{N}_+$, set $G_q=\exp(C_q/q)$ where

$$C_q = q\cdot(H_q-\gamma-1), \tag{14}$$

and H_{q-1} is the $(q-1)$st harmonic number with $H_0=0$ and $\gamma\approx 0.57722$ is the Euler-Mascheroni constant. Also set

$$W_q = \left(\frac{\alpha}{\alpha+1}\cdot\frac{\Gamma(q+1+1/\alpha)}{\Gamma(q+1)}\right)^{\alpha} \text{ and } F_q = \left(\frac{\alpha}{\alpha-1}\cdot\frac{\Gamma(q+1-1/\alpha)}{\Gamma(q+1)}\right)^{-\alpha}, \tag{15}$$

where W_q is defined for $\alpha>0$ and C_q is defined for $\alpha>1$. Our approximations to $J^{\star}_{\mathrm{OFF}}(q,\lambda t)$ in the large market regime are presented in Table 1. Their derivation for the Weibull and Frechet domains of attraction are as follows (see Abdallah and Reed (2025) for the Gumbel domain of attraction).

Weibull	$qF^{-1}\left(1-\frac{W_q}{\lambda t}\right)$
Gumbel	$qF^{-1}\left(1-\frac{G_q}{\lambda t}\right)$
Frechet	$qF^{-1}\left(1-\frac{F_q}{\lambda t}\right)$

Table 1 Approximations to the optimal offline value function in the large market regime depending on the domain of attraction that F lies in.

- Weibull approximation: First note by Proposition A1 that if the customer item valuation distribution F lies in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (6), then for each $q \in \mathbb{N}_+$,

$$J^{\star}_{\mathrm{OFF}}(q, \lambda t) = \frac{x_U}{\Gamma(q)} \int_0^{\lambda t} \Gamma(q, v) \cdot dv - \frac{\alpha}{\alpha+1} \cdot \frac{\Gamma(q+1+1/\alpha)}{\Gamma(q)} a(\lambda t) + o(a(\lambda t)) \tag{16}$$

as $\lambda t \to \infty$. Next, integrating-by-parts it follows that

$$\int_0^{\lambda t} \Gamma(q, v) \cdot dv = \Gamma(q+1) + (\lambda t) \cdot \Gamma(q, \lambda t) - \Gamma(q+1, \lambda t). \tag{17}$$

Using standard inequalities for the incomplete gamma function (Olver et al. 2010) and (8) one then obtains that

$$\int_0^{\lambda t} \Gamma(q, v) \cdot dv = \Gamma(q+1) + o(a(\lambda t)) \text{ as } \lambda t \to \infty. \tag{18}$$

It then follows by (16) after some algebra that

$$J^{\star}_{\mathrm{OFF}}(q, \lambda t) = q \left(x_U - \frac{\alpha}{\alpha+1} \cdot \frac{\Gamma(q+1+1/\alpha)}{\Gamma(q+1)} a(\lambda t) \right) + o(a(\lambda t)) \text{ as } \lambda t \to \infty. \tag{19}$$

Finally, recalling that L_1 in (8) is slowly varying and using the definition of W_q in (15), one obtains that

$$\frac{\alpha}{\alpha+1} \cdot \frac{\Gamma(q+1+1/\alpha)}{\Gamma(q+1)} a(\lambda t) = a(\lambda t / W_q) + o(a(\lambda t)) \text{ as } \lambda t \to \infty, \tag{20}$$

and so the Weibull approximation in Table 1 is implied by (7) .

- Frechet approximation: First note by Proposition A2 that if the customer item valuation distribution F lies in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (10), then for each $q \in \mathbb{N}_+$,

$$J^{\star}_{\mathrm{OFF}}(q, \lambda t) = q \cdot \frac{\alpha}{\alpha-1} \cdot \frac{\Gamma(q+1-1/\alpha)}{\Gamma(q+1)} a(\lambda t) + o(a(\lambda t)) \text{ as } \lambda t \to \infty. \tag{21}$$

Next, recalling that L_1 in (12) is slowly varying and using the definition of F_q in (15), it follows that

$$\frac{\alpha}{\alpha-1} \cdot \frac{\Gamma(q+1-1/\alpha)}{\Gamma(q+1)} a(\lambda t) = a(\lambda t / F_q) + o(a(\lambda t)) \text{ as } \lambda t \to \infty. \tag{22}$$

The Frechet approximation in Table 1 then follows by (11).

We next have the following result bounding the constants W_q, G_q and F_q appearing in Table 1.

PROPOSITION 1. *For each* $q \in \mathbb{N}_+$,

- *in the Weibull case* $W_q < ((\alpha + 1/q)/(\alpha+1))^{\alpha} q$ *if* $\alpha > 1$,
- *in the Gumbel case* $G_q < \exp(1/2q - 1) q$,

- *in the Frechet case* $F_q < ((\alpha-1)/\alpha)^{\alpha} q$ *if* $\alpha > 1$.

Now recall by the results of Gallego and Van Ryzin (1994) that in the online setting and for sufficiently large λt the expected revenue of the firm is upper bounded by $qF^{-1}(1-q/\lambda t)$, regardless of the customer item valuation distribution F. On the other hand, it is straightforward to verify by Proposition 1 that for each $q \in \mathbb{N}_+$, the constants W_q, G_q, and F_q appearing in Table 1 are all strictly less than q. Since F^{-1} is non-decreasing, it follows that regardless of the domain of attraction that the customer item valuation distribution lies in, our first-order approximations to the optimal offline value function are always greater than the corresponding upper bound of Gallego and Van Ryzin (1994) in the online setting. This can be attributed to the fact that in the offline setting the firm is given additional information beyond that available in the online setting.

7. The Online Problem

We now proceed to study the online single-item revenue management problem described in Section 3. First, in Section 7.1 we provide a system of equations whose solutions appear in our approximations. Next, in Section 7.2 we assume that the customer item valuation distribution lies in the Weibull domain of attraction, in which case we provide large market regime approximations to the optimal online value function, pricing policy and purchasing probability policy. This is followed by Section 7.3 where we provide corresponding large market regime approximations assuming item valuation distributions in the Frechet domain of attraction. In Section 7.4, we recall from Abdallah and Reed (2025) the definition of the regret of the optimal online value function relative to the optimal offline value function. We then provide large market regime approximations to the minimal regret for customer item valuation distributions in each of the three extreme value domains of attraction. Finally, in Section 7.5 we define the notion of asymptotic optimality in the large market regime for each of the three domains of attraction. We then provide a family of generalized run-out rate pricing policies that are easy to implement and asymptotically optimal.

7.1. System of Equations for v_q

For each $\kappa > 0$, consider the system of equations

$$v_q^{\kappa-1} = \frac{1}{\kappa}(v_q^{\kappa} - v_{q-1}^{\kappa}) \text{ for } q \in \mathbb{N}_+, \tag{23}$$

where $v_0 = 0$. We then have the following result.

PROPOSITION 2. *For each* $\kappa > 0$*, there exists a unique positive solution* $\{v_q(\kappa), q \in \mathbb{N}_+\}$ *to the system of equations* (23)*. Moreover, for each* $q \in \mathbb{N}_+$,

$$v_q(\kappa) = q + \frac{\kappa-1}{2}\ln q + o(\ln q) \text{ as } q \to \infty. \tag{24}$$

Finally, for fixed $q \in \mathbb{N}_+$ *it follows that* $v_q(\kappa)$ *is continuous and strictly increasing in* $\kappa > 0$.

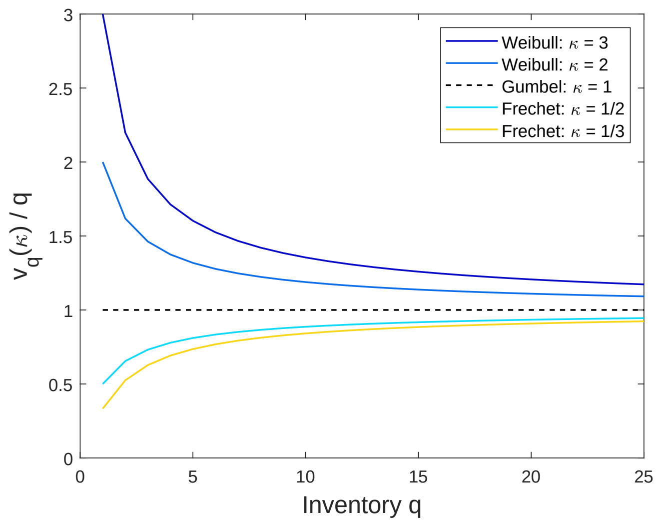


Figure 3 **The values of $\{v_q(\kappa)/q, q \geq 1\}$ for different values of κ.**

If $\kappa = 1$, it is straightforward to verify that $v_q = q$ for $q \in \mathbb{N}_+$ is the unique positive solution to (23). Recalling by Proposition 2 that $v_q(\kappa)$ is strictly increasing in κ for each $q \in \mathbb{N}_+$, it then follows that

$$\begin{cases} v_q(\kappa) & < & q \text{ if } 0 < \kappa < 1, \\ v_q(\kappa) & = & q \text{ if } \kappa = 1, \\ v_q(\kappa) & > & q \text{ if } \kappa > 1. \end{cases} \tag{25}$$

As is shown in our main results of Sections 7.2 and 7.3 below, the asymptotic behavior of the optimal value function, pricing policy and purchasing probability policy in the large market regime depends on the sequence $\{v_q(\kappa), q \in \mathbb{N}_+\}$. Loosely speaking, $\kappa > 1$ corresponds to item valuation distributions in the Weibull domain of attraction and $0 < \kappa < 1$ corresponds to item valuation distributions in the Frechet domain of attraction. Interestingly, the results of Abdallah and Reed (2025) for the Gumbel domain of attraction can be viewed as corresponding to the special case of $\kappa = 1$. In Figure 3, we provide a graph of $\{v_q(\kappa), q \in \mathbb{N}_+\}$ normalized by $\{q, q \in \mathbb{N}_+\}$ for different values of κ.

The system of equations (23) first appeared in McAfee and te Velde (2008) in which the single-item dynamic pricing problem is studied where at each point in time the demand function has a constant elasticity $\varepsilon > 0$. Specifically, suppose that with $t \geq 0$ units of time left on the selling horizon the demand as a function of price is given by $\lambda(t, p) = a(t)p^{-\varepsilon}$ for $p > 0$. It then turns out that the HJB equation (4) may be explicitly solved. Specifically, letting

$$A(t) = \int_0^t a(s)ds \text{ for } t \geq 0$$

and setting $\kappa = (\varepsilon - 1)/\varepsilon$, it follows that $J^\star(q, t) = v_q^\kappa(\kappa)(A(t))^{1-\kappa}$ for $q \in \mathbb{N}_+$.

It may at first seem that demand functions having constant elasticity have little to do with item valuation distributions in the Weibull and Frechet domains of attraction. However, it turns out that demand functions induced by item valuation distributions in these two domains of attraction are asymptotically similar to those of McAfee and te Velde (2008). Although the results of McAfee and te Velde (2008) cannot be directly applied, they are still helpful in putting our results in context.

7.2. Approximations for Item Valuation Distributions in the Weibull Domain of Attraction

We now provide the large market regime asymptotics of the optimal online value function, pricing policy and purchasing probability policy assuming customer item valuation distributions in the Weibull domain of attraction. Given a fixed $q \in \mathbb{N}_+$, our asymptotics provide expressions for $J^\star(q,\lambda t)$ and $p^\star(q,\lambda t)$ up to an $o(a(\lambda t))$ remainder term, where a is the norming function defined in (8). We also provide an expression for $\pi^\star(q,\lambda t)$ up to an $o(1/\lambda t)$ remainder term. The following is our main result. Using the definitions of the norming functions a and b in (7) , it follows in a straightforward manner by Proposition A3 in the appendix.

Theorem 3. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (6), then for each $q \in \mathbb{N}_+$ as $\lambda t \to \infty$,*

$$J^\star(q,\lambda t) = qF^{-1}\left(1 - \left(\frac{w_q(\alpha)}{q}\right)^{\alpha} \frac{w_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{26}$$

and

$$p^\star(q,\lambda t) = F^{-1}\left(1 - \frac{w_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{27}$$

and

$$\pi^\star(q,\lambda t) = \frac{w_q(\alpha)}{\lambda t} + o(1/\lambda t), \tag{28}$$

where $w_q(\alpha) = v_q((\alpha+1)/\alpha)$.

Note by (8) that $a(\lambda t) \to 0$ as $\lambda t \to \infty$ and so the $o(a(\lambda t))$ remainder terms in (26) and (27) vanish in the limit of the large market regime. Moreover, a lower bound on the rate of decay of the $o(a(\lambda t))$ term is obtained by recalling that $a(\lambda t) = (\lambda t)^{-1/\alpha} L_1(\lambda t)$ where L_1 is a slowly varying function. Next, by (26) it is evident that in the large market regime and for item valuation distributions in the Weibull domain of attraction and fixed $q \in \mathbb{N}_+$, $J^\star(q,t) \to q x_U < \infty$ as $\lambda t \to \infty$. In other words, as $\lambda t \to \infty$ the optimal value function converges to its upper bound of $q x_U$. Also recall from the statement of the theorem that $w_q(\alpha) = v_q((\alpha+1)/\alpha)$ where $\{v_q((\alpha+1)/\alpha), q \in \mathbb{N}_+\}$

is the unique positive solution to the system of equations (23) with κ set equal to $(\alpha+1)/\alpha$. Thus, since $(\alpha+1)/\alpha > 1$, it follows by Proposition 2 that $w_q(\alpha) > q$. This then implies that

$$F^{-1}\left(1-\left(\frac{w_q(\alpha)}{q}\right)^{\alpha}\frac{w_q(\alpha)}{\lambda t}\right) < F^{-1}\left(1-\frac{q}{\lambda t}\right),$$

as expected by (26) since $qF^{-1}(1-q/\lambda t)$ is an upper bound (Gallego and Van Ryzin 1994) on the optimal expected revenue.

We next observe by (27) that $p^\star(q,\lambda t) \to x_U < \infty$ as $\lambda t \to \infty$ and so the optimal price in the large market regime converges to the upper limit of the support of F. The first-order term in the expression (27) for $p^\star(q,\lambda t)$ is similar to the form of the asymptotically optimal run-out rate policies of the fluid regime of Gallego and Van Ryzin (1994). However, the usual inventory level q has been replaced by $w_q(\alpha) > q$. This implies that given an inventory level q and market size λt, the optimal pricing policy in the large market regime prices lower than the optimal pricing policy in the fluid regime. Presumably this is because the upper bound of the item valuation distributions in the Weibull domain of attraction makes avoiding leftover inventory at the end of the selling horizon more important than marginal increases in price. Similar insights may also be deduced for the optimal purchasing probability $\pi^\star(q,\lambda t)$ using the expression (28). Section 8 includes a further discussion on these points.

We now provide two examples illustrating the application of Theorem 3 to different item valuation distributions in the Weibull domain of attraction.

Example 1 (The Uniform Distribution). Consider first the case of items valued according to the Uniform$[a,b]$ distribution. It is straightforward to verify that this distribution is in the Weibull domain of attraction with an index α equal to 1 and slowly varying function L equal to the constant $1/(b-a)$. The norming functions a and b are then explicitly given for $t \geq 1$ by $b(t) = b$ and

$$a(t) = \frac{b-a}{t}.$$

It then follows from Theorem 3 and since $a(t) = o(1/t)$ that for fixed $q \in \mathbb{N}_+$, as $\lambda t \to \infty$,

$$J^\star(q,\lambda t) = qb - \frac{w_q^2(1)(b-a)}{\lambda t} + o(1/\lambda t) \tag{29}$$

and

$$p^\star(q,\lambda t) = b - \frac{w_q(1)(b-a)}{\lambda t} + o(1/\lambda t) \tag{30}$$

and

$$\pi^\star(q,\lambda t) = \frac{w_q(1)}{\lambda t} + o(1/\lambda t).$$

Moreover, in this case $w_q(1) = v_q(2)$ and so by (23), $\{w_q(1), q \in \mathbb{N}_+\}$ satisfies the system of equations

$$2w_q(1) = w_q^2(1) - w_{q-1}^2(1) \text{ for } \ q \in \mathbb{N}_+,$$

where $w_0(1) = 0$. Using the quadratic formula then yields that $w_q(1) = 1 + \sqrt{1 + w_{q-1}^2(1)}$ for $q \in \mathbb{N}_+$. In Section 9, we investigate the accuracy and performance of the approximations (29)-(30).

Example 2 (The Beta Distribution). Next, consider the case of items valued according to the beta distribution with shape parameters $\alpha, \beta > 0$. The support of the beta distribution is the interval $[0,1]$ and its CDF is given by

$$F_{\alpha,\beta}(x) = \frac{1}{B_{\alpha,\beta}} \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt \ \text{ for } \ 0 \le x \le 1,$$

where $B_{\alpha,\beta}$ is the beta function. In this case, it can be shown that for $t \ge 1$, $b(t) = 1$ and

$$a(t) = L(t) t^{-1/\beta},$$

where the slowly varying function L converges to the constant $(B_{\alpha,\beta}\beta)^{1/\beta}$. As in the example of the Uniform$[a,b]$ distribution above, Theorem 3 can then be used to obtain the asymptotics in the large market regime of the optimal value function, pricing policy and purchasing probability policy. We also note that the special case of $\alpha = \beta = 1$ corresponds to the Uniform$[0,1]$ distribution.

7.3. Approximations for Item Valuation Distributions in The Frechet Domain of Attraction

We next consider customer item valuation distributions in the Frechet domain of attraction. As in Section 7.2, we provide the asymptotics of $J^\star(q,\lambda t)$, $p^\star(q,\lambda t)$ and $\pi^\star(q,\lambda t)$ in the large market regime. The following is our main result. Using the definition of the norming functions a and b in (11), it follows in a straightforward manner by Proposition A4 in the appendix.

Theorem 4. *If F is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (10), then for each $q \in \mathbb{N}_+$ as $\lambda t \to \infty$,*

$$J^\star(q,\lambda t) = qF^{-1}\left(1 - \left(\frac{q}{\phi_q(\alpha)}\right)^\alpha \frac{\phi_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{31}$$

and

$$p^\star(q,\lambda t) = F^{-1}\left(1 - \frac{\phi_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{32}$$

and

$$\pi^\star(q,\lambda t) = \frac{\phi_q(\alpha)}{\lambda t} + o(1/\lambda t),$$

where $\phi_q(\alpha) = v_q((\alpha-1)/\alpha)$.

Note by (31) that the optimal expected revenue grows to ∞ in the large market regime for item valuation distributions in the Frechet domain of attraction. Specifically, using the expression (12) for the norming function $a(t)$, the optimal expected revenue grows at a rate proportional to $(\lambda t)^{1/\alpha}L_1(\lambda t)$ where L_1 is a slowly varying function. The $o(a(\lambda t))$ remainder term in (31) grows at a rate strictly less than $L_1(\lambda t)(\lambda t)^{1/\alpha}$. Next, recall that $\phi_q(\alpha) = v_q((\alpha-1)/\alpha)$ where $\{v_q((\alpha-1)/\alpha), q \in \mathbb{N}_+\}$ is the unique positive solution to the system of equations (23) with κ set equal to $(\alpha-1)/\alpha$. Thus, since $\alpha > 1$ and $0 < (\alpha-1)/\alpha < 1$, it follows from Proposition 2 that $\phi_q(\alpha) < q$. This then implies that

$$F^{-1}\left(1-\left(\frac{q}{\phi_q(\alpha)}\right)^{\alpha}\frac{\phi_q(\alpha)}{\lambda t}\right) < F^{-1}\left(1-\frac{q}{\lambda t}\right), \tag{33}$$

as expected by (31) since $qF^{-1}(1-q/\lambda t)$ is an upper bound (Gallego and Van Ryzin 1994) on the optimal expected revenue.

We next observe from (32) that $p^\star(q,\lambda t)$ diverges to ∞ as $\lambda t \to \infty$. Specifically, $p^\star(q,\lambda t)$ grows at a rate proportional to $(\lambda t)^{1/\alpha}L_1(\lambda t)$ where L_1 is a slowly varying function. Similar to the case of item valuation distributions in the Weibull domain of attraction, the optimal pricing policy is again of a run-out rate form. In this case however $\phi_q(\alpha) < q$ is substituted in place of the usual q used in the fluid regime. This implies that given an inventory level q and market size λt, the optimal pricing policy in the large market regime prices higher than the optimal pricing policy in the fluid regime. Presumably this is due to the heaviness of the tails of item valuation distributions in the Frechet domain of attraction which makes selling at a higher price worth the risk of holding leftover inventory at the end of the selling horizon. Similar insights can be obtained from the expression (33) for the optimal purchasing probability $\pi^\star(q,\lambda t)$. For further discussion on these points, see Section 8.

We now provide two examples illustrating the application of Theorem 4.

Example 3 (The Pareto Distribution). Consider first the case of customer item valuations following a Pareto distribution with scale parameter $x_m > 0$ and shape parameter $\alpha > 0$. The support of the item valuation distribution is then given by $[x_m, \infty)$ and its CDF is

$$F(x) = 1-\left(\frac{x_m}{x}\right)^{\alpha} \text{ for } x \geq x_m.$$

In this case, the slowly varying function in (9) is simply the constant x_m^α and it is straightforward to identify the norming function $a(t) = x_m t^{1/\alpha}$. Consequently, it follows from Theorem 4 that for $q \in \mathbb{N}_+$ as $\lambda t \to \infty$,

$$J^\star(q,\lambda t) = x_m\phi_q(\alpha)(\lambda t/\phi_q(\alpha))^{1/\alpha} + o((\lambda t)^{1/\alpha}) \tag{34}$$

and

$$p^\star(q,\lambda t) = x_m(\lambda t/\phi_q(\alpha))^{1/\alpha} + o((\lambda t)^{1/\alpha}) \tag{35}$$

and

$$\pi^\star(q,\lambda t) = \frac{\phi_q(\alpha)}{\lambda t} + o(1/\lambda t).$$

In Section 9, we investigate the accuracy and performance of the approximations (34)-(35) assuming $\alpha = 2$.

Example 4 (The Log-Gamma Distribution). Next, consider the case of items valued according to the log-gamma distribution with parameters $\alpha, \beta > 0$. The support of the log gamma distribution is $(1,\infty)$ and its pdf is given by

$$f_{\alpha,\beta}(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1}x^{-\alpha-1}.$$

In this case, the norming function $a(t)$ cannot be solved for explicitly but we have the tail equivalence

$$a(t) \sim \left(\frac{(\ln t)^{\beta-1}t}{\Gamma(\beta)}\right)^{1/\alpha} \text{ as } t \to \infty,$$

which is sufficient to use in applying Theorem 4.

7.4. Regret Analysis

We now study the regret of the optimal online value function $J^\star(q,\lambda t)$ relative to its upper bound given by the optimal offline value function $J^\star_{\text{OFF}}(q,\lambda t)$. It turns out that the large market regime asymptotics of the regret depends on which domain of attraction that the customer item valuation distribution lies in. We proceed as follows.

Recall from Abdallah and Reed (2025) that for each admissible pricing policy $p \in \mathcal{V}$, we may define its regret relative to the optimal offline policy for each customer arrival rate $\lambda > 0$, initial inventory level $q \in \mathbb{N}_+$, and selling horizon $t > 0$, by setting

$$\text{Regret}_p(\lambda; q, t) = J^\star_{\text{OFF}}(q,\lambda t) - J_p(\lambda; q, t) \ \geq \ 0. \tag{36}$$

It then turns out that the minimal regret is a function of only the initial inventory level $q \in \mathbb{N}_+$ and market size $\lambda t \geq 0$. Specifically,

$$\text{Regret}^\star(q,\lambda t) = J^\star_{\text{OFF}}(q,\lambda t) - J^\star(q,\lambda t).$$

Using the results of Sections 7.2 and 7.3, one can next show that for a fixed initial inventory level $q \in \mathbb{N}_+$ and for customer item valuation distributions in either the Weibull or Frechet domains of attraction, the minimal regret in the large market regime is of the form

$$\text{Regret}^\star(q, \lambda t) = c(q)a(\lambda t) + o(a(\lambda t)) \text{ as } \lambda t \to \infty, \tag{37}$$

where a is the norming function corresponding to the extreme value domain of attraction that the customer item valuation distribution belongs to. The function $c(q)$ is given in Table 2 and also depends on which domain of attraction the customer item valuation distribution lies in. The asymptotic (37) was also shown in Abdallah and Reed (2025) to hold for customer item valuation distributions in the Gumbel domain of attraction.

Weibull	$(w_q(\alpha))^{(\alpha+1)/\alpha} - q \cdot \dfrac{\alpha}{\alpha+1} \cdot \dfrac{\Gamma(q+1+1/\alpha)}{\Gamma(q+1)}$
Gumbel	$\ln\left(\dfrac{e^q q!}{e^{C_q}}\right)$
Frechet	$q \cdot \dfrac{\alpha}{\alpha-1} \cdot \dfrac{\Gamma(q+1-1/\alpha)}{\Gamma(q+1)} - (\phi_q(\alpha))^{(\alpha-1)/\alpha}$

Table 2 The function $c(q)$ depending on the domain of attraction that F lies in.

We now analyze the righthand side of (37) in further detail. Recall first from (7) that for item valuation distributions in the Weibull domain of attraction, the norming function $a(t) = t^{-1/\alpha} L_1(t)$ where L_1 is a slowly varying function. It then follows from (37) that in the large market regime the minimal regret decreases polynomially (up to a slowly varying function) as $\lambda t \to \infty$. Next assume that $\alpha > 1$. Then, by the results of Merkle (1998),

$$\frac{\Gamma(q+1+1/\alpha)}{\Gamma(q+1)} = q^{1/\alpha} + O(q^{(1-\alpha)/\alpha}) \text{ as } q \to \infty. \tag{38}$$

Using Proposition 2 it is then straightforward to show that

$$c(q) = \frac{1}{\alpha+1} \cdot q^{(\alpha+1)/\alpha} + \frac{1}{2} \cdot \frac{\alpha+1}{\alpha} \cdot q^{1/\alpha} \ln q^{1/\alpha} + o(q^{1/\alpha} \ln q) \text{ as } q \to \infty,$$

from which we see that $c(q)$ diverges as q grows large.

In the case of item valuation distributions in the Frechet domain of attraction, the norming function $a(t) = t^{1/\alpha} L_1(t)$ where L_1 is slowly varying. Thus, by (37) the minimal regret grows polynomially (up to a slowly varying function) in the large market regime as $\lambda t \to \infty$. Moreover, regarding $c(q)$, in a similar manner to the Weibull case using (38) and Proposition 2 it can be shown that for $\alpha > 1$,

$$c(q) = \frac{1}{\alpha - 1} \cdot q^{(\alpha-1)/\alpha} + \frac{1}{2} \cdot \frac{\alpha - 1}{\alpha} \cdot q^{-1/\alpha} \ln q^{1/\alpha} + o(q^{-1/\alpha} \ln q) \text{ as } q \to \infty,$$

and so $c(q)$ again diverges as q grows large.

In summary, we find that in the large market regime the minimal regret decreases polynomially (up to a slowly varying function) for item valuation distributions in the Weibull domain of attraction and increases polynomially (up to a slowly varying function) for item valuation distributions in the Frechet domain of attraction. We also recall from Abdallah and Reed (2025) that in the case of item valuation distributions in the Gumbel domain of attraction, the minimal regret in the large market regime is always slowly varying.

7.5. Asymptotic Optimality

We now complete Section 7 by constructing easy to implement pricing policies whose regret in the large market regime have the same first-order asymptotics as the optimal pricing policies in Theorems 3 and 4. Specifically, we have the following definition of an asymptotically optimal pricing policy in the large market regime.

Definition 4. An admissible pricing policy $p \in \mathcal{V}$ is said to be asymptotically optimal in the large market regime if for each $q \in \mathbb{N}_+$,

$$\lim_{\lambda t \to \infty} \frac{\text{Regret}_p(\lambda; q, t)}{a(\lambda t)} = c(q), \tag{39}$$

where $c(q)$ is given by Table 2 and the limit above holds for any sequence of (λ, t) such that $\lambda t \to \infty$.

Note that both the norming function $a(\lambda t)$ and the limiting function $c(q)$ in Definition 4 depend on the extreme value domain of attraction that the customer item valuation distribution lies in. In this sense, Definition 4 generalizes the definition of asymptotic optimality in Abdallah and Reed (2025) where customer item valuation distributions were restricted to lie in the Gumbel domain of attraction.

The results of Section 7.4 imply that an admissible pricing policy $p \in \mathcal{V}$ is asymptotically optimal in the large market regime if on a scale of $a(\lambda t)$, it achieves the same regret with respect to the optimal offline value function as does the optimal pricing policy $p^\star$. Using the relationship (37) between the minimal regret and the optimal value function, another way to state this is to say that $p \in \mathcal{V}$ is asymptotically optimal if

$$\lim_{\lambda t \to \infty} \frac{J^\star(q, \lambda t) - J_p(\lambda; q, t)}{a(\lambda t)} = 0, \tag{40}$$

for any sequence of (λ, t) such that $\lambda t \to \infty$.

Now consider the subset of pricing policies in $\mathcal{V}$ such that the family of functions $\{p(\lambda; q, \cdot), q \in \mathbb{N}_+\}$ are of the form

$$p(\lambda; q, t) = F^{-1}\left(1 - \frac{R_q}{\lambda t}\right) \text{ for } t > 0,\ q \in \mathbb{N}_+, \tag{41}$$

and a sequence of positive constants $\{R_q, q \in \mathbb{N}_+\}$. We refer to such policies as generalized run-out rate policies. Note that if $R_q = q$ for $q \in \mathbb{N}_+$, we obtain the classical run-out rate policy which is known (Maglaras and Meissner 2006) to be asymptotically optimal in the fluid regime. It turns out however that for item valuation distributions in the Weibull and Frechet domains of attraction, generalized run-out rate policies different from the classical run-out rate policy are asymptotically optimal in the large market regime. In order to prove this, we first need the following result on the asymptotics of the expected revenue for any generalized run-out policy.

Lemma 1. *If $p \in \mathcal{V}$ is a generalized run-out rate policy of the the form (41), then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t \to \infty} \frac{J_p(q, \lambda t) - q b(\lambda t)}{a(\lambda t)} = \begin{cases} -\xi_q(1/\alpha), & \text{if } F \in \text{Weibull domain of attraction with index } \alpha > 0 \text{ and} \\ & \qquad R_q > 1/\alpha \text{ for } q \in \mathbb{N}_+, \\ -\ln\left(\prod_{k=1}^{q} R_k\right) - \sum_{k=1}^{q} \frac{k}{R_k}, & \text{if } F \in \text{Gumbel domain of attraction}, \\ \xi_q(-1/\alpha), & \text{if } F \in \text{Frechet domain of attraction with index } \alpha > 1, \end{cases} \tag{42}$$

where for each $\gamma \in \mathbb{R}$, $\{\xi_q(\gamma), q \in \mathbb{N}_+\}$ is the solution to the system of equations

$$\xi_q(\gamma) = \frac{R_q}{R_q - \gamma}\left(R_q^{\gamma} + \xi_{q-1}(\gamma)\right) \text{ for } q \in \mathbb{N}_+, \tag{43}$$

with $\xi_0(\gamma) = 0$.

Note that in the above result we assume that F satisfies the von-Mises condition corresponding to its domain of attraction.

Now comparing the recursion (43) above for $\{\xi_q(\gamma), q \in \mathbb{N}_+\}$ to the system of equations (23) for $\{v_q(\kappa), q \in \mathbb{N}_+\}$ and then using Propositions A3 and A4 in the appendix, the result below follows in a straightforward manner from the definition of asymptotic optimality.

Theorem 5. *If $p \in \mathcal{V}$ is a generalized run-out rate policy of the the form (41) where for each $q \in \mathbb{N}_+$,*

$$R_q = \begin{cases} w_q(\alpha), & \text{if } F \in \text{Weibull domain of attraction with index } \alpha > 0, \\ q, & \text{if } F \in \text{Gumbel domain of attraction}, \\ \phi_q(\alpha), & \text{if } F \in \text{Frechet domain of attraction with index } \alpha > 1, \end{cases}$$

then p is asymptotically optimal in the large market regime.

By Theorems 3 and 4, it is evident that the asymptotically optimal generalized run-out rate policies mimic the optimal pricing policies in the large market regime up to $o(a(\lambda t))$. In particular, by (25), given an inventory level $q \in \mathbb{N}_+$ and remaining selling horizon of length $t \geq 0$, for item valuation distributions in the Weibull domain of attraction the asymptotically optimal generalized run-out rate policy prices slightly lower than the classical run-out rate policy. On the other hand, for item valuation distributions in the Frechet domain of attraction it prices slightly higher. For item valuation distributions in the Gumbel domain of attraction, the asymptotically optimal generalized run-out rate policy is exactly the classical run-out rate policy.

8. Discussion

We now provide additional discussion on the optimality results of Section 7. Specifically, in Section 8.1 we provide the intuition behind the generalized run-out rate policies of Section 7.5. Then, in Section 8.2 we analyze the role of the Mills ratio of the customer item valuation distribution in determining the large market regime asymptotics of the minimal regret provided in Section 7.4.

8.1. Generalized Run-Out Rate Policies Interpretation

Suppose that $p \in \mathcal{V}$ is a generalized run-out rate policy of the form (41) where $R_q = c$ for each $q \in \mathbb{N}_+$ and some constant $c > 0$. That is,

$$p(\lambda; q, t) = F^{-1}\left(1 - c \cdot \frac{q}{\lambda t}\right) \text{ for } \lambda t > 0,\ q \in \mathbb{N}_+. \tag{44}$$

In this section, we analyze in expectation the dynamics of the inventory process and price process under the policy p above. We then make an intuitive connection between our analysis and the price process under the asymptotically optimal pricing policies of Section 7.5.

Suppose first for convenience that the customer item valuation distribution F has a positive density on the entirety of its support. This implies that $F(F^{-1}(\pi)) = \pi$ for $\pi \in [0,1)$. Now denote by $Q(p)$ the inventory process corresponding to the pricing policy (44), assuming an initial inventory level of $Q_0 \in \mathbb{N}_+$. Also denote by $t > 0$ the length of the selling horizon. It is then straightforward to show that up until some time s close to t, $Q(p)$ is equal in distribution to the unique solution to the equation

$$Q_s = Q_0 - N\left(\int_0^s c \cdot \frac{Q_s}{\lambda(t-s)} ds\right) \text{ for } 0 \leq s \leq t, \tag{45}$$

where N is a Poisson process with rate $\lambda > 0$. Note that the integrand in the above is greater than 1 for s close to t which technically cannot occur in our model. However, since this only occurs very close to the end of the selling horizon and its effect is minor in the large market regime, we ignore it moving forward in order to streamline the analysis.

Now taking expectations on both sides of (45) and then differentiating with respect to s yields (Brémaud 1981) an ODE for $E[Q_s]$ given by

$$\frac{dE[Q_s]}{ds} = -c \cdot \frac{E[Q_s]}{t-s} \text{ for } 0 \le s < t,$$

with initial condition $E[Q_0] = Q_0$. The solution to this ODE is explicit and given by

$$E[Q_s] = Q_0 \left(1 - \frac{s}{t}\right)^c \text{ for } 0 \le s \le t. \tag{46}$$

The expected inventory level therefore decreases polynomially with respect to the remaining time on the selling horizon. The precise rate of decrease is determined by the choice of the constant c

The expected purchasing probability at each point in time can also be obtained in closed form. Specifically, note by (44) that up until some time s close to t the probability an arriving customer purchases the item given an inventory level of Q_s may be written as

$$\pi(Q_s, t-s) = c \cdot \frac{Q_s}{\lambda(t-s)}.$$

Taking expectations on both sides of the above and ignoring the boundary effects for s close t, it follows from (46) and after some algebra that given an initial inventory level of Q_0 the expected purchasing probability for a customer arriving at time s is given by

$$E[\pi(Q_s, t-s)] = c \cdot \frac{Q_0}{\lambda t} \left(1 - \frac{s}{t}\right)^{c-1} \text{ for } 0 \le s \le t. \tag{47}$$

Note in the above that depending on the choice of the constant $c > 0$, the expected purchasing probability either decreases or increases polynomially (or remains constant) with respect to the remaining time on the selling horizon. This fact is used in the paragraphs below to provide intuition behind the asymptotically optimal pricing policies of Section 7.5.

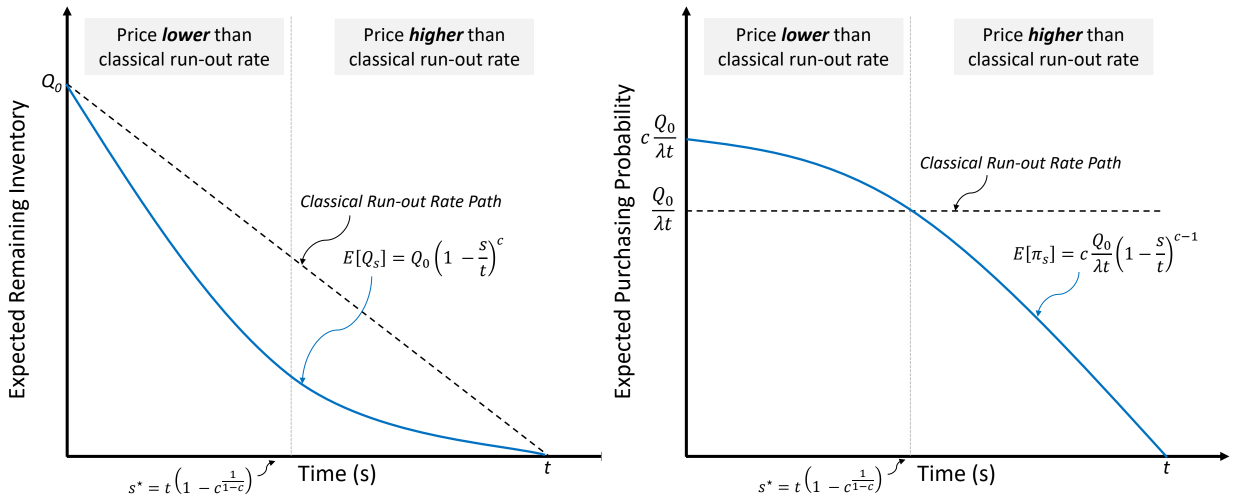
First note that in the case of $c = 1$, the pricing policy (44) is precisely the classical continuous run-out rate pricing policy which is known (Maglaras and Meissner 2006) to be asymptotically optimal in the fluid regime. In particular, substituting $c = 1$ into (46) we recover the optimal expected inventory process of the fluid regime where the expected inventory level decreases linearly from Q_0 at time 0 to 0 at time t. Moreover, assuming that $Q_0 < \lambda t$, it follows from (47) that over the course of the selling horizon the expected purchasing probability is always equal to the static run-out rate purchasing probability of $Q_0/\lambda t$. In terms of price, this suggests that the firm closely follows a fixed price policy.

Now suppose that $c > 1$ in (44). In this case by (46), the expected inventory process lies entirely below the optimal path in the fluid regime corresponding to $c = 1$. In particular, note that the shape of the expected inventory process is convex. See Figure 4a. The intuition behind this is explained by the graph of the expected purchasing probability process. See Figure 4b. Since $c > 1$, it follows from (47) that the expected purchasing probability decreases over the course of the selling horizon. In particular, the expected purchasing probability starts out higher than the static run-out rate purchasing probability but by the end of the selling horizon the expected purchasing probability is lower than it. In terms of price, this suggests that when $c > 1$ in (44) the expected price increases over time. In other words, in expectation the firm follows a markup pricing policy.

When $0 < c < 1$ in (44), the dynamics are reversed. The expected inventory process now lies entirely above the optimal path in the fluid regime. In particular, note that the shape of the expected inventory process is concave. See Figure 5a. Moreover, the expected purchasing probability increases over the course of the selling horizon. See Figure 5b. In particular, at the beginning of the selling horizon the expected purchasing probability is lower than the static run-out rate purchasing probability while towards the end of the horizon the expected purchasing probability is higher than it. This suggests in terms of price that when $0 < c < 1$ in (44) the expected price decreases over time. In other words, in expectation the firm closely follows a markdown pricing policy.

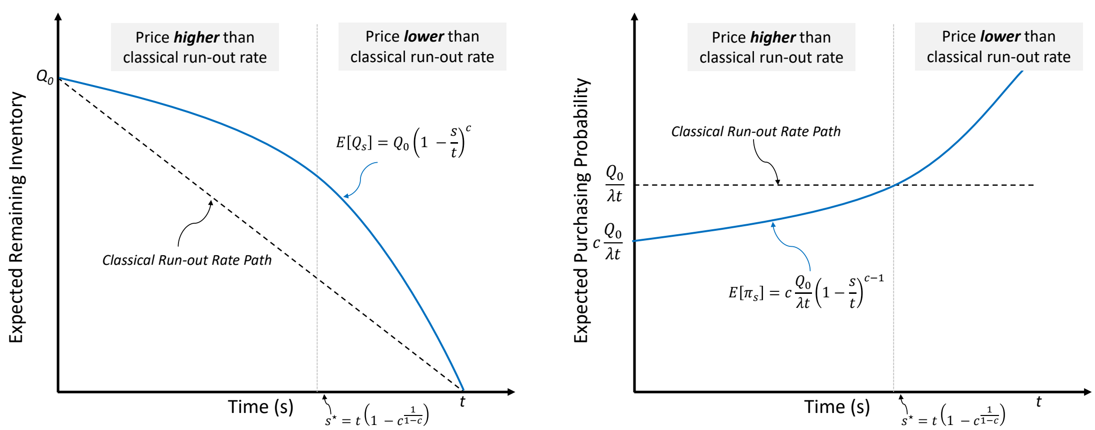
We now connect the observations above with the asymptotically optimal generalized run-out rate policies of Theorem 5. First note by Theorem 5 and (25) that for item valuation distributions in the Weibull domain of attraction, the asymptotically optimal generalized run-out rate policy has run-out rate parameters $\{R_q, q \in \mathbb{N}_+\}$ such that $R_q > q$ for each $q \in \mathbb{N}_+$. This loosely corresponds to $c > 1$ in the policy (44), which by the analysis above suggests that the firm should in expectation follow a mark-up pricing policy. The intuition here is that early on in the selling horizon the firm conservatively prices lower than the static run-out rate price of $F^{-1}(1 - Q_0/\lambda t)$ in order to decrease its odds of having leftover inventory at the end of the selling horizon. Such a policy makes sense given that in the Weibull domain of attraction there is an upper bound on the amount of revenue that can be collected from any one particular item. Later on, as the selling season progresses and inventory levels significantly decrease, the firm is in a position to markup its price above the static run-out rate price.

For item valuation distributions in the Frechet domain of attraction, the asymptotically optimal generalized run-out rate policy has run-out rate parameters $\{R_q, q \in \mathbb{N}_+\}$ such that that $R_q < q$ for $q \in \mathbb{N}_+$. This corresponds to $0 < c < 1$ in (44), which by the analysis above suggests the firm should approximately follow a markdown pricing policy. The intuition in this case is that early on in the selling horizon the firm aggressively prices higher than the static run-out rate price in an attempt to sell to customers with item valuations in the right tail of the valuation distribution. This is sensible given that item valuation distributions in the Frechet domain of attraction are heavy-tailed. Towards the end of the selling horizon, the firm may be left with an excess of inventory and so it lowers its price below the static run-out rate price in order to avoid leftover inventory.



(a) Inventory (b) Purchase Probability

Figure 4 **The expected inventory level and purchasing probability at time $s \in [0, t]$ in the case of $c > 1$.**



(a) Inventory (b) Purchase Probability

Figure 5 **The expected inventory level and purchasing probability at time $s \in [0, t]$ in the case of $0 < c < 1$.**

8.2. Role of the Mills Ratio in Interpreting the Minimal Regret

Recall from Section 5 as well as Abdallah and Reed (2025) that regardless of which of the 3 extreme value domains of attraction the customer item valuation distribution lies in, if F is a von-Mises function, it possesses a density f in a neighborhood of the upper limit of its support. Hence, for sufficiently large x we may define its Mills ratio $m(x) = (1 - F(x))/f(x)$. It is then straightforward to verify that for t sufficiently large, the relevant extreme value theory norming function $a(t)$ may be written in terms of the Mills ratio as

$$a(t) = \alpha(t) \cdot m(F^{-1}(1 - 1/t)), \tag{48}$$

where $\alpha(t)$ is identically 1 for F in the Gumbel domain of attraction, and $\alpha(t) \to \alpha$ as $t \to \infty$ for F in the Weibull and Frechet domains of attraction with index α.

The representation (48) provides an interesting way to interpret the minimal regret asymptotic (37) of Section 7.4. In particular, we have that in the large market regime for each $q \in \mathbb{N}_+$,

$$\text{Regret}^\star(q,t) = c(q) \cdot \alpha(\lambda t) \cdot m(F^{-1}(1 - 1/\lambda t)) + o(a(\lambda t)) \text{ as } \lambda t \to \infty. \tag{49}$$

The Mills ratio always vanishes as $x \to x_U$ for item valuation distributions in the Weibull domain of attraction. Since F^{-1} is always monotonically increasing to x_U, we then recover from (49) the observation previously made in Section 7.4 that in the large market regime the minimal regret vanishes for all F in the Weibull domain of attraction. On the other hand, the Mills ratio always diverges to ∞ as $x \to x_U$ for item valuation distributions in the Frechet domain of attraction. This implies by (49) that the minimal regret always diverges to ∞ for F in the Frechet domain of attraction.

For item valuation distributions in the Gumbel domain of attraction, the defining feature of the Mills ratio is that $m'(x) \to 0$ as $x \to x_U$. This incorporates a variety of limiting behaviors such as m converging to a constant, increasing or decreasing (albeit slowly), and possibly even oscillating. In each of these cases, however, one may attempt to gain information about the asymptotic behavior of the minimal regret from (49).

In the case that F does not lie in any of the 3 extreme value domains attraction, it is tempting to still use (49) as a guide to understanding the behavior of the minimal regret. In particular, the Mills ratio is defined anywhere that F has a positive density. The $c(q)$ and $\alpha(\lambda t)$ terms in (49) unfortunately do not appear to be generalizable in any sort of straightforward way. Nevertheless, it is reasonable to conjecture from (49) that if the Mills ratio of F is vanishing, then so too is the minimal regret and, likewise that if the Mills ratio of F diverges, then the minimal regret does as well.

9. Numerical Experiments

In this section, we present the results of several numerical experiments. In particular, in Section 9.1 we test the large market regime accuracy of our approximations to the optimal value function from Sections 7.2 and 7.3. Then, in Section 9.2 we use our asymptotically optimal pricing policies from Section 7.5 to assist in numercially determining the boundaries between the small market, fluid and large market regimes.

Throughout this section, we use two canonical distributions to represent customer item valuation distributions in the Weibull and Frechet domains of attraction. Specifically, to represent item valuation distributions in the Weibull domain of attraction, we use the Uniform$[0,1]$ distribution, and for item valuation distributions in the Frechet domain of attraction, we use the Pareto distribution $F(x) = 1 - 1/x^2$ for $x \geq 1$. Note that the Uniform$[0,1]$ distribution has an index of $\alpha = 1$ and the Pareto distribution we have selected has an index of $\alpha = 2$.

9.1. Accuracy of Optimal Value Function Approximations

We now proceed to test the accuracy of using the first-order terms Theorems 3 and 4 to approximate the optimal online value function in the large market regime. The fluid upper bound $qF^{-1}(1-q/\lambda t)$ of Gallego and Van Ryzin (1994) is also commonly used to approximate the optimal online value function and so we evaluate its accuracy as well. For each approximation, we calculate its predicted gap relative to the optimal value function as the market size increases with the initial inventory level held fixed. The predicted gap is defined to be the difference between the optimal value function and its value predicted by the approximation. Note that this gap may be either positive or negative. Positive values represent an overprediction and negative values an underprediction. In order to obtain the value of the optimal value function, we numerically ran several experiments. The results are presented in Figure 6. For both the Uniform$[0,1]$ and Pareto customer item valuation distributions, we assume an initial inventory level of $Q_0 = 5$ and vary the the market size from 25 to 250 customers.

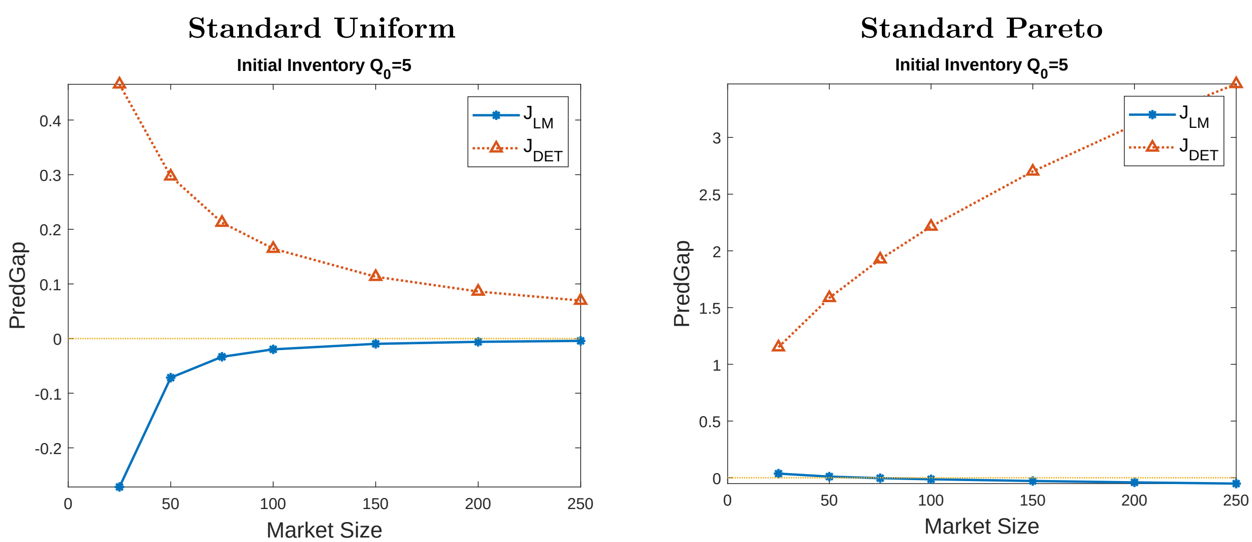


Figure 6 Predicted Gap from the optimal policy for valuations distributed according to Uniform$[0,1]$ and standard Pareto distribution.

We first discuss the case of customer item valuations having a Uniform$[0,1]$ distribution. Recall by (8) that for all item valuation distributions in the Weibull domain of attraction, the norming function $a(\lambda t)$ vanishes as $\lambda t \to \infty$. The predicted gap of the large market regime approximation using the first-order term of (26) should therefore vanish as well. This is indeed the case in the lefthand graph of Figure 6. It also appears to be the case that for the Uniform$[0,1]$ item valuation distribution, the large market regime approximation using (26) serves as a lower bound for the optimal value function. It is an interesting question as to whether this extends to all item valuation distributions in the Weibull domain of attraction. The fluid upper bound in this case also has a predicted gap that vanishes as the market size increases but at a significantly slower rate than the large market regime approximation using (26).

We next consider the case of customer item valuations having a Pareto distribution. In this case, by (12) the norming function $a(\lambda t)$ grows polynomially as $\lambda t \to \infty$, which provides an upper bound on the rate of growth of the predicted gap using the large market regime approximation (31). Viewing the righthand graph in Figure 6 corresponding to the Pareto case, it appears that the first-order term of (31) starts off as an upper bound to the optimal value function but eventually becomes a lower bound when the market size is large. The rate of change of the predicted gap is however quite low. Finally, the predicted gap of the fluid upper bound increases with the market size and possibly diverges to ∞.

9.2. Regime Boundaries

We now numerically determine the boundaries between the small market, fluid and large market regimes in the $(\lambda t, q)$ parameter space. In order to do so, we evaluate the performance of the following 3 approximations to the optimal pricing policy in each regime.

1. A static pricing policy which uses the optimal monopoly price $p^\star = \arg\max_{p>0} p(1 - F(p-))$.
2. A classical fluid run-out rate pricing policy (ReOpt Fluid) as in Maglaras and Meissner (2006), where

$$p(\lambda; q, t) = F^{-1}\left(1 - \frac{q}{\lambda t}\right).$$

3. The asymptotically optimal large market regime pricing policy (LM) of Theorem 5. Specifically,

$$p(\lambda; q, t) = F^{-1}\left(1 - \frac{v_q}{\lambda t}\right),$$

where $v_q = w_q$ or ϕ_q depending on the domain of attraction of F.

We also test the performance of a proposed transition pricing policy between the fluid and large market regimes. Specifically, the following.

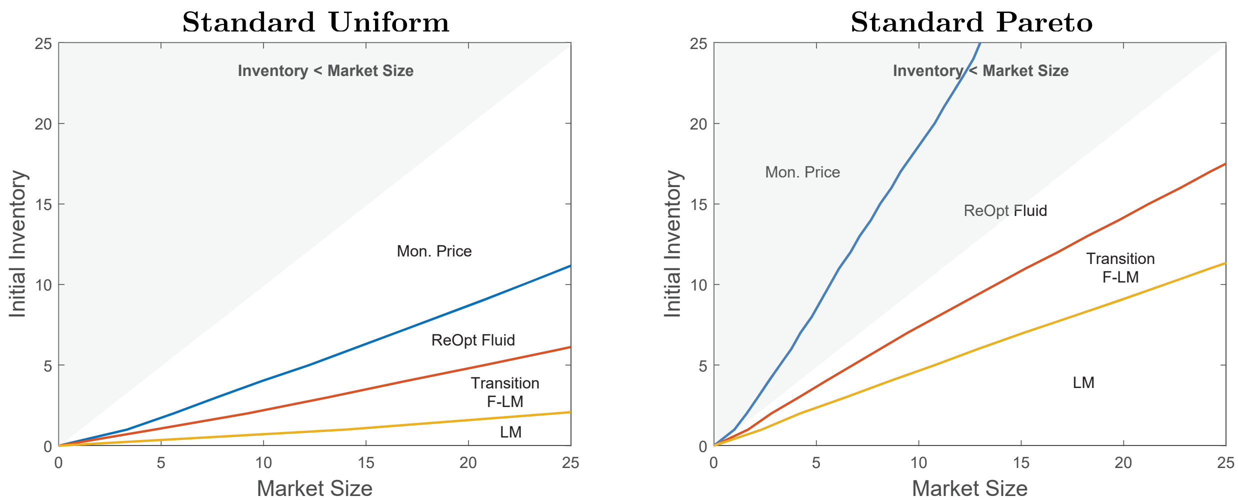


Figure 7 **Phase diagrams for item valuations distributed according to the Uniform$[0,1]$ and Pareto distributions.**

1. A transition pricing policy (F-LM) where either w_q or ϕ_q are approximated using the first two terms in the asymptotic expansion (24). That is,

$$p(\lambda;q,t) = F^{-1}\left(1 - \frac{q + [(\kappa-1)/2]\ln q}{\lambda t}\right),$$

where $\kappa = (\alpha+1)/\alpha$ or $(\alpha-1)/\alpha$ depending on the domain of attraction of F.

Each of the pricing policies above can be improved in practice by bounding their price from below by the optimal monopoly price $p^\star$. This however makes the policies indistinguishable from one another when the inventory level is high relative to the market size. Because of this, in our numerical experiments we implement each pricing policy without the monopoly price lower bound.

We now numerically estimate the expected revenue of each of the 4 pricing policies above over a range of market sizes and initial inventory levels ranging from 1 to 25. We then determine for each pair of parameters which of the 4 policies has the highest expected revenue. Our results are presented in the phase diagrams in Figure 7. The lefthand diagram corresponds to customer item valuations with a Uniform$[0,1]$ distribution and the righthand side a Pareto distribution.

Note that for both the Uniform$[0,1]$ and Pareto distributions, the general structure of the phase diagram is the same. The parameter space is divided into 4 distinct regimes. Holding the market size fixed, the static monopoly pricing policy starts out as the best performing policy when the initial inventory level is high. As the initial inventory level decreases, the fluid pricing policy next becomes the best performing policy. This occurs at the point where inventory first becomes scarce relative to the market size and a static price is no longer optimal. Dropping the inventory level slightly further, the transition pricing policy next becomes the best performing policy. This occurs at the point where the fluid regime begins its transition into the large market regime. Eventually, the transition is complete and the large market regime pricing policy is the best policy to use. At this point, the firm is pricing in the right tail of the item valuation distribution. We also note that the structure of the phase diagrams in Figure 7 is the same as those in Abdallah and Reed (2025) where customer item valuation distributions in the Gumbel domain of attractin were considered.

It also turns out that the boundaries between each of the 4 regimes can be approximated in a simple and intuitive way. Recall from Abdallah and Reed (2025) the definition of the inventory-to-market size ratio (IMR),

$$\text{IMR} = \frac{\text{initial inventory}}{\text{market size}}.$$

Intuitively, for values less than or equal to 100% the IMR represents the percentage of the market to which a sale will be made, assuming there is no leftover inventory at the end of the selling horizon. It is apparent from the phase diagrams in Figure 7 that as the initial inventory level decreases the transitions from one regime to the next approximately occur at specific thresholds of the IMR which do not depend on the market size. These thresholds may be computed and are given in Table 3 below. The small market regime begins at an initital inventory level of ∞. Notice

Regime	Uniform$[0,1]$	Pareto
Fluid	44%	200%
Transition	24%	70%
Large Market	8%	44%

Table 3 Upper thresholds of the IMR for the large market, transition and fluid regimes.

that the upper threshold of the large market regime is much larger for the Pareto item valuation distribution than when items are valued according to a Uniform$[0,1]$ distribution. This is likely due to the fact that the Pareto distribution has a heavy tail which the firm can take advantage of in the large market regime. The fluid regime similarly is larger in the Pareto case too.

References

Abdallah, Tarek, Josh Reed. 2025. Regime dependent approximations for the single-item dynamic pricing problem. *Submitted for publication* .

Arnold, Vladimir I. 1992. *Ordinary differential equations*. Springer Science & Business Media.

Atar, Rami, Martin I Reiman. 2013. Asymptotically optimal dynamic pricing for network revenue management. *Stochastic Systems* **2**(2) 232–276.

Balseiro, Santiago R, Omar Besbes, Dana Pizarro. 2023. Survey of dynamic resource-constrained reward collection problems: Unified model and analysis. *Operations Research* .

Bimpikis, Kostas, Mihalis G Markakis. 2016. Inventory pooling under heavy-tailed demand. *Management Science* **62**(6) 1800–1813.

Bitran, Gabriel, René Caldentey. 2003. An overview of pricing models for revenue management. *Manufacturing & Service Operations Management* **5**(3) 203–229.

Brémaud, Pierre. 1981. *Point processes and queues: martingale dynamics*, vol. 50. Springer.

Chen, Qi, Stefanus Jasin, Izak Duenyas. 2016. Real-time dynamic pricing with minimal and flexible price adjustment. *Management Science* **62**(8) 2437–2455.

Correa, José, Dana Pizarro, Victor Verdugo. 2021. Optimal revenue guarantees for pricing in large markets. *Algorithmic Game Theory: 14th International Symposium, SAGT 2021, Aarhus, Denmark, September 21–24, 2021, Proceedings 14*. Springer, 221–235.

Das, Bikramjit, Anulekha Dhara, Karthik Natarajan. 2021. On the heavy-tail behavior of the distributionally robust newsvendor. *Operations Research* **69**(4) 1077–1099.

den Boer, Arnoud V. 2021. How long does it take to sell a product? *Available at SSRN 3943319* .

Embrechts, Paul, Claudia Klüppelberg, Thomas Mikosch. 2013. *Modelling extremal events: for insurance and finance*, vol. 33. Springer Science & Business Media.

Gabaix, Xavier. 2009. Power laws in economics and finance. *Annu. Rev. Econ.* **1**(1) 255–294.

Gallego, Guillermo, Huseyin Topaloglu. 2019. *Revenue management and pricing analytics*, vol. 209. Springer.

Gallego, Guillermo, Garrett Van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science* **40**(8) 999–1020.

Ibragimov, Rustam, Johan Walden. 2010. Optimal bundling strategies under heavy-tailed valuations. *Management Science* **56**(11) 1963–1976.

Jasin, Stefanus. 2014. Reoptimization and self-adjusting price control for network revenue management. *Operations Research* **62**(5) 1168–1178.

Jiang, Jiashuo, Will Ma, Jiawei Zhang. 2022. Degeneracy is ok: Logarithmic regret for network revenue management with indiscrete distributions. *arXiv preprint arXiv:2210.07996* .

Maglaras, Constantinos, Joern Meissner. 2006. Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing & Service Operations Management* **8**(2) 136–148.

McAfee, R Preston, Vera te Velde. 2008. Dynamic pricing with constant demand elasticity. *Production and operations Management* **17**(4) 432–438.

Merkle, Milan. 1998. Conditions for convexity of a derivative and some applications to the gamma function. *Aequationes Mathematicae* **55**(3) 273–280.

Nair, Jayakrishnan, Adam Wierman, Bert Zwart. 2022. *The fundamentals of heavy tails: Properties, emergence, and estimation*, vol. 53. Cambridge University Press.

Natalini, Pierpaolo, Biagio Palumbo. 2000. Inequalities for the incomplete gamma function. *Mathematical Inequalities & Applications* **3**(1) 69–77.

Olver, Frank W, Daniel W Lozier, Ronald F Boisvert, Charles W Clark. 2010. *NIST handbook of mathematical functions*. Cambridge university press.

Resnick, Sidney I. 2013. *Extreme values, regular variation and point processes*. Springer.

Rudin, Walter. 1953. *Principles of mathematical analysis*.

Wang, Yining, He Wang. 2022. Constant regret resolving heuristics for price-based revenue management. *Operations Research* **70**(6) 3538–3557.

Wendel, James G. 1948. Note on the gamma function. *The American Mathematical Monthly* **55**(9) 563.

Wierman, Adam, Bert Zwart. 2012. Is tail-optimal scheduling possible? *Operations Research* **60**(5) 1249–1257.

Zhao, Wen, Yu-Sheng Zheng. 2000. Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management science* **46**(3) 375–388.

Appendix. Proofs

In the appendix, we provide the proofs of our main result from the paper.

A. Proof of Expansion for v_q

In this section, we provide the proof of Proposition 2.

Proof of Proposition 2. Let $\kappa > 0$ and note that (23) may be rewritten as

$$v_{q-1}^{\kappa} = v_q^{\kappa} - \kappa v_q^{\kappa-1} \text{ for } q \in \mathbb{N}_+, \tag{A1}$$

with $v_0 = 0$. We first show that there exists a unique positive solution $\{v_q(\kappa), q \in \mathbb{N}_+\}$ to (23) or, equivalently, (A1). We proceed by induction on $q \in \mathbb{N}_+$. In the base case of $q = 1$, recalling that $v_0 = 0$, it follows that $v_1(\kappa) = \kappa$, which is positive. Next, let $q > 1$ and suppose that $v_{q-1}(\kappa)$ is unique and positive. From (A1) and the fact that $v_{q-1}(\kappa)$ is positive, it follows that any positive solution $v_q(\kappa)$ to (A1) must satisfy $v_q^{\kappa} - \kappa v_q^{\kappa-1} \geq 0$, or, equivalently, $v_q \geq \kappa$. Hence, we may write $v_{q-1}(\kappa) = f(v_q(\kappa), \kappa)$, where the function $f(\cdot, \kappa) : [\kappa, \infty) \mapsto \mathbb{R}_+$ is defined by

$$f(x, \kappa) := \left(x^{\kappa} - \kappa x^{\kappa-1}\right)^{1/\kappa} = x\left(1 - \frac{\kappa}{x}\right)^{1/\kappa}.$$

Next, note that $\lim_{x \downarrow \kappa} f(x, \kappa) = 0$ and $\lim_{x \to \infty} f(x, \kappa) = \infty$. Moreover, $f(\cdot, \kappa)$ is continuous and strictly increasing in x. Therefore, the equation $v_{q-1}(\kappa) = f(x, \kappa)$ yields a unique positive solution $x = v_q(\kappa)$.

Next we prove by induction that for fixed $q \in \mathbb{N}_+$, $v_q(\kappa)$ is strictly increasing in $\kappa > 0$. The base case of $q = 1$ follows immediately by the above. Now let $q > 1$ and suppose that $v_{q-1}(\kappa)$ is strictly increasing in κ. We then prove by contradiction that $v_q(\kappa)$ is strictly increasing in κ as well. Suppose for two positive values $\kappa_1 < \kappa_2$ that $v_q(\kappa_1) \geq v_q(\kappa_2)$ and recall that for any given $\kappa > 0$, $f(x, \kappa)$ is increasing in $x > \kappa$. Also note that for any given $x > 0$, $f(x, \kappa)$ is strictly decreasing in $0 < \kappa < x$. Hence,

$$v_{q-1}(\kappa_1) = f(v_q(\kappa_1), \kappa_1) \geq f(v_q(\kappa_2), \kappa_1) > f(v_q(\kappa_2), \kappa_2) = v_{q-1}(\kappa_2),$$

which is in contradiction with the induction hypothesis that $v_{q-1}(\kappa)$ is strictly increasing in κ. This shows that $v_q(\kappa_1) < v_q(\kappa_2)$ and establishes that $v_q(\kappa)$ is stricly increasing in κ.

Finally, we prove that for fixed $q \in \mathbb{N}_+$, $v_q(\kappa)$ is continuous in $\kappa > 0$. We again proceed by induction on $q \in \mathbb{N}_+$. The base case of $q = 1$ follows immediately by the above. Next, let $q > 1$ and suppose that $v_{q-1}(\kappa)$ is continuous in κ. Then, fix $\bar{\kappa} > 0$ and note that since by the preceding $v_q(\kappa)$ is increasing in $\kappa > 0$, it follows that there exists a $\bar{v} > 0$ such that $v_q(\kappa) \to \bar{v}$ as $k \uparrow \bar{\kappa}$. Moreover, since by the induction hypothesis $v_{q-1}(\kappa)$ is continuous in κ, it follows by (A1) and the uniqueness portion of the proof above that it must be the case that $\bar{v} = v_q(\bar{\kappa})$. Similarly, it must be the case that $v_q(\kappa) \to v_q(\bar{\kappa})$ as $k \downarrow \bar{\kappa}$. Since $\bar{\kappa} > 0$ was arbitrary, the continuity of $v_q(\kappa)$ is proved.

In order to complete the proof, we now must show that the asymptotic expansion (24) holds. In the case of $\kappa = 1$, it is clear that the unique positive solution to (23) is $\{q, q \in \mathbb{N}_+\}$ and hence (24) holds. Thus, suppose that $0 < \kappa < 1$ or $\kappa > 1$. For ease of notation, for the remainder of the proof we refer to the sequence $\{v_q(\kappa), q \in \mathbb{N}_+\}$ simply as $\{v_q, q \in \mathbb{N}_+\}$. Next, let $f : (0, \infty) \mapsto \mathbb{R}_+$ be the function defined by $f(q) = v_{\lceil q \rceil}^{\kappa-1}$ for

$q>0$, and denote by $F:\mathbb{R}_+\mapsto\mathbb{R}_+$ the antiderivative of f with $F(0)=0$. Then, letting $\gamma(q)=f^{\kappa/(\kappa-1)}(q)/F(q)$ for $q>0$, it is straightforward to verify that

$$(f(q))^{1/(\kappa-1)} = \frac{1}{\kappa}(\gamma(q))^{\frac{1}{\kappa}}\int_0^q(\gamma(u))^{\frac{\kappa-1}{\kappa}}\,du \text{ for } q>0. \tag{A2}$$

Moreover, noting that $f^{1/(\kappa-1)}(q)=v_{\lceil q\rceil}$, it follows subtracting q from both sides of (A2) that after a little bit of algebra we obtain

$$\begin{aligned} v_{\lceil q\rceil}-q = &\left(\frac{1}{\kappa}\right)^{1/\kappa} q\left((\gamma(q))^{1/\kappa}-\kappa^{1/\kappa}\right) \\ &+\frac{1}{\kappa}(\gamma(q))^{1/\kappa}\int_0^q\left((\gamma(u))^{\frac{\kappa-1}{\kappa}}-\kappa^{(\kappa-1)/\kappa}\right)du. \end{aligned} \tag{A3}$$

Hence, in order to complete the proof it suffices to show that

$$\lim_{q\to\infty}\frac{q}{\ln q}\left((\gamma(q))^{1/\kappa}-\kappa^{1/\kappa}\right) = 0 \tag{A4}$$

and

$$\lim_{q\to\infty}\frac{(\gamma(q))^{1/\kappa}}{\ln q}\int_0^q\left((\gamma(u))^{\frac{\kappa-1}{\kappa}}-\kappa^{(\kappa-1)/\kappa}\right)du = \kappa\cdot\frac{\kappa-1}{2}. \tag{A5}$$

In order to show that (A4) holds, we first prove that $\gamma(q)\to\kappa$ and $v_{\lceil q\rceil}/q\to 1$ as $q\to\infty$. In order to see this, first note that by the definition of F and its density function f, it follows that

$$F(q) = \sum_{k=1}^{\lceil q\rceil-1} v_k^{\kappa-1}+(1-(\lceil q\rceil-q))v_{\lceil q\rceil}^{\kappa-1} \text{ for } q\in\mathbb{R}_+. \tag{A6}$$

Next, summing over both sides of (23), we obtain that

$$\frac{v_q^\kappa}{\sum_{k=1}^q v_k^{\kappa-1}} = \kappa \text{ for } q\in\mathbb{N}_+. \tag{A7}$$

Thus, from (A6) and (A7) it follows that

$$\frac{f(q)^{\kappa/(\kappa-1)}}{F(q)} = \kappa \text{ for } q\in\mathbb{N}_+. \tag{A8}$$

Next, noting that $f(q)=f(\lceil q\rceil)$ and $F(q)\le F(\lceil q\rceil)$, it follows from (A7) that $f(q)^{\kappa/(\kappa-1)}/F(q)\ge\kappa$ for $q>0$. Moreover, setting $Q=\lceil q\rceil$, it follows that

$$\frac{f(q)^{\kappa/(\kappa-1)}}{F(q)}-\kappa < v_q^\kappa\cdot\left(\frac{1}{\sum_{k=1}^{Q-1}v_k^{\kappa-1}}-\frac{1}{\sum_{k=1}^{Q}v_k^{\kappa-1}}\right) = \kappa^2\cdot\frac{v_Q^{\kappa-1}}{v_{Q-1}^\kappa}. \tag{A9}$$

Now, using the fact that $\{v_q,q\in\mathbb{N}_+\}$ is a positive and increasing sequence, it is straightforward to show by (23) that $v_Q^{\kappa-1}/v_{Q-1}^\kappa\to 0$ as $Q\to\infty$. Hence, by (A9) we obtain that

$$\lim_{q\to\infty}\frac{f(q)^{\kappa/(\kappa-1)}}{F(q)}-\kappa = 0.$$

The above implies that $\gamma(q)\to\kappa$ as $q\to\infty$. Then, dividing both sides of (A3) by q, it follows that $v_{\lceil q\rceil}/q\to 1$ as $q\to\infty$.

We now show that (A4) holds. First note that by the definition of $\gamma(q)$ and (A8) above, it follows after some algebra that

$$\gamma(q)-\kappa = \frac{f^{\kappa/(\kappa-1)}(\lceil q\rceil)(F(\lceil q\rceil)-F(q))}{F(q)F(\lceil q\rceil)} \text{ for } q>0.$$

Next, note that $F(\lceil q \rceil) - F(q) = v_{\lceil q \rceil}^{\kappa-1}(\lceil q \rceil - q)$ and so since $f(q) = v_{\lceil q \rceil}^{\kappa-1}$, it follows after some further algebra that

$$\gamma(q) - \kappa = C_q \frac{\lceil q \rceil - q}{\lceil q \rceil}, \tag{A10}$$

where

$$C_q = \frac{v_{\lceil q \rceil}^{1+2(\kappa-1)}/(\lceil q \rceil)^{1+2(\kappa-1)}}{(F(q)/(\lceil q \rceil)^{\kappa})(F(\lceil q \rceil)/(\lceil q \rceil)^{\kappa})} \to \kappa^2 \text{ as } q \to \infty, \tag{A11}$$

where the above convergence follows since $\gamma(q) \to \kappa$ and $v_{\lceil q \rceil}/q \to 1$ as $q \to \infty$. Hence, since $(\lceil q \rceil - q)/\lceil q \rceil < 1/\lceil q \rceil$, it follows by an application of Taylor's theorem (Rudin 1953) that (A4) holds.

We now show that (A5) holds, which will complete the proof. Since by the preceding, $\gamma(q) \to \kappa$ as $q \to \infty$, it suffices to show that

$$\lim_{q \to \infty} \frac{1}{\ln q} \int_0^q \left((\gamma(u))^{(\kappa-1)/\kappa} - \kappa^{(\kappa-1)/\kappa} \right) du = \frac{1}{2} \cdot \left(\frac{1}{\kappa} \right)^{1/\kappa} \cdot \kappa \cdot (\kappa - 1).$$

First, note that we may write

$$\int_0^q \left((\gamma(u))^{(\kappa-1)/\kappa} - \kappa^{(\kappa-1)/\kappa} \right) du = \int_0^q \eta_u \frac{\lceil u \rceil - u}{\lceil u \rceil} du \text{ for } q \in \mathbb{R}_+, \tag{A12}$$

where using (A10) together with (A11) and the definition of the derivative, it follows that

$$\eta_u = \frac{\lceil u \rceil}{\lceil u \rceil - u} \left((\gamma(u))^{(\kappa-1)/\kappa} - \kappa^{(\kappa-1)/\kappa} \right) \to \left(\frac{1}{\kappa} \right)^{1/\kappa} \cdot \kappa \cdot (\kappa - 1) \text{ as } u \to \infty. \tag{A13}$$

Now note by (A13) and the fact that

$$\int_q^{\lceil q \rceil} \frac{\lceil u \rceil - u}{\lceil u \rceil} du < \frac{1}{\lceil q \rceil},$$

a straightforward argument shows that in order to complete the proof it suffices to show that

$$\lim_{\substack{q \to \infty \\ q \in \mathbb{N}_+}} \frac{1}{\ln q} \int_1^q \frac{\lceil u \rceil - u}{\lceil u \rceil} du = \frac{1}{2}.$$

In order to see that this is the case first note that for $q = 2, 3, \ldots,$

$$\int_1^q \frac{\lceil u \rceil - u}{\lceil u \rceil} du = \sum_{n=2}^{q} \int_{n-1}^{n} \frac{\lceil u \rceil - u}{\lceil u \rceil} du.$$

Moreover, for each $n = 2, 3, \ldots, q$,

$$\int_{n-1}^{n} \frac{\lceil u \rceil - u}{\lceil u \rceil} du = \frac{1}{n} \int_0^1 (1-u) du = \frac{1}{2n}.$$

Thus

$$\int_1^q \frac{\lceil u \rceil - u}{\lceil u \rceil} du = \frac{1}{2} \sum_{n=2}^{q} \frac{1}{n}$$

and so the result follows since

$$\frac{1}{\ln q} \sum_{n=2}^{q} \frac{1}{n} \to 1 \text{ as } q \to \infty.$$

□

B. The Offline Problem

In this section, we provide the proofs for Section 6.

A2.1. The Weibull Domain of Attraction

We have the following result.

PROPOSITION A1. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (6), then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t \to \infty} \frac{1}{a(\lambda t)} \left(\frac{x_U}{\Gamma(q)} \int_0^{\lambda t} \Gamma(q,v) \cdot dv - J^{\star}_{\mathrm{OFF}}(q, \lambda t) \right) = \frac{\alpha}{\alpha+1} \cdot \frac{\Gamma(q+1+1/\alpha)}{\Gamma(q)}. \tag{A14}$$

Proof of Proposition A1. Recalling from (7) the definition of the norming function a, it follows from (8) and (13) that for each $q \in \mathbb{N}_+$ and $\lambda t > 0$,

$$\frac{1}{a(\lambda t)} \left(\frac{x_U}{\Gamma(q)} \int_0^{\lambda t} \Gamma(q,v) \cdot dv - J^{\star}_{\mathrm{OFF}}(q, \lambda t) \right) = \frac{1}{\Gamma(q)} \int_0^{\lambda t} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv, \tag{A15}$$

where L_1 is a slowly varying function. Now let $0 < \varepsilon < 1$ and write

$$\int_0^{\lambda t} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv = \int_0^{(\lambda t)^{\varepsilon}} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv \tag{A16}$$

$$+ \int_{(\lambda t)^{\varepsilon}}^{\lambda t} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv. \tag{A17}$$

We now analyze each of the terms on the righthand side of (A16) separately.

Regarding the first term on the righthand side of (A16), first note that since L_1 is slowly varying, it follows by Definition 1 that $L_1(\lambda t/v)/L_1(\lambda t) \to 1$ as $\lambda t \to \infty$ for each $v > 0$. Now let $0 < \delta < 1/\alpha$. It then follows by Proposition 0.8 in Resnick (2013) that λt sufficiently large, $L_1(\lambda t/v)/L_1(\lambda t) < v^{-\delta}$ for $0 < v < 1$ and $L_1(\lambda t/v)/L_1(\lambda t) < v^{-\delta}$ for $1 < v < (\lambda t)^{\varepsilon}$. Now recalling by Natalini and Palumbo (2000) the inequality $\Gamma(q,v) < C v^{q-1} e^{-v}$ for $v > 0$ and some constant $C > 0$, it follows by the dominated convergence theorem that

$$\int_0^{(\lambda t)^{\varepsilon}} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv \to \int_0^{\infty} v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv \text{ as } \lambda t \to \infty. \tag{A18}$$

Regarding the second term on the righthand side of (A16), since L_1 is slowly varying, it follows by Definition 1 that $L_1(\lambda t/v)/L_1(\lambda t) \to 1$ as $\lambda t \to \infty$ for each $v > 0$. It thus follows by the dominated convergence theorem that

$$\int_{(\lambda t)^{\varepsilon}}^{\lambda t} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv \to 0 \text{ as } \lambda t \to \infty. \tag{A19}$$

Combining (A18) and (A19), it now follows that

$$\int_0^{\lambda t} \frac{L_1(\lambda t/v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q,v) \cdot dv \to \int_0^{\infty} v^{-1/\alpha} \cdot \Gamma(q,v) \cdot dv \text{ as } \lambda t \to \infty. \tag{A20}$$

Finally, since $\partial \Gamma(q,v)/\partial v = -v^{q-1} e^{-v}$ and using the inequality $\Gamma(q,v) < C v^{q-1} e^{-v}$, integrating-by-parts yields that

$$\int_0^{\infty} v^{1/\alpha} \cdot \Gamma(q,v) \cdot dv = \frac{\alpha}{\alpha+1} \cdot \int_0^{\infty} v^{q+1/\alpha} \cdot e^{-v} dv = \frac{\alpha}{\alpha+1} \cdot \Gamma\left(q+1+\frac{1}{\alpha}\right). \tag{A21}$$

Using (A15), the result now follows. $\square$

A2.2. The Frechet Domain of Attraction

We next have the following result.

PROPOSITION A2. *If F is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (10), then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t \to \infty} \frac{J^\star_{\mathrm{OFF}}(q, \lambda t)}{a(\lambda t)} = \frac{\alpha}{\alpha - 1} \cdot \frac{\Gamma(q + 1 - 1/\alpha)}{\Gamma(q)}. \tag{A22}$$

Proof of Proposition A2. Recalling from (11) the definition of the norming function a, it follows from (12) and (13) that for each $q \in \mathbb{N}_+$ and $\lambda t > 0$,

$$\frac{J^\star_{\mathrm{OFF}}(q, \lambda t)}{a(\lambda t)} = \frac{1}{\Gamma(q)} \int_0^{\lambda t} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv, \tag{A23}$$

where L_1 is a slowly varying function. Letting $0 < \varepsilon < 1$, it follows that we may write

$$\int_0^{\lambda t} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv = \int_0^{\varepsilon} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv \tag{A24}$$

$$+ \int_{\varepsilon}^{\lambda t} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv. \tag{A25}$$

We now analyze each of the terms on the righthand side of (A24) separately.

Regarding the first term on the righthand side of (A24), by Proposition 0.8 in Resnick (2013) it follows that for λt sufficiently large, $L_1(\lambda t / v)/L_1(\lambda t) < v^{1/2\alpha}$ for $0 < v < \varepsilon$. Since $\Gamma(q, \cdot) \leq \Gamma(q)$, we then obtain that for λt sufficiently large,

$$\int_0^{\varepsilon} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv < \Gamma(q) \cdot \left(1 - \frac{1}{2\alpha}\right)^{-1} \cdot \varepsilon^{1 - 1/2\alpha} \to 0 \text{ as } \varepsilon \to 0, \tag{A26}$$

where the final convergence follows since by assumption $\alpha > 1$.

Regarding the second term on the righthand side of (A24), first note that since L_1 is slowly varying, it follows by Definition 1 that $L_1(\lambda t / v)/L_1(\lambda t) \to 1$ as $\lambda t \to \infty$ for each $v > 0$. Next, by (12) we have for each $\lambda t > 0$ and $v > 0$ the equality

$$\frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} = \frac{a(\lambda t / v)}{a(\lambda t)}. \tag{A27}$$

Moreover, since $a(\cdot)$ is an increasing function, it follows for each fixed $\lambda t > 0$ that $(L_1(\lambda t / v)/L_1(\lambda t))v^{-1/\alpha}$ is decreasing in $v > 0$. Now recalling by Natalini and Palumbo (2000) the inequality $\Gamma(q, v) < C v^{q-1} e^{-v}$ for $v > 0$ and some constant $C > 0$, it follows by the dominated convergence theorem that

$$\int_{\varepsilon}^{\lambda t} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv \to \int_{\varepsilon}^{\infty} v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv \text{ as } \lambda t \to \infty. \tag{A28}$$

Combining (A26) and (A28), a straightforward argument now yields that

$$\int_0^{\lambda t} \frac{L_1(\lambda t / v)}{L_1(\lambda t)} \cdot v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv \to \int_0^{\infty} v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv \text{ as } \lambda t \to \infty. \tag{A29}$$

Then, since $\partial \Gamma(q, v)/\partial v = -v^{q-1} e^{-v}$ and using the inequality $\Gamma(q, v) < C v^{q-1} e^{-v}$, integrating-by-parts yields that

$$\int_0^{\infty} v^{-1/\alpha} \cdot \Gamma(q, v) \cdot dv = \frac{\alpha}{\alpha - 1} \cdot \int_0^{\infty} v^{q - 1/\alpha} \cdot e^{-v} dv = \frac{\alpha}{\alpha - 1} \cdot \Gamma\left(q + 1 - \frac{1}{\alpha}\right). \tag{A30}$$

Using (A23), the result now follows. □

A2.3. Bounds on Constants in Table 1

We have the following.

Proof of Proposition 1. First note for the Weibull case that if $\alpha > 1$ and $q \geq 1$, then

$$\frac{\Gamma(q+1+1/\alpha)}{\Gamma(q+1)} \leq (q+1)^{1/\alpha} < q^{1/\alpha}\left(1+\frac{1}{\alpha q}\right) \leq q^{1/\alpha}\cdot\frac{\alpha+1}{\alpha}, \tag{A31}$$

where the first inequality follows by the results of Wendel (1948) and the second inequality can be seen by a Taylor expansion. Substituting into the expression (15) for W_q, one then obtains that $W_q < ((\alpha+1/q)/(\alpha+1))^\alpha q$.

Next, for the Gumbel case first note that $H_q = H_{q-1} + 1/q$ for each $q \in \mathbb{N}_+$. Now let ψ denote the digamma function (Olver et al. 2010) in which case we have the identity $\psi(q) = H_{q-1} - \gamma$. Moreover, $\psi(q) \leq \ln q - 1/2q$. It then follows from (14) that $C_q/q \leq \ln q + 1/2q - 1$ and so $G_q \leq q\exp(1/2q-1)$.

Finally, for the Frechet case note that if $\alpha > 1$ and $q \geq 1$, then

$$\frac{\Gamma(q+1-1/\alpha)}{\Gamma(q+1)} = \frac{\Gamma(q+1-1/\alpha)}{q\Gamma(q)} \leq q^{-1/\alpha}, \tag{A32}$$

where the second inequality follows by the results of Wendel (1948). Substituting into the expression (15) for F_q, one then obtains that $F_q < ((\alpha-1)/\alpha)^\alpha q$. □

C. Proofs of Main Results

Let a and b be the norming functions of Section 5 for item valuation distributions in either the Weibull or Frechet domains of attraction. Then, for each $q \in \mathbb{N}_+$ and $\lambda t > 0$, define the centered and scaled optimal value function

$$\tilde{J}^\star(q,\lambda t) = \frac{J^\star(q,\lambda t) - qb(\lambda t)}{a(\lambda t)},$$

centered and scaled optimal selling price

$$\tilde{p}^\star(q,\lambda t) = \frac{p^\star(q,\lambda t) - b(\lambda t)}{a(\lambda t)},$$

and scaled optimal purchasing probability

$$\tilde{\pi}^\star(q,\lambda t) = \lambda t \pi^\star(q,\lambda t).$$

In this section, we prove Propositions A3 and A4 below.

A3.1. The Weibull Domain of Attraction

In this subsection, we prove the following.

Proposition A3. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (6), then for each $q \in \mathbb{N}_+$,*

$$\lim_{\lambda t\to\infty} \tilde{J}^\star(q,\lambda t) = -w_q^{(\alpha+1)/\alpha} \tag{A33}$$

and

$$\lim_{\lambda t\to\infty} \tilde{p}^\star(q,\lambda t) = -w_q^{1/\alpha} \tag{A34}$$

and

$$\lim_{\lambda t\to\infty} \tilde{\pi}^\star(q,\lambda t) = w_q, \tag{A35}$$

where $w_q = v_q((\alpha+1)/\alpha)$.

Proof of Proposition A3. It suffices to prove the result for the case of $\lambda = 1$ and $t \to \infty$. Also note that since F is assumed to lie in the Webull domain of attraction, it follows that $x_U < \infty$. Now for each $q \in \mathbb{N}_+$ and $t \geq 0$ set

$$\hat{J}^\star(q,t) = qx_U - J^\star(q,t) \text{ and } \hat{p}^\star(q,t) = x_U - p^\star(q,t). \tag{A36}$$

Also set $\Delta\hat{J}^\star(q,t) = \hat{J}^\star(q,t) - \hat{J}^\star(q-1,t)$, where $\hat{J}^\star(0,\cdot) = 0$. The HJB equation (4) then implies that

$$-\frac{\partial \hat{J}^\star(q,t)}{\partial t} = \sup_{\hat{p}\in[0,x_U]} \left\{(1 - F(x_U - \hat{p}))(\Delta\hat{J}^\star(q,t) - \hat{p})\right\}. \tag{A37}$$

Moreover, by (5) of Theorem 1, $\hat{p}^\star(q,t)$ in (A36) is equal to a maximizer on the righthand side of (A37).

Next, it is straightforward to show that $0 < \Delta\hat{J}^\star(q,t) < x_U$ for $q \in \mathbb{N}_+$ and $t > 0$ with $\Delta\hat{J}^\star(q,t) \to 0$ as $t \to \infty$. Moreover, the objective function on the righthand side of (A37) equals zero when $\hat{p} = 0$, is positive for $0 < \hat{p} < \Delta\hat{J}^\star(q,t)$, equals zero when $\hat{p} = \hat{J}^\star(q,t)$, and is negative for $\Delta\hat{J}^\star(q,t) < \hat{p} \leq x_U$. Thus, $0 < \hat{p}^\star(q,t) < \Delta\hat{J}^\star(q,t)$ and so since $\Delta\hat{J}^\star(q,t) \to 0$ as $t \to \infty$, there exists a $t^{'}$ such that $x_U - \hat{p}^\star(q,t) > x_0$ for $t > t^{'}$. Then, recalling by the von-Mises condition (6) that F is absolutely continuous on (x_0, x_U) with density f, it follows after some algebra that for $t > t^{'}$, $\hat{p}^\star(q,t)$ must satisfy the first-order condition

$$\frac{\hat{p}^\star(q,t)}{\Delta\hat{J}^\star(q,t)} = C_0(q,t) = \left(1 + \frac{1 - F(x_U - \hat{p}^\star(q,t))}{\hat{p}^\star(q,t) f(x_U - \hat{p}^\star(q,t))}\right)^{-1} \to \frac{\alpha}{\alpha+1} \text{ as } t \to \infty, \tag{A38}$$

where the convergence follows by the von-Mises condition (6) and since $\hat{p}^\star(q,t) \to 0$ as $t \to \infty$.

Now define $\check{J}^\star(q,t) = \hat{J}^\star(q,t)/a(t)$ and set $\Delta\check{J}^\star(q,t) = \check{J}^\star(q,t) - \check{J}^\star(q-1,t)$. Also note from the von-Mises condition (6) that a is absolutely continuous on (t_0, ∞) where $t_0 = 1/(1 - F(x_0))$. Thus, by (A37) it follows after some algebra that for $t > t_0$,

$$\frac{d\check{J}^\star(q,t)}{dt} = \frac{1}{t}\left(C_1(t)\check{J}^\star(q,t) - C_2(q,t)\left(\Delta\check{J}^\star(q,t)\right)^{\alpha+1}\right), \tag{A39}$$

where

$$C_1(t) = -\frac{ta^{'}(t)}{a(t)} \to \frac{1}{\alpha} \text{ as } t \to \infty \tag{A40}$$

and

$$C_2(q,t) = (C_0(q,t))^\alpha (1 - C_0(q,t)) L\left(\frac{1}{C_0(q,t)\Delta\hat{J}^\star(q,t)}\right)\left(L\left(\frac{1}{a(t)}\right)\right)^{-1}. \tag{A41}$$

The convergence in (A40) follows from the inverse function theorem (Rudin 1953) and the von-Mises condition (6).

We now prove that for each $q \in \mathbb{N}_+$,

$$\check{J}^\star(q,t) \to w_q^{(\alpha+1)/\alpha} \text{ as } t \to \infty. \tag{A42}$$

Since $\check{J}^\star(q,t) = -\tilde{J}^\star(q,t)$, this proves (A33). (A34) and (A35) then follow from (A33) combined with (A38) and Proposition 2. Thus, proving (A42) is sufficient to complete the proof.

We proceed by induction on q. Let $q \in \mathbb{N}_+$ and suppose that (A42) holds for $q - 1$ (note this is automatic in the base case of $q = 1$). Moreover, suppose that (to be proven below)

$$0 < \liminf_{t\to\infty} \Delta\check{J}^\star(q,t) \leq \limsup_{t\to\infty} \Delta\check{J}^\star(q,t) < \infty. \tag{A43}$$

Then, since L is slowly varying and $a(t) \to 0$ as $t \to \infty$, it follows by (A38), (A41) and Proposition 0.5 of Resnick (2013) that

$$C_2(q,t) \to \frac{1}{\alpha} \cdot \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \text{ as } t \to \infty. \tag{A44}$$

Next, note that (A43) and the induction hypothesis imply that $\grave{J}^\star(q,t) > w_{q-1}^{(\alpha+1)/\alpha}$ for t sufficienlty large. Then, by (A39) for sufficiently large t we may write

$$\frac{d\grave{J}^\star(q,t)}{dt} = \frac{1}{\alpha t}\left(\varepsilon(q,t) + g(\grave{J}^\star(q,t))\right), \tag{A45}$$

where

$$g(x) = x - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} (x - w_{q-1}^{(\alpha+1)/\alpha})^{\alpha+1} \text{ for } x \in [w_{q-1}^{(\alpha+1)/\alpha}, \infty),$$

and by (A40), (A43), (A44) and the induction hypothesis, $\varepsilon(q,t) \to 0$ as $t \to \infty$. Next, since by assumption $\alpha > 0$, it follows by Proposition 2 that the function g has a unique root on $[w_{q-1}^{(\alpha+1)/\alpha}, \infty)$ which is given by $w_q^{(\alpha+1)/\alpha}$. Moreover, it is straightforward to verify that $g(x)$ is positive for $x \in [w_{q-1}^{(\alpha+1)/\alpha}, w_q^{(\alpha+1)/\alpha})$ and negative for $x \in (w_q^{(\alpha+1)/\alpha}, \infty)$ with $g(x) \to -\infty$ as $x \to \infty$. Hence, since $\varepsilon(t) \to 0$ as $t \to \infty$, it can be shown using the comparison theorem (Arnold 1992) that $\grave{J}^\star(q,t) \to w_q^{(\alpha+1)/\alpha}$ as $t \to \infty$, and so (A42) holds.

In order to complete the proof it remains to prove (A43). We begin with the lim inf result. First recall that $\Delta\hat{J}^\star(q,t) \to 0$ and $a(t) \to 0$ as $t \to \infty$. Next, let $\varepsilon > 0$. Then, by (A38) and since the function L is slowly varying, it follows by Proposition 0.8 of Resnick (2013) that there exists a $t_\varepsilon > 0$ such that for $t > t_\varepsilon$,

$$L\left(\frac{1}{C_0(q,t)\Delta\hat{J}^\star(q,t)}\right)\left(L\left(\frac{1}{a(t)}\right)\right)^{-1} \le (C_0(q,t))^{-\varepsilon}(\Delta\grave{J}^\star(q,t))^{-\varepsilon} \text{ if } \Delta\grave{J}^\star(q,t) < 1/C_0(q,t).$$

Subtracting $d\grave{J}^\star(q-1,t)/dt$ from both sides of (A39) and recalling that $\grave{J}^\star(q,t) > \grave{J}^\star(q-1,t)$, it follows that

$$\frac{d\Delta\grave{J}^\star(q,t)}{dt} \ge \frac{1}{t}\left(C_1(t)\grave{J}^\star(q-1,t) - t\frac{d\grave{J}^\star(q-1,t)}{dt} - (C_0(q,t))^{\alpha-\varepsilon}(1 - C_0(q,t))\left(\Delta\grave{J}^\star(q,t)\right)^{\alpha+1-\varepsilon}\right) \tag{A46}$$

for t sufficiently large if $\Delta\grave{J}^\star(q,t) < 1/C_0(q,t)$. Now recall the convergences of $C_0(q,t)$ and $C_1(t)$ in (A38) and (A40), respectively. Also recall by the induction hypothesis that $\grave{J}^\star(q-1,t) \to w_{q-1}^{(\alpha+1)/\alpha} > 0$ as $t \to \infty$. By (A45) and the induction hypothesis this also implies that

$$t\frac{d\grave{J}^\star(q-1,t)}{dt} \to 0 \text{ as } t \to \infty.$$

Now select ε such that $\alpha + 1 - \varepsilon > 0$. It then follows by (A46) that there exists a $\bar{\Delta} > 0$ such that for t sufficiently large, $d\Delta\grave{J}^\star(q,t)/dt > 0$ if $\Delta\grave{J}^\star(q,t) < \bar{\Delta}$. This then proves the lim inf result.

Next, we prove the lim sup result. By the induction hypothesis it suffices to show that $\limsup_{t\to\infty} \grave{J}^\star(q,t) < \infty$. Now recall that $\Delta\hat{J}^\star(q,t), a(t) \to 0$ as $t \to \infty$ and let $\varepsilon > 0$. Then, by (A38) and since the function L is slowly varying, it follows by Proposition 0.8 of Resnick (2013) that there exists a $t_\varepsilon > 0$ such that for $t > t_\varepsilon$,

$$L\left(\frac{1}{C_0(q,t)\Delta\hat{J}^\star(q,t)}\right)\left(L\left(\frac{1}{a(t)}\right)\right)^{-1} \ge (C_0(q,t))^{-\varepsilon}(\Delta\grave{J}^\star(q,t))^{-\varepsilon} \text{ if } \Delta\grave{J}^\star(q,t) > 1/C_0(q,t).$$

By (A39), this implies that

$$\frac{d\grave{J}^\star(q,t)}{dt} \leq \frac{1}{t}\left(C_1(t)\grave{J}^\star(q,t) - (C_0(q,t))^{\alpha-\varepsilon}(1-C_0(q,t))\left(\Delta\grave{J}^\star(q,t)\right)^{\alpha-\varepsilon+1}\right) \tag{A47}$$

for t sufficiently large if $\Delta\grave{J}^\star(q,t) > 1/C_0(q,t)$. Now recall the convergences of $C_0(q,t)$ and $C_1(t)$ in (A38) and (A40), respectively. Also recall by the induction hypothesis that $\grave{J}^\star(q-1,t) \to w_{q-1}^{(\alpha+1)/\alpha}$ as $t\to\infty$. Now select ε such that $\alpha-\varepsilon+1>1$. It then follows by (A47) that there exists a $\bar{J}>0$ such that for t sufficiently large, $d\grave{J}^\star(q,t)/dt<0$ if $\grave{J}^\star(q,t)>\bar{J}$. This then proves the lim sup result. □

A3.2. The Frechet Domain of Attraction

In this subsection, we prove the following.

Proposition A4. *If F is in the Frechet domain of attraction with index $\alpha>1$ and satisfies the von-Mises condition* (10)*, then for each $q\in\mathbb{N}_+$,*

$$\lim_{\lambda t\to\infty}\tilde{J}^\star(q,\lambda t) = \phi_q^{(\alpha-1)/\alpha} \tag{A48}$$

and

$$\lim_{\lambda t\to\infty}\tilde{p}^\star(q,\lambda t) = \phi_q^{-1/\alpha} \tag{A49}$$

and

$$\lim_{\lambda t\to\infty}\tilde{\pi}^\star(q,\lambda t) = \phi_q, \tag{A50}$$

where $\phi_q = v_q((\alpha-1)/\alpha)$.

Proof of Proposition A4. It suffices to prove the result for the case of $\lambda=1$ and $t\to\infty$. Next, note that for each $q\in\mathbb{N}_+$ the objective function on the righthand side of the HJB equation (4) equals zero when $p=0$, is positive for $p>0$ and, since $\alpha>1$, tends to 0 as $p\to\infty$. Moreover, it is straightforward to show using the monotonicity (Zhao and Zheng 2000) of $p^\star(q,\cdot)$ that $p^\star(q,t)\to\infty$ as $t\to\infty$. Thus, there exists a t' such that $p^\star(q,t)>x_0$ for $t>t'$. Then, recalling by the von-Mises condition (10) that F is absolutely continuous on (x_0,∞) with density f, it follows after some algebra that for $t>t'$, $p^\star(q,t)$ must satisfy the first-order condition

$$\frac{p^\star(q,t)}{\Delta J^\star(q,t)} = C_0(q,t) = \left(1-\frac{1-F(p^\star(q,t))}{p^\star(q,t)f(p^\star(q,t))}\right)^{-1} \to \frac{\alpha}{\alpha-1} \text{ as } t\to\infty, \tag{A51}$$

where the convergence follows by the von-Mises condition (10) and since $p^\star(q,t)\to\infty$ as $t\to\infty$.

Now set $\Delta\tilde{J}^\star(q,t)=\tilde{J}^\star(q,t)-\tilde{J}^\star(q-1,t)$. Also note from the von-Mises condition (10) that the norming function a is absolutely continuous on (t_0,∞) where $t_0=1/(1-F(x_0))$. Thus, by the HJB equation (4) it follows after some algebra that for $t>t_0$,

$$\frac{d\tilde{J}^\star(q,t)}{dt} = \frac{1}{t}\left(C_1(q,t)\left(\Delta\tilde{J}^\star(q,t)\right)^{1-\alpha} - C_2(t)\tilde{J}^\star(q,t)\right), \tag{A52}$$

where

$$C_1(q,t) = (C_0(q,t))^{-\alpha}(C_0(q,t)-1)\frac{L(C_0(q,t)\Delta J^\star(q,t))}{L(a(t))} \tag{A53}$$

and

$$C_2(t) \;=\; \frac{ta^{'}(t)}{a(t)} \;\rightarrow\; \frac{1}{\alpha} \text{ as } t\rightarrow\infty. \tag{A54}$$

The convergence in (A54) follows from the inverse function theorem (Rudin 1953) and the von-Mises condition (10).

We now prove that (A48) holds for each $q\in\mathbb{N}_+$. (A49) and (A50) then follow from (A48) combined with (A51) and Proposition 2. Thus, proving (A48) is sufficient to complete the proof.

We proceed by induction on q. Let $q\in\mathbb{N}_+$ and suppose that (A48) holds for $q-1$ (note this is automatic in the base case of $q=1$). Moreover, suppose that (to be proven below)

$$0 \;<\; \liminf_{t\rightarrow\infty}\Delta\tilde{J}^{\star}(q,t) \leq \limsup_{t\rightarrow\infty}\Delta\tilde{J}^{\star}(q,t) < \infty. \tag{A55}$$

Then, since L is slowly varying and $a(t)\rightarrow\infty$ as $t\rightarrow\infty$, it follows by (A51), (A53) and Proposition 0.5 of Resnick (2013) that

$$C_1(q,t) \;\rightarrow\; \frac{1}{\alpha}\cdot\left(\frac{\alpha}{\alpha-1}\right)^{1-\alpha} \quad \text{as } t\rightarrow\infty. \tag{A56}$$

Next, note that (A55) and the induction hypothesis imply that $\tilde{J}^{\star}(q,t) > \phi_{q-1}^{(\alpha-1)/\alpha}$ for t sufficienlty large. Then, by (A52) for sufficiently large t we may write

$$\frac{d\tilde{J}^{\star}(q,t)}{dt} \;=\; \frac{1}{\alpha t}\left(\varepsilon(t) + g(\tilde{J}^{\star}(q,t))\right), \tag{A57}$$

where

$$g(x) \;=\; \left(\frac{\alpha}{\alpha-1}\right)^{1-\alpha}(x-\phi_{q-1}^{(\alpha-1)/\alpha})^{1-\alpha} - x \text{ for } x\in(\phi_{q-1}^{(\alpha-1)/\alpha},\infty),$$

and by (A54), (A55), (A56) and the induction hypothesis, $\varepsilon(t)\rightarrow 0$ as $t\rightarrow\infty$. Since by assumption $\alpha>1$, it follows by Proposition 2 that the function g has a unique root on $(\phi_{q-1}^{(\alpha-1)/\alpha},\infty)$ which is given by $\phi_{q}^{(\alpha-1)/\alpha}$. Moreover, it is straightforward to verify that $g(x)$ is positive for $x\in(\phi_{q-1}^{(\alpha-1)/\alpha},\phi_{q}^{(\alpha-1)/\alpha})$ with $g(x)\rightarrow\infty$ as $x\rightarrow\phi_{q-1}^{(\alpha-1)/\alpha}$, and negative for $x\in(\phi_{q}^{(\alpha-1)/\alpha},\infty)$ with $g(x)\rightarrow-\infty$ as $x\rightarrow\infty$. Hence, since $\varepsilon(t)\rightarrow 0$ as $t\rightarrow\infty$, it can be shown using the comparison theorem(Arnold 1992) that $\tilde{J}^{\star}(q,t)\rightarrow\phi_{q}^{(\alpha-1)/\alpha}$ as $t\rightarrow\infty$, and so (A48) holds.

In order to complete the proof it remains to prove (A55). We begin with the lim inf result. First recall by (12) that $a(t)\rightarrow\infty$ as $t\rightarrow\infty$. Also, since $p^{\star}(q,t)\rightarrow\infty$ at $t\rightarrow\infty$, it is straightforward to show using (5) that it must be the case that $\Delta J^{\star}(q,t)\rightarrow\infty$ as $t\rightarrow\infty$ as well. Next, let $\varepsilon>0$. Then, by (A51) and since the function L is slowly varying, it follows by Proposition 0.8 of Resnick (2013) that there exists a $t_{\varepsilon}>0$ such that for $t>t_{\varepsilon}$,

$$L\left(C_0(q,t)\Delta J^{\star}(q,t)\right)\left(L\left(a(t)\right)\right)^{-1} \;\geq\; (C_0(q,t))^{\varepsilon}(\Delta\tilde{J}^{\star}(q,t))^{\varepsilon} \text{ if } \Delta\tilde{J}^{\star}(q,t) < 1/C_0(q,t).$$

Subtracting $d\tilde{J}^{\star}(q-1,t)/dt$ from both sides of (A52), we then obtain that

$$\frac{d\Delta\tilde{J}^{\star}(q,t)}{dt} \;\geq\; \frac{1}{t}\left((C_0(q,t))^{\varepsilon-\alpha}\left(C_0(q,t)-1\right)\left(\Delta\tilde{J}^{\star}(q,t)\right)^{1+\varepsilon-\alpha} - t\frac{d\tilde{J}^{\star}(q-1,t)}{dt} - C_2(t)\tilde{J}^{\star}(q,t)\right) \tag{A58}$$

for t sufficiently large if $\Delta\tilde{J}^\star(q,t) < 1/C_0(q,t)$. Now recall the convergences of $C_0(q,t)$ and $C_2(t)$ in (A51) and (A54), respectively. Also recall by the induction hypothesis that $\tilde{J}^\star(q-1,t) \to \phi_{q-1}^{(\alpha-1)/\alpha}$ as $t \to \infty$. By (A57) and the induction hypothesis this also implies that

$$t\frac{d\tilde{J}^\star(q-1,t)}{dt} \to 0 \text{ as } t \to \infty.$$

Now select ε such that $1+\varepsilon-\alpha < 0$. Such a ε exists since by assumption $\alpha > 1$. It then follows by (A58) that there exists a $\bar{\Delta} > 0$ such that for t sufficiently large, $d\Delta\tilde{J}^\star(q,t)/dt > 0$ if $\Delta\tilde{J}^\star(q,t) < \bar{\Delta}$. This then proves the $\liminf$ result.

Next, we prove the $\limsup$ result. By the induction hypothesis, it suffices to show that $\limsup_{t\to\infty} \tilde{J}^\star(q,t) < \infty$. Recall as above that $\Delta J^\star(q,t), a(t) \to \infty$ as $t \to \infty$. Next, let $\varepsilon > 0$. Then, by (A51) and since the function L is slowly varying, it follows by Proposition 0.8 of Resnick (2013) that there exists a $t_\varepsilon > 0$ such that for $t > t_\varepsilon$,

$$L\left(C_0(q,t)\Delta J^\star(q,t)\right)\left(L\left(a(t)\right)\right)^{-1} \le (C_0(q,t))^\varepsilon(\Delta\tilde{J}^\star(q,t))^\varepsilon \text{ if } \Delta\tilde{J}^\star(q,t) > 1/C_0(q,t).$$

By (A52), this implies that

$$\frac{d\tilde{J}^\star(q,t)}{dt} \le \frac{1}{t}\left((C_0(q,t))^{\varepsilon-\alpha}\left(C_0(q,t)-1\right)\left(\Delta\tilde{J}^\star(q,t)\right)^{1+\varepsilon-\alpha} - C_2(t)\tilde{J}^\star(q,t)\right) \tag{A59}$$

for t sufficiently large if $\Delta\tilde{J}^\star(q,t) > 1/C_0(q,t)$. Now recall the convergences of $C_0(q,t)$ and $C_1(t)$ in (A51) and (A54), respectively. Also recall by the induction hypothesis that $\tilde{J}^\star(q-1,t) \to \phi_{q-1}^{(\alpha-1)/\alpha}$ as $t \to \infty$. Now select ε such that $1+\varepsilon-\alpha < 0$. Such a ε exists since by assumption $\alpha > 1$. It then follows by (A59) that there exists a $\bar{J} > 0$ such that for t sufficiently large, $d\grave{J}^\star(q,t)/dt < 0$ if $\grave{J}^\star(q,t) > \bar{J}$. This then proves the $\limsup$ result. $\square$

D. Asymptotic Optimality Proof

In this section, we provide the proof of Lemma 1.

Proof of Lemma 1 It suffices to prove the result for the case of $\lambda = 1$ and $t \to \infty$. The proof for the case of F lying in the Gumbel domain of attraction follows from a straightforward modification to the results of Abdallah and Reed (2025). Next, assume that F lies in either the Weibull or Frechet domain of attraction and satisfies either the von-Mises condition (6) or (10), respectively. It then follows that F is absolutely continuous on (x_0, x_U). This then implies that $1 - F(F^{-1}(1-r)) = r$ for any $r < F(1-x_0) = r_0$. Now let $p \in \Pi$ be a generalized run-out rate policy with run-out rate parameters $\{R_q, q \in \mathbb{N}_+\}$. It then follows by standard theory that for each $q \in \mathbb{N}_+$ and $t > c_q = R_q/r_0$,

$$\frac{\partial J_p(q,t)}{\partial t} = \frac{R_q}{t}(F^{-1}(1-R_q/t) - J_p(q,t) + J_p(q-1,t)).$$

The above is a linear, first-order ODE and its solution for $t > c_q$ is given by

$$J_p(q,t) - \left(\frac{c_q}{t}\right)^{R_q} J_p(q,c_q) = \frac{R_q}{t^{R_q}}\int_{c_q}^t s^{R_q-1}F^{-1}(1-R_q/s)ds + \frac{R_q}{t^{R_q}}\int_{c_q}^t s^{R_q-1}J_p(q-1,s)ds. \tag{A60}$$

Now suppose that F is in the Weibull domain of attraction. We proceed to prove (42) by induction on $q \in \mathbb{N}_+$. Let $q \in \mathbb{N}_+$ and suppose that (42) holds for $q-1$. Clearly, this is true in the base case of $q = 1$.

Now recall from (7) that $b(t) = x_U$ and $F^{-1}(1-1/t) = x_U - a(t)$ where $a(t) = L_1(t)t^{-1/\alpha}$ and L_1 is a slowly varying function. It then follows from (A60) after some algebra that for $t > c_q$,

$$qx_U - J_p(q,t) = (c_q/t)^{R_q}(qx_U - J_p(q,c_q)) + \frac{R_q^{(\alpha+1)/\alpha}}{t^{R_q}} \int_{c_q}^{t} s^{R_q-(\alpha+1)/\alpha} L_1(s/R_q)ds \qquad \text{(A61)}$$

$$+\frac{R_q}{t^{R_q}} \int_{c_q}^{t} s^{R_q-1}((q-1)x_U - J_p(q-1,s))ds.$$

Next, if $R_q > 1/\alpha$, then since $a(t) = t^{-1/\alpha}L_1(t)$, it follows that

$$\lim_{t\to\infty} \frac{1}{a(t)}(c_q/t)^{R_q}(qx_U - J_p(q,c_q)) = 0.$$

Also if $R_q > 1/\alpha$, it follows by Karamata's theorem (Resnick 2013) that

$$\lim_{t\to\infty} \frac{1}{a(t)} \frac{R_q^{(\alpha+1)/\alpha}}{t^{R_q}} \int_{c_q}^{t} s^{R_q-(\alpha+1)/\alpha} L_1(s/R_q)ds = \frac{R_q^{(1+\alpha)/\alpha}}{R_q - (1/\alpha)}.$$

inally, by the induction hypothesis it follows that $((q-1)x_U - J_p(q-1,t)) = t^{-1/\alpha}L_2(t)$ where L_2 is a slowly varying function and $L_2(t)/L_1(t) \to \xi_{q-1}(1/\alpha)$ as $t\to\infty$. Therefore, for $R_q > 1/\alpha$ it follows by Karamata's theorem that

$$\lim_{t\to\infty} \frac{1}{a(t)} \frac{R_q}{t^{R_q}} \int_{c_q}^{t} s^{R_q-1}((q-1)x_U - J_p(q-1,s))ds = \frac{R_q}{R_q-(1/\alpha)}\xi_{q-1}(1/\alpha).$$

Now dividing both sides of (A61) by $a(t)$ and taking the limit as $t\to\infty$ yields the desired result.

Finally, suppose that F is in the Frechet domain of attraction. We again prove (42) by induction on $q \in \mathbb{N}_+$. Let $q \in \mathbb{N}_+$ and suppose that (42) holds for $q-1$. Clearly this is true in the base case of $q = 1$. Now recall from (11) that $b(t) = 0$ and $F^{-1}(1-1/t) = a(t)$ where $a(t) = L_1(t)t^{1/\alpha}$ and L_1 is a slowly varying function. It then follows from (A60) after some algebra that for $t > c_q$,

$$J_p(q,t) = \left(\frac{c_q}{t}\right)^{R_q} J_p(q,c_q) + \frac{R_q^{(\alpha-1)/\alpha}}{t^{R_q}} \int_{c_q}^{t} s^{R_q-(\alpha-1)/\alpha} L_1(s/R_q)ds \qquad \text{(A62)}$$

$$+\frac{R_q}{t^{R_q}} \int_{c_q}^{t} s^{R_q-1} J_p(q-1,s)ds.$$

Next, if $R_q > -1/\alpha$ it follows by Karamata's theorem that

$$\lim_{t\to\infty} \frac{1}{a(t)} \frac{R_q^{(\alpha-1)/\alpha}}{t^{R_q}} \int_{c_q}^{t} s^{R_q-(\alpha-1)/\alpha} L_1(s/R_q)ds = \frac{R_q^{(\alpha-1)/\alpha}}{R_q + (1/\alpha)}.$$

Finally, by the induction hypothesis it follows that $J_p(q-1,t) = t^{1/\alpha}L_2(t)$ where L_2 is a slowly varying function and $L_2(t)/L_1(t) \to \xi_{q-1}(-1/\alpha)$ as $t\to\infty$. Therefore, for $R_q > -1/\alpha$ it follows by Karamata's theorem that

$$\lim_{t\to\infty} \frac{1}{a(t)} \frac{R_q}{t^{R_q}} \int_{c_q}^{t} s^{R_q-1} J_p(q-1,s)ds = \frac{R_q}{R_q+(1/\alpha)}\xi_{q-1}(-1/\alpha).$$

Now recalling by (11) that $b(t) = 0$ and dividing both sides of (A62) by $a(t)$ and taking the limit as $t\to\infty$ yields the desired result. $\square$