Modeling a Two-sided Limit Order Book

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Abstract

We study a limit order book which is modeled as a pair of coupled measure-valued stochastic processes representing the bid and ask sides of the order book. Limit and market orders arrive to both sides of the book according to independent Poisson processes and the distribution of the prices at which limit buy and sell orders are placed depends on the best bid and ask at the time of their arrival. We study the asymptotic behavior of this model in a high frequency regime where the arrival rate of incoming orders is large and limit buy and ask orders are placed in close vicinity to the current best prices. Our first main result provides a pair of coupled measure-valued stochastic differential equations as the formal limit of the bid and ask sides of the properly scaled order book in the high frequency regime. We then proceed to study the solution to this SDE for varying parameter regimes of the pre-limit model.

1 Introduction

Equity exchanges have experienced a fundamental change over the past couple of decades. Instead of historical quote-driven markets, trading is now commonly organized around order-driven markets making use of limit order books. In more recent developments, many cryptocurrency exchanges have also made use of limit order books to facilitate their trades, see below for a snapshot of the Coinbase Dogecoin-US Dollar limit order book from April 2022. In a limit order book driven market, any agent who wants to buy or sell can post their

Figure 1: A snapshot of the Coinbase Dogecoin-US Dollar limit order book on April 16th, 2022 at 12:57:23 EDT.

order at whatever prices and quantities they choose. One of the main challenges in tractably modeling the microstructure of such markets is the high-dimensional state-space of the order book and its complex evolution.

Limit order book (LOB) modeling has been substantially studied in the literature, and at least two different modeling approaches are taken. Economists model the order flow as a static process where fully rational agents submit limit orders based on the history of the order book and their objective is to maximize their personal utility. On the other hand, econophysicists treat the evolution of the order book as a stochastic process, where traders are assumed to have zero-intelligence and the arrivals and cancellations of orders are random.

Parlour [1998] presented a one-tick dynamic model where agents can only submit orders at some specified price. All traders know that their orders will affect other participants' order strategies. Parlour [1998] provided the optimal order submitting strategy and derived the order flows in equilibrium. Parlour's model fails to incorporate cancellations of active orders and the pricing grid is restricted to a single value. Hollifield et al. [2004] showed that empirically the above model fails to describe the behavior of traders trading Ericsson stock on the Stockholm Stock Exchange and suggested that modeling cancellations in equity markets might be important. Goettler et al. [2006] considered a model where an agent randomly enters a market with a single asset and leaves the market forever after the order is executed. They studied traders' willingness to purchase information on the fundamental value (or true value) of an asset and discovered that the value of such information to a trader decreases as the trader's desire to trade increases. One drawback of Goettler et al.'s model is that it relies solely on numerical analysis and fails in analytical tractability. Roşu [2009] studied an order book model without the effect of asymmetric information where traders can freely place, modify or cancel their orders. Their model is the first perfect-rationality LOB model to reflect the full range of actions that are available in real LOBs. They showed that their model admits a unique Markov equilibrium and provided an optimal strategy for new traders. In Cohen and Szpruch [2012], the authors discuss a LOB model with two investors with different speeds of trade execution. They showed that the faster trader may construct a strategy to gain a risk-free profit. They derived the faster trader's optimal behavior when he has only distributional knowledge of the slower trader's actions, with few restrictions on the possible prior distributions. They showed that the introduction of a "Tobin tax" can eliminate such arbitrage and increase market efficiency.

The zero-intelligence model takes a different approach to modeling, regarding order arrivals and cancellations as pure stochastic processes in nature. Bak et al. [1997] first introduced a diffusion model which models the state of the LOB by the movement of particles. Several authors studied this diffusion model by simulation and used it to explain regularities observed in real data [Bak et al., 1997, Eliezer and Kogan, 1998, Tang and Tian, 1999]. However, the diffusion of active orders predicted by the model is not observed in real data. Several discrete-time zero-intelligence models were proposed in [Maslov, 2000, Slanina, 2001, Challet and Stinchcombe, 2001] before the first continuous-time model was given in Daniels et al. [2003]. Daniels et al. proposed a master equation $\mathcal{L}(t)$ by assuming that order arrivals and cancellations are governed by Poisson processes. They assumed that orders arrive in fixed amounts of shares, and that limit orders are placed at a constant rate uniformly over a semi-infinite interval. By assuming i.i.d. random order flow, Smith et al. [2003] solved the master equation and developed a microscopic dynamical, statistical model for the continuous double auction. Smith et al. [2003]'s model makes testable predictions based on properties of the LOB that can be directly estimated. In this model, it is discovered that the order size is a more significant determinant of the market than tick size. It is also shown that like perfect rationality models, zero-intelligence models can be used to make strong predictions. Farmer et al. [2005] showed that this model performs well against empirical data.

Mike and Farmer [2008] developed a behavioral model for liquidity and volatility and revealed several empirical regularities in trading order flow. They assumed that the relative price of incoming limit orders follows students' t-distribution and they constructed a complex model to describe order cancellations. Their model predicts the distribution of mid-price returns quite well for small tick size and low volatility stocks, but is less effective for other stocks. Gu and Zhou [2009] carefully studied the Mike and Farmer [2008] model and found that the volatility simulated from the model doesn't exhibit long memory which is inconsistent with the observed stylized fact of volatility clustering. They proposed a modified version of the Mike and Farmer [2008] model where volatility shows long memory properties.

Cont et al. [2010] introduced a continuous-time stochastic model of LOBs as a variant of Daniels et al. [2003] and Smith et al. [2003]'s model which can be estimated easily from the data, captures key empirical properties and is analytically and computationally tractable. They assumed that the relative price of limit orders follows a power-law distribution and the parameter is estimated from data. Simulation of the model displays a hump-shaped depth profile which agrees with the empirical data. Cont et al. [2010] followed a queueing theory approach and used Laplace transforms to calculate several probabilities related to the limit order book. Their main assumption was that the interarrival times of orders and time until cancellations are independent exponential random variables. Zhao [2010] and Toke [2011] pointed out that this assumption might not hold based on an empirical study of crude oil futures traded at the International Petroleum Exchange. Zhao [2010] and Toke [2011] proposed using a Hawkes process instead of a Poisson process to model the arrival rates of orders and cancellations. Cont and De Larrard [2013] proposed tracking only the best bid and ask prices rather than the entire limit order book. They assume that when the number of orders at the best bid or ask reaches zero, the depth of the second best price is a random variable with a certain distribution. This assumption reduces the dimension of the state space of their model and allows one to obtain analytical expressions of certain quantities of interest. Cont and Bouchaud [2000] studied a model in a financial market where a random communication structure exists between N agents, and agents who communicate will imitate each other. Cont and Bouchaud [2000] found that such a model results in heavy tails in the distribution of stock price variations in the form of an exponentially truncated power law.

In this paper, our focus is on the dynamics of a limit order book where each side of the book is modeled as a measure-valued process. We consider a high frequency regime similar to the one proposed by Lakner et al. [2016], such that the arrival rate of limit and market orders is high, and limit orders are placed relatively close to the best price. In the high frequency regime, the asymptotic dynamics of the limit order book can be approximated by a pair of coupled measure-valued stochastic differential equations. The structure of the solution to this SDE is dependent on the average placement of limit buy and limit sell orders. If on average limit buy orders are placed below the best bid, and on average limit ask orders are placed above the best ask, then the solution to the pair of coupled SDEs can be represented by a pair of measures where the best bid and best ask are themselves the solution to a pair of coupled ODEs. On the other hand, if on average both limit buy and limit sell orders are placed inside the spread, then the best bid and best ask are the solution to a pair of coupled integral equations where the integrands are functions of the measure-valued state of the book. For mixtures of the above two cases, the solution to the limiting SDE is a combination of the above two solutions.

The remainder of the paper is organized as follows. Section 2 provides the details of our model. In Section 3, we state the definition of the high frequency regime and in Section 4, we formally derive its limiting measure-valued SDE. In Section 5, we provide our main results in the case that on average limit buy and sell orders are placed outside the spread, and in Section 6 we provide our main results in the case that on average limit buy and sell orders are placed inside the spread. Section 7 provides a result on the mixture of the above two cases. The proofs of our main results may be found in the appendix.

1.1 Notation

The following notation will be used throughout the paper. We assume that all random variables are defined on the common probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{M}_F(\mathbb{R})$ be the set of all finite, non-negative measures on $\mathbb{R}$, and let $C_b(\mathbb{R})$ be the set of all bounded continuous functions on $\mathbb{R}$. For any $\mu \in \mathcal{M}_F(\mathbb{R})$ and $\phi \in C_b(\mathbb{R})$ we adopt the inner product notation

$$\langle \mu, \phi \rangle = \int_{\mathbb{R}} \phi(u) d\mu(u).$$

We endow $\mathcal{M}_F(\mathbb{R})$ with the weak topology. Specifically, a sequence of elements $\{\mu_n, n \geq 1\}$ in $\mathcal{M}_F(\mathbb{R})$ weakly converges to an element $\mu \in \mathcal{M}_F(\mathbb{R})$ if and only if $\langle \mu_n, \phi \rangle \to \langle \mu, \phi \rangle$ as $n \to \infty$ for every $\phi \in C_b(\mathbb{R})$. Moreover, let $B(\mathcal{M}_F(\mathbb{R}))$ be the Borel sigma field on $\mathcal{M}_F(\mathbb{R})$

generated by the weak topology. We shall say that an $\mathcal{M}_F(\mathbb{R})$ valued random variable is measurable if it is $\mathcal{F}/B(\mathcal{M}_F(\mathbb{R}))$ measurable. Similarly, a mapping $f : \mathcal{M}_F(\mathbb{R}) \mapsto \mathbb{R}$ will be called measurable if it is $B(\mathcal{M}_F(\mathbb{R}))/B(\mathbb{R})$ measurable. Let $\mathcal{S}$ be a separable and complete metric space, and define $D([0,\infty),\mathcal{S})$ to be the Skorokhod space of all functions on $[0,\infty)$ that are right-continuous with left limits, and taking values in $\mathcal{S}$. We equip $D([0,\infty),\mathcal{S})$ with the standard Skorokhod topology (see, for instance, Ethier and Kurtz [2009]) and its associated Borel σ-algebra. We also assume that the product of a finite number of metric spaces is equipped with the product topology.

2 The LOB Model

In this paper, we consider a two-sided limit order book which could be used to trade equities, cryptocurrencies or any other relevant security. The specific dynamics of the order book are given below, but at a high level the bid side of the book is modeled by the measure-valued process $\mu_B \in D([0,\infty),\mathcal{M}_F(\mathbb{R}))$, and the ask side of the book is modeled by the interrelated measure-valued process $\mu_A = D([0,\infty),\mathcal{M}_F(\mathbb{R}))$. The pair (μ_B,μ_A) therefore provides the evolution of the entire book and is viewed as a process with sample functions in the space $D([0,\infty),\mathcal{M}_F^2(\mathbb{R}))$.

For each $t \geq 0$ and $\mathcal{A} \in \mathcal{B}(\mathbb{R})$, the number of bid orders on the book at time t with prices in the set $\mathcal{A}$ is given by $\mu_B(t)(\mathcal{A})$, and, similarly, the number of ask orders on the book at time t with prices in the set $\mathcal{A}$ is given by $\mu_A(t)(\mathcal{A})$. We assume for simplicity that all orders are of unit size of the security being traded. This assumption can also be relaxed without too much difficulty. As will become evident in the discussion below, both μ_B and μ_A turn out to be càdlàg processes taking values in the subspace of $\mathcal{M}_F(\mathbb{R})$ consisting of all finite counting measures on $\mathbb{R}$.

The best bid and best ask prices at time $t \geq 0$ are given respectively by

$$p_B(t) \;=\; \sup\{x \in \mathbb{R} : \mu_B(t)([x,\infty)) > 0\}$$

and

$$p_A(t) \;=\; \inf\{x \in \mathbb{R} : \mu_A(t)((-\infty,x]) > 0\}.$$

These definitions naturally correspond to the best bid being the highest price that someone is willing to buy the security for and the best ask being the lowest price that someone is willing to sell it for. We will also be interested in the best bid and best ask processes. These processes have sample functions included in the space $D([0,\infty),\mathbb{R})$ and are denoted respectively by p_B and p_A.

The dynamics of the order book process are as follows. Market sell orders arrive to the bid side of the book according to the Poisson process $N_{BM} = \{N_{BM}(t), t \geq 0\}$ with a rate of $\lambda_{BM} > 0$, and market buy orders arrive to the ask side of the book according to the Poisson process $N_{AM} = \{N_{AM}(t), t \geq 0\}$ with a rate of $\lambda_{AM} > 0$. We make no assumptions at the moment on the relationship between N_{BM} and N_{AM}. For each $i \geq 1$, we denote by τ_{BM}^i the time at which the ith market sell order arrives, and by τ_{AM}^i the time at which the ith market buy order arrives. We assume that each time a market sell order arrives, it is matched with a single limit buy order at the current best bid on the bid side of book. Similarly, each time a market buy order arrives, it is matched with a single limit sell order at the current best ask on the ask side of the book. Each of these transactions decreases the total number of orders on the book by 1 and, depending on the state of book, may or may not change the best bid or best ask price as well. See Figures 2 and 3 below for illustrations of 2 consecutive market sell orders arriving.

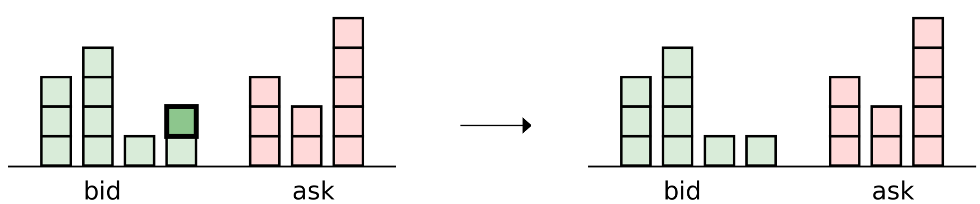


Figure 2: A market sell order arrives and is matched with one out of two limit buy orders at the best bid. The best bid price does not change.

We next discuss the role played by limit orders in our model. Limit buy orders arrive to the bid side of the book according to the Poisson process $N_{BL} = \{N_{BL}(t), t \geq 0\}$ with

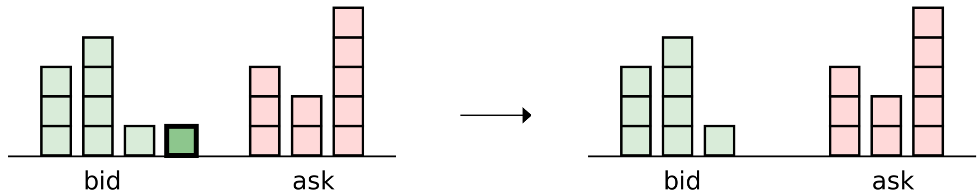


Figure 3: A market sell order arrives and is matched with the only limit buy order at the best bid. The best bid price decreases.

a rate of $\lambda_{BL} > 0$. Similarly, limit sell orders arrive to the ask side of the book according to the Poisson process $N_{AL} = \{N_{AL}(t), t \geq 0\}$ with a rate of $\lambda_{AL} > 0$. For each $i \geq 1$, we denote by τ^i_{BL} the time at which the ith limit buy order arrives and we denote by τ^i_{AL} the time at which the ith limit sell order arrives. Each time a limit order arrives, it places a corresponding order of unit size. The price at which the order is placed is random but has some dependency on the prevailing state of the book as we next describe.

Next we shall discuss the placement of the limit orders. Recall the definition of τ^i_{BL} as the time at which the ith limit buy order arrives. The prevailing best bid at the time at which limit buy order i arrives is given by $p_B(\tau^i_{BL}-)$, which is the left limit of p_B at time τ^i_{BL}. Similarly, the prevailing best ask at the time at which limit buy order i arrives is given by $p_A(\tau^i_{BL}-)$. The actual price at which limit buy order i is placed is given by

$$\text{price of limit buy order } i \quad = \quad p_A(\tau^i_{BL}-)\left(\frac{p_B(\tau^i_{BL}-)}{p_A(\tau^i_{BL}-)}\right)^{X^B_i}, \tag{1}$$

where X^B_i is a positive random variable with CDF F_B. Moreover, we assume that $\{X^B_i, i \geq 1\}$ is i.i.d.

Assumption (1) implies that so long as $p_B(0) < p_A(0)$, then in our model the spread (p_B, p_A) will always be non-empty. In order to see this, we proceed by induction and suppose that $p_B(\tau^i_{LB}-) < p_A(\tau^i_{LB}-)$. Then, since by assumption X^B_i is positive, it is straightforward to verify by (1) that limit buy order i will be placed at a price less than the best ask. In other words, limit buy order i will not cross the spread. More precisely, the following may be verified as well. If $0 < X^B_i < 1$, then limit buy order i will be placed inside the spread.

This will result in an increase of the best bid to the new price set by limit buy order i. On the other hand, if $X_i^B = 1$, then limit buy order i will be placed exactly at the prevailing best bid price. Finally, if $X_i^B > 1$, then limit buy order i will be placed at a price less than the best bid price and therefore deeper into the bid side of the book. See Figures 4 and 5 below for illustrations of 2 consecutive limit buy orders arriving.

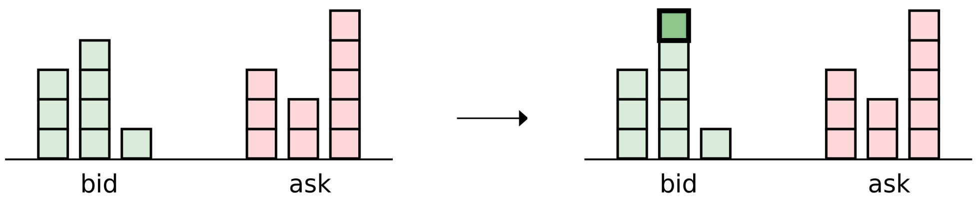


Figure 4: A limit buy order is placed at a price less than the best bid. The best bid price does not change.

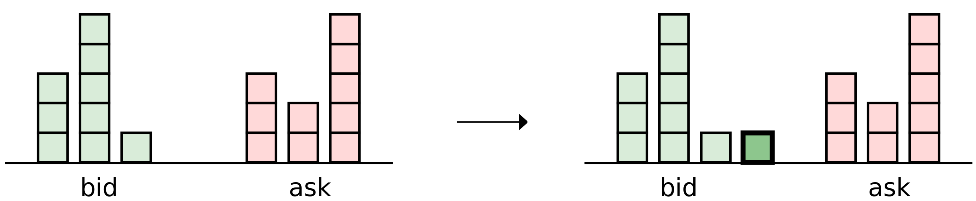


Figure 5: A limit buy order is placed at a price higher than the best bid. The best bid price increases.

The placement of limit sell orders proceeds in a symmetrical fashion to the placement of limit buy orders. Specifically, the price at which limit sell order i is placed is given by

$$\text{price of limit sell order } i \quad = \quad p_B(\tau_{AL}^i-) \left(\frac{p_A(\tau_{AL}^i-)}{p_B(\tau_{AL}^i-)} \right)^{X_i^A}, \tag{2}$$

where X_i^A is a positive random variable with CDF F_A. Moreover, we assume that $\{X_i^A, i \geq 1\}$ is i.i.d. Similar to the case of limit buy orders, one may show that (2) preserves the fact that the spread remains non-empty. Moreover, if $0 < X_i^A < 1$, then limit sell order i is placed inside the spread and the best ask price decreases to the price of limit sell order i. If $X_i^A = 1$, then limit sell order i is placed at the best ask price. Finally, $X_i^A > 1$, then limit sell order i is placed at a price higher than the best ask price and deeper into the ask side of the book.

We also assume the existence of a market maker who ensures that the number of orders on either side of the book never drops below a designated threshold. Specifically, let

$$S_B(t) \;=\; \mu_B(t)(\mathbb{R}) \;\text{ and }\; S_A(t) \;=\; \mu_A(t)(\mathbb{R}) \tag{3}$$

denote the number of orders on the bid and ask sides of the book, respectively, at each point in time $t \geq 0$. We also define the respective processes $S_B = \{S_B(t), t \geq 0\}$ and $S_A = \{S_A(t), t \geq 0\}$. The market maker then keeps track of the sizes of both sides of the book. Whenever a market sell order arrives and decreases the number of limit buy orders on the bid side of the book below the threshold $a_B > 0$, the market maker immediately places a new limit buy order on the bid side of the book at the same price at which the previous order was removed. In an identical fashion, the market maker ensures that the number of limit sell orders on the ask side of the book does not drop below the threshold $a_A > 0$.

Now for each point in time $t \geq 0$, denote by $L_B(t)$ and $L_A(t)$ the number of limit buy and limit sell orders, respectively, that the market maker has placed on the order book by time t. Also let L_B and L_A represent their respective counting processes. It is straightforward to show that

$$L_B(t) \;=\; -\inf_{0\leq s\leq t} \min(S_B(0) + N_{BL}(s) - N_{BM}(s) - a_B, 0) \tag{4}$$

and

$$L_A(t) \;=\; -\inf_{0\leq s\leq t} \min(S_A(0) + N_{AL}(s) - N_{AM}(s) - a_A, 0). \tag{5}$$

Moreover, by the preceding discussion the total number of orders on the bid and ask sides of the book at time t given by

$$S_B(t) = S_B(0) + N_{BL}(t) - N_{BM}(t) + L_B(t) \tag{6}$$

and

$$S_A(t) = S_A(0) + N_{AL}(t) - N_{AM}(t) + L_A(t). \tag{7}$$

We are now in a position to write the equations governing the evolution of the order book process (μ_B, μ_A). Let $\mu_B(0), \mu_A(0) \in \mathcal{M}_F(\mathbb{R})$ denote the initial bid and ask sides of the order book. We always assume that $\mu_B(0)$ and $\mu_A(0)$ are finite counting measures whose supports are a subset of $(0, \infty)$. Moreover, we make the assumption that $p_B(0) < p_A(0)$ so that the initial spread is positive. Next, for each $x \in \mathbb{R}$, let $\delta(x)$ denote the Dirac measure concentrated at x. Then, the order book processes μ_B and μ_A described above may be characterized as the unique solution to the pair of equations

$$\begin{aligned} \mu_B(t) &= \mu_B(0) + \sum_{i=1}^{N_{BL}(t)} \delta\left(p_A(\tau_{BL}^i-) \left(\frac{p_B(\tau_{BL}^i-)}{p_A(\tau_{BL}^i-)} \right)^{X_i^B} \right) - \int_0^t \delta(p_B(s-)) dN_{BM}(s) \\ &\quad + \int_0^t \delta(p_B(s-)) dL_B(s) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mu_A(t) &= \mu_A(0) + \sum_{i=1}^{N_{AL}(t)} \delta\left(p_B(\tau_{AL}^i-) \left(\frac{p_A(\tau_{AL}^i-)}{p_B(\tau_{AL}^i-)} \right)^{X_i^A} \right) - \int_0^t \delta(p_A(s-)) dN_{AM}(s) \\ &\quad + \int_0^t \delta(p_A(s-)) dL_A(s) \end{aligned} \tag{9}$$

for $t \geq 0$.

For convenience of analysis and in order to state some of our main results, it will be helpful to rewrite the equations above by taking their inner product with the set of test functions $\phi \in C_b(\mathbb{R})$. In this way, μ_B and μ_A may be characterized as the unique solution to the pair of equations

$$\begin{aligned} \langle \mu_B(t), \phi \rangle &= \langle \mu_B(0), \phi \rangle + \sum_{i=1}^{N_{BL}(t)} \phi\left(p_A(\tau_{BL}^i-) \left(\frac{p_B(\tau_{BL}^i-)}{p_A(\tau_{BL}^i-)} \right)^{X_i^B} \right) \\ &\quad - \int_0^t \phi(p_B(s-)) dN_{BM}(s) + \int_0^t \phi(p_B(s-)) dL_B(s) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \langle \mu_A(t), \phi \rangle &= \langle \mu_A(0), \phi \rangle + \sum_{i=1}^{N_{AL}(t)} \phi\left(p_B(\tau_{AL}^i-) \left(\frac{p_A(\tau_{AL}^i-)}{p_B(\tau_{AL}^i-)} \right)^{X_i^A} \right) \\ &\quad - \int_0^t \phi(p_A(s-)) dN_{AM}(s) + \int_0^t \phi(p_A(s-)) dL_A(s) \end{aligned} \tag{11}$$

for $t \geq 0$ and $\phi \in C_b(\mathbb{R})$.

3 The High Frequency Regime

The order book equations (8)-(9) are difficult to solve and so in this paper, we consider them in the high frequency regime introduced in Lakner et al. [2016]. Loosely speaking, in this regime the arrival rates of both limit and market orders are high, and limit buy orders are placed close to the best bid while limit sell orders are placed close to the best ask. The mathematical details of the high frequency regime is as follows.

Consider a sequence of order book models described above and indexed by $n \geq 1$. All quantities associated with the nth model are denoted by a superscript n. The definition of the high frequency regime then consists of several assumptions on how the model parameters scale with n.

Assumption 1 of the high frequency regime is that the arrival rates of limit and market orders scale roughly proportional to n. Moreover, it is assumed that the arrival rates of limit and market orders on each side of the book are closely matched. Technically speaking, this is accomplished by assuming that both

$$\frac{\lambda_{BL}^n}{n}, \frac{\lambda_{BM}^n}{n} \to \lambda_B > 0 \quad \text{and} \quad \frac{\lambda_{AL}^n}{n}, \frac{\lambda_{AB}^n}{n} \to \lambda_A > 0 \quad \text{as} \quad n \to \infty, \tag{12}$$

and

$$\sqrt{n}\left(\frac{\lambda_{BL}^n}{n} - \frac{\lambda_{BM}^n}{n}\right) \to \theta_B \in \mathbb{R} \quad \text{and} \quad \sqrt{n}\left(\frac{\lambda_{AL}^n}{n} - \frac{\lambda_{AM}^n}{n}\right) \to \theta_A \in \mathbb{R} \quad \text{as} \quad n \to \infty. \tag{13}$$

Assumption 2 of the high frequency regime is that loosely speaking limit buy orders are placed close to the best bid, and limit sell orders are placed close to the best ask. Technically speaking, this is accomplished by assuming that for each $n \geq 1$,

$$X_i^{B,n} = (X_i^B)^{1/\sqrt{n}} \quad \text{and} \quad X_i^{A,n} = (X_i^A)^{1/\sqrt{n}} \quad \text{for } i = 1, 2, ... \tag{14}$$

By Taylor's theorem it follows from (14) that

$$X_i^{B,n} \quad = \quad 1 + \frac{\ln X_i^B}{\sqrt{n}} + o(1/\sqrt{n}), \tag{15}$$

and so using (1) and applying Taylor's theorem again the price of limit buy order i is given by

$$p_B^n(\tau_{BL}^{i,n}-) + \frac{\ln X_i^B}{\sqrt{n}} p_B^n(\tau_{BL}^{i,n}-) \ln\left(\frac{p_B^n(\tau_{BL}^{i,n}-)}{p_A^n(\tau_{BL}^{i,n}-)}\right) + o(1/\sqrt{n}). \tag{16}$$

Similarly, using (14) and (2) it may be shown that the price of limit sell order i is given by

$$p_A^n(\tau_{AL}^{i,n}-) + \frac{\ln X_i^A}{\sqrt{n}} p_A^n(\tau_{AL}^{i,n}-) \ln\left(\frac{p_A^n(\tau_{AL}^{i,n}-)}{p_B^n(\tau_{AL}^{i,n}-)}\right) + o(1/\sqrt{n}). \tag{17}$$

Finally, assumption 3 of the high frequency regime is that the market maker keeps the number of orders on each side of the book above a level that is roughly proportional to the imbalance between the arrival rate of limit and market orders. Technically speaking, this assumption is given by

$$a_B^n/\sqrt{n} \to \tilde{a}_B > 0 \quad \text{and} \quad a_A^n/\sqrt{n} \to \tilde{a}_A > 0 \text{ as } n \to \infty. \tag{18}$$

3.1 Derivation of the Limiting SDE

In Lakner et al. [2016], it was rigorously shown that for the case of a one-sided limit order book, the order book process after proper normalization converges in the high frequency regime to the solution of a measure-valued stochastic differential equation (SDE). It appears to be the case that the techniques of Lakner et al. [2016] can be applied to rigorsouly prove a similar result in the present situation of a two-sided order book. The proof of the result in Lakner et al. [2016] is however somewhat long and tedious. Thus, rather than proceeding with a detailed proof, we have chosen to outline the key steps in formally deriving a limiting SDE for the properly normalized two-sided order book process in the high frequency regime..

First note that using the identities (6)-(7) for the total number of orders on each side of the book, the system equations (10)-(11) may be simplified by writing

$$\begin{aligned} \langle \mu_B(t), \phi \rangle &= \langle \mu_B(0), \phi \rangle + \sum_{i=1}^{N_{BL}(t)} \left[\phi\left(p_A(\tau_{BL}^i-) \left(\frac{p_B(\tau_{BL}^i-)}{p_A(\tau_{BL}^i-)} \right)^{X_i^B} \right) - \phi(p_B(\tau_{BL}^i-)) \right] \\ &\quad + \int_0^t \phi(p_B(s-)) dS_B(s) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \langle \mu_A(t), \phi \rangle &= \langle \mu_A(0), \phi \rangle + \sum_{i=1}^{N_{AL}(t)} \left[\phi\left(p_B(\tau_{AL}^i-) \left(\frac{p_A(\tau_{AL}^i-)}{p_B(\tau_{AL}^i-)} \right)^{X_i^A} \right) - \phi(p_A(\tau_{AL}^i-)) \right] \\ &\quad + \int_0^t \phi(p_A(s-)) dS_A(s) \end{aligned} \tag{20}$$

for $t \geq 0$ and $\phi \in C_b(\mathbb{R})$.

Next, we scale the mass of the order book process by $1/\sqrt{n}$. Thus, for each $n \geq 1$ define the normalized order book processes $\tilde{\mu}_B^n = \{\tilde{\mu}_B^n(t), t \geq 0\}$ and $\tilde{\mu}_A^n = \{\tilde{\mu}_A^n(t), t \geq 0\}$ by setting

$$\tilde{\mu}_B^n(t)(\mathcal{A}) = \frac{1}{\sqrt{n}}\mu_B^n(t)(\mathcal{A}) \quad \text{and} \quad \tilde{\mu}_A^n(t)(\mathcal{A}) \;\; = \;\; \frac{1}{\sqrt{n}}\mu_A^n(t)(\mathcal{A})$$

for $\mathcal{A} \in B(\mathbb{R})$ and $t \geq 0$. The choice of this normalization is in anticipation of Proposition 1 below. Similarly, we define the normalized versions of the processes tracking the number of orders on each side of the book by setting

$$\tilde{S}_B^n(t) = \frac{1}{\sqrt{n}}S_B^n(t) \quad \text{and} \quad \tilde{S}_A^n(t) = \frac{1}{\sqrt{n}}S_A^n(t)$$

for $t \geq 0$, and we let $\tilde{S}_B^n = \{\tilde{S}_B^n(t), t \geq 0\}$ and $\tilde{S}_A^n = \{\tilde{S}_A^n(t), t \geq 0\}$ denote their corresponding processes.

It then follows after some algebra that from (19)-(20) we obtain the normalized equations

$$\begin{aligned}\langle \tilde{\mu}_B^n(t), \phi \rangle \;\; &= \;\; \langle \tilde{\mu}_B^n(0), \phi \rangle + \frac{1}{\sqrt{n}} \sum_{i=1}^{N_{BL}^n(t)} \left[\phi\left(p_A^n(\tau_{BL}^{i,n}-) \left(\frac{p_B^n(\tau_{BL}^{i,n}-)}{p_A^n(\tau_{BL}^{i,n}-)} \right)^{X_i^{B,n}} \right) - \phi(p_B^n(\tau_{BL}^{i,n}-)) \right] \\ &\quad + \int_0^t \phi(p_B^n(s-)) d\tilde{S}_B^n(s) \end{aligned} \tag{21}$$

and

$$\begin{aligned}\langle \tilde{\mu}_A^n(t), \phi \rangle \;\; &= \;\; \langle \tilde{\mu}_A^n(0), \phi \rangle + \frac{1}{\sqrt{n}} \sum_{i=1}^{N_{AL}^n(t)} \left[\phi\left(p_B^n(\tau_{AL}^{i,n}-) \left(\frac{p_A^n(\tau_{AL}^{i,n}-)}{p_B^n(\tau_{AL}^{i,n}-)} \right)^{X_i^{A,n}} \right) - \phi(p_A^n(\tau_{AL}^{i,n}-)) \right] \\ &\quad + \int_0^t \phi(p_A^n(s-)) d\tilde{S}_A^n(s) \end{aligned} \tag{22}$$

for $t \geq 0$ and $\phi \in C_b(\mathbb{R})$.

Now recall by Taylor's theorem and (16) that for each $\phi \in C_b^2(\mathbb{R})$, we have

$$\phi\left(p_A^n(\tau_{BL}^{i,n}-) \left(\frac{p_B^n(\tau_{BL}^{i,n}-)}{p_A^n(\tau_{BL}^{i,n}-)} \right)^{X_i^{B,n}} \right) - \phi(p_B^n(\tau_{BL}^{i,n}-)) = \phi'(p_B^n(\tau_{BL}^{i,n}-))\Delta_n + o(1/\sqrt{n}), \tag{23}$$

where

$$\Delta_n = \frac{\ln X_i^B}{\sqrt{n}} p_B^n(\tau_{BL}^{i,n}-) \ln\left(\frac{p_B^n(\tau_{BL}^{i,n}-)}{p_A^n(\tau_{BL}^{i,n}-)} \right).$$

A similar expression for the ask side of the book may be obtained using Taylor's theorem and (17). Substituting (23) into (21), and its counterpart for the ask side of the book into (22), it follows that

$$\begin{aligned}\langle \tilde{\mu}_B^n(t), \phi \rangle &= \langle \tilde{\mu}_B^n(0), \phi \rangle \\ &+ \frac{1}{n} \sum_{i=1}^{N_{BL}^n(t)} \ln(X_i^B) \left[\phi'(p_B^n(\tau_{BL}^{i,n}-)) p_B^n(\tau_{BL}^{i,n}-) \ln\left(\frac{p_B^n(\tau_{BL}^{i,n}-)}{p_A^n(\tau_{BL}^{i,n}-)} \right) + o(1) \right] \\ &+ \int_0^t \phi(p_B^n(s-)) d\tilde{S}_B^n(s),\end{aligned} \tag{24}$$

and

$$\begin{aligned}\langle \tilde{\mu}_A^n(t), \phi \rangle &= \langle \tilde{\mu}_A^n(0), \phi \rangle \\ &+ \frac{1}{n} \sum_{i=1}^{N_{AL}^n(t)} \left[\phi'(p_A^n(\tau_{AL}^{i,n}-)) p_A^n(\tau_{AL}^{i,n}-) \left(\ln\left(\frac{p_A^n(\tau_{AL}^{i,n}-)}{p_B^n(\tau_{AL}^{i,n}-)} \right) \ln(X_i^A) \right) + o(1) \right] \\ &+ \int_0^t \phi(p_A^n(s-)) d\tilde{S}_A^n(s)\end{aligned} \tag{25}$$

for $t \geq 0$ and $\phi \in C_b^2(\mathbb{R})$. Equations (24)-(25) are now already fairly close to the limiting SDE we obtain below.

4 Limiting SDE

We now proceed to informally take weak limits in (24)-(25) in order to a obtain a limiting measure-valued SDE for the properly normalized order book process in the high frequency regime. In order for this limit to exist, we first must require that after proper normalization the initial state of the order book weakly converges as $n \to \infty$. That is, we assume that

$$(\tilde{\mu}_B^n(0), \tilde{\mu}_A^n(0)) \quad \Rightarrow \quad (\tilde{\mu}_B(0), \tilde{\mu}_A(0)) \text{ as } n \to \infty. \tag{26}$$

Recall that by assumption $\tilde{\mu}_B(0)$ and $\tilde{\mu}_A(0)$ are discrete measures. This need not however be the case for the limiting measures $\tilde{\mu}_B(0)$ and $\tilde{\mu}_A(0)$). In fact, some of our results below require us to make continuity assumptions on $(\tilde{\mu}_B(0), \tilde{\mu}_A(0))$.

Next, note that by (3), assumption (26) also implies weak convergence of the properly normalized initial sizes of the two sides of the order book. That is, (26) implies the weak convergence

$$(\tilde{S}_B^n(0), \tilde{S}_A^n(0)) \;\;\Rightarrow\;\; (\tilde{S}_B(0), \tilde{S}_A(0)) \text{ as } n \to \infty. \tag{27}$$

Now assume that the order arrival processes N_{AM}, N_{AL}, N_{BM} and N_{BL} are independent of one another. It then follows from standard results that aside from their initial conditions, the normalized order book size processes $\tilde{S}_B^n$ and $\tilde{S}_A^n$ weakly converge as $n \to \infty$ to independent reflected Brownian motions. Specifically, we have the following result.

Proposition 1. *If* (27) *holds, then*

$$(\tilde{S}_B^n, \tilde{S}_A^n) \;\;\Rightarrow\;\; (\tilde{S}_B, \tilde{S}_A) \text{ as } n \to \infty, \tag{28}$$

where $\tilde{S}_B$ is a Brownian motion reflected at $\tilde{a}_B > 0$ with constant drift θ_B and infinitesimal variance $2\lambda_B$, and $\tilde{S}_A$ is a Brownian motion reflected at $\tilde{a}_A > 0$ with constant drift θ_A and infinitesimal variance $2\lambda_A$. Moreover, conditional on $(\tilde{S}_B(0), \tilde{S}_A(0))$, the processes $\tilde{S}_B$ and $\tilde{S}_A$ are independent of one another.

Recall by standard results that if $\tilde{S}$ is a Brownian motion reflected at some level $\tilde{a}$, and with a drift θ and infinitesimal variance σ^2, then we have the standard construction

$$\tilde{S}(t) \;\;=\;\; \tilde{S}(0) + \theta t + \sigma \tilde{B}(t) + \tilde{L}(t), \tag{29}$$

where $\tilde{B} = \{\tilde{B}(t), t \geq 0\}$ is a standard Brownian motion, and

$$\tilde{L}(t) \;\;=\;\; -\inf_{0 \leq s \leq t} \min(\tilde{S}(0) + \theta t + \sigma \tilde{B}(t) - \tilde{a}, 0). \tag{30}$$

Further references on such processes may be found in Harrison [2002] .

Now taking the limit as $n \to \infty$ in (24)-(25), we formally obtain the limiting measure-valued stochastic differential equation

$$\begin{aligned} \langle \tilde{\mu}_B(t), \phi \rangle \;\;=\;\; & \langle \tilde{\mu}_B(0), \phi \rangle + \lambda_B \mathbb{E}[\ln(X^B)] \int_0^t \left[\phi'(p_B(s)) p_B(s) \ln\left(\frac{p_B(s)}{p_A(s)} \right) \right] ds \\ & + \int_0^t \phi(p_B(s)) d\tilde{S}_B(s) \end{aligned} \tag{31}$$

and

$$\begin{aligned} \langle \tilde{\mu}_A(t), \phi \rangle \;\;=\;\; & \langle \tilde{\mu}_A(0), \phi \rangle + \lambda_A \mathbb{E}[\ln(X^A)] \int_0^t \left[\phi'(p_A(s)) p_A(s) \ln\left(\frac{p_A(s)}{p_B(s)} \right) \right] ds \\ & + \int_0^t \phi(p_A(s)) d\tilde{S}_A(s) \end{aligned} \tag{32}$$

for $t \geq 0$ and $\phi \in C_b^2(\mathbb{R})$, where

$$p_B(t) \;=\; \sup\{x \in \mathbb{R} : \tilde{\mu}_B(t)([x,\infty)) > 0\}$$

and

$$p_A(t) \;=\; \inf\{x \in \mathbb{R} : \tilde{\mu}_A(t)((-\infty,x]) > 0\}.$$

The structure of the solution to (31)-(32) depends on the signs of $E[\ln X^B]$ and $E[\ln X^A]$. There are four possible combinations corresponding to each of $E[\ln X^B]$ and $E[\ln X^A]$ being either positive or negative. Each of these combinations are discussed in the sections below and have a natural interpretation with respect to the placement of limit buy and sell orders relative to the best bid and best ask. The cases of either $E[\ln X^B]$ or $E[\ln X^A]$ being equal to zero are degenerate as can be seen from above. In particular, if $E[\ln X^B] = 0$, then $\tilde{\mu}_B \equiv \tilde{\mu}_B(0)$. Similarly, if $E[\ln X^A] = 0$, then $\tilde{\mu}_A \equiv \tilde{\mu}_A(0)$.

5 The case of $\mathbb{E}(\ln X^B)$ and $\mathbb{E}(\ln X^A)$ being positive

In this section, we provide the solution to (31)-(32) when both $\mathbb{E}(\ln X^B)$ and $\mathbb{E}(\ln X^A)$ are positive. Using (16)-(17), it is straightforward to see that this case loosely corresponds to limit buy and sell orders being placed on average outside the spread. Our main result is the following. Its proof may be found in the appendix.

Theorem 1. *If $\mathbb{E}(\ln X^B), \mathbb{E}(\ln X^A) \geq 0$ and the functions $x \mapsto \tilde{\mu}_B(0)((0,x))$ and $x \mapsto \tilde{\mu}_A(0)((0,x))$ are Lipschitz continuous on $(0, p_B(0))$ and $(p_A(0),\infty)$, resp., and $\tilde{\mu}_B(0)(\{p_B(0)\}) > 0$ and $\tilde{\mu}_A(0)(\{p_A(0)\}) > 0$, then a solution $(\tilde{\mu}_B, \tilde{\mu}_A)$ to the SDE (31)-(32) is given by*

$$\tilde{\mu}_B(t)([0,x]) = \begin{cases} \tilde{\mu}_B(0)([0,x]) & \text{if } 0 \leq x < p_B(t), \\ \tilde{S}_B(t) & \text{if } x \geq p_B(t), \end{cases} \tag{33}$$

and

$$\tilde{\mu}_A(t)([x,\infty)) = \begin{cases} \tilde{S}_A(t) & \text{if } 0 \leq x \leq p_A(t), \\ \tilde{\mu}_A(0)([x,\infty)) & \text{if } x > p_A(t), \end{cases} \tag{34}$$

for $t \geq 0$, where (p_B, p_A) is the unique solution to the pair of coupled equations

$$\ln p_B(t) \;=\; \ln p_B(0) - \lambda_B \mathbb{E}(\ln X^B) \int_0^t \ln\left(\frac{p_A(s)}{p_B(s)}\right) \frac{ds}{\alpha_B(s)} \tag{35}$$

and

$$\ln p_A(t) \;=\; \ln p_A(0) + \lambda_A \mathbb{E}(\ln X^A) \int_0^t \ln\left(\frac{p_A(s)}{p_B(s)}\right) \frac{ds}{\alpha_A(s)}, \tag{36}$$

where

$$\alpha_B(t) \;=\; \tilde{S}_B(t) - \tilde{\mu}_B(0)((0, p_B(t))) \quad \text{and} \quad \alpha_A(t) \;=\; \tilde{S}_A(t) - \tilde{\mu}_A(0)((p_A(t), \infty)). \tag{37}$$

Since by assumption both $\mathbb{E}(\ln X^B), \mathbb{E}(\ln X^A) > 0$ and $p_B(0) < p_A(0)$, it is straightforward to show by (35)-(36) that the best bid price in Theorem 1 is monotonically decreasing and the best ask price is monotonically increasing. Thus, the spread is continually widening. The order book itself is given by (33)-(34) and is comprised at each point in time of an atom of size $\alpha_B(t)$ and $\alpha_A(t)$ at the best bid and best ask, respectively, with the rest of the book consisting of any remaining limit orders leftover from the initial conditions.

Note also that the rates of change of the best bid and best ask are faster when the spread is larger and slower when there is a high number of orders at either the best bid or best ask, respectively. This is natural since in order for the spread to increase, orders at either the best bid or best ask need to first be removed. This type of behavior matches an empirical study by Cont et al. [2014], which finds that changes in price are inversely proportional to market depth over short periods of time.

5.1 An Example

The equations (35)-(36) for the limiting price processes (p_B, p_A) cannot in general be solved in closed form. There exists however a special case in which an exact solution may be found. Suppose that both $\tilde{\mu}_B(0) = \tilde{S}_B(0)\delta_{p_B(0)}$ and $\tilde{\mu}_A(0) = \tilde{S}_A(0)\delta_{p_A(0)}$. In order words, the initial state of the order book is such that its mass is concentrated solely at the best bid and best ask. Then, since p_B is decreasing and p_A is increasing, it is straightforward to verify that $\alpha_B(t) = \tilde{S}_B(t)$ and $\alpha_A(t) = \tilde{S}_A(t)$ for $t \geq 0$. In other words, the mass of the book remains concentrated at the best bid and best ask for all $t \geq 0$. The equations (35)-(36) for (p_B, p_A) then simplify considerably.

Specifically, note that subtracting (35) from (36) we obtain that

$$\ln\left(\frac{p_A(t)}{p_B(t)}\right) \;=\; \ln\left(\frac{p_A(0)}{p_B(0)}\right) + \int_0^t \ln\left(\frac{p_A(s)}{p_B(s)}\right)\left(\frac{\lambda_A \mathbb{E}(\ln X^A)}{\tilde{S}_A(s)} + \frac{\lambda_B \mathbb{E}(\ln X^B)}{\tilde{S}_B(s)}\right) ds \tag{38}$$

for $t \geq 0$. The solution to this equation is given by

$$\ln\left(\frac{p_A(t)}{p_B(t)}\right) = \ln\left(\frac{p_A(0)}{p_B(0)}\right)\exp\left(\int_0^t \left(\frac{\lambda_A \mathbb{E}(\ln X^A)}{\tilde{S}_A(s)} + \frac{\lambda_B \mathbb{E}(\ln X^B)}{\tilde{S}_B(s)}\right) ds\right). \tag{39}$$

Substituting (39) back into (35)-(36), one then obtains closed form expressions for (p_B, p_A).

6 The case of $\mathbb{E}(\ln X^B)$ and $\mathbb{E}(\ln X^A)$ being negative

In this section, we provide the solution to the SDE (31)-(32) when $\mathbb{E}(\ln X^B)$ and $\mathbb{E}(\ln X^A)$ are negative. Using (16)-(17), it is straightforward to see that this case loosely corresponds to limit buy and sell orders being placed on average inside the spread. We begin by providing the dynamics of the price processes in Section 6.1 and then move on to the order book itself in Section. 6.2.

6.1 The Price Process

For each $t \geq 0$, let

$$\underline{m}_B(t) = \inf_{0 \leq s \leq t} \tilde{S}_B(s) \text{ and } \underline{m}_A(t) = \inf_{0 \leq s \leq t} \tilde{S}_A(s) \tag{40}$$

denote the minimum bid and ask order book sizes, respectively, up until time t. Next, for each $\underline{m}_B(t) \leq x \leq \tilde{S}_B(t)$ set

$$\tau^t_{B,x} = \sup\{0 \leq s \leq t : \tilde{S}_B(s) = x\} \tag{41}$$

to be the last visit time of $\tilde{S}_B$ at the level x before time t. Similarly, for each $\underline{m}_A(t) \leq x \leq \tilde{S}_A(t)$ set

$$\tau^t_{A,x} = \sup\{0 \leq s \leq t : \tilde{S}_A(s) = x\} \tag{42}$$

to be the last visit time of $\tilde{S}_A$ at the level x before time t. By the continuity of $\tilde{S}_B$ and $\tilde{S}_A$, one can see that (41)-(42) are well-defined and strictly increasing in x. Finally, let $G_B : \mathbb{R}_+ \mapsto [0, \tilde{S}_B(0)]$ and $G_A : \mathbb{R}_+ \mapsto [0, \tilde{S}_A(0)]$ be the functions defined by

$$G_B(x) = \tilde{\mu}_B(0)(x, \infty) \text{ and } G_A(x) = \tilde{\mu}_A(0)(x, \infty). \tag{43}$$

Also denote by G_B^{-1} and G_A^{-1} the respective right-continuous inverses of G_B and G_A.

Now consider the system of equations

$$\ln p_B(t) = \ln G_B^{-1}(\underline{m}_B(t)) - \mathbb{E}(\ln X^B) \int_{\underline{m}_B(t)}^{\tilde{S}_B(t)} \ln\left(\frac{p_A(\tau_{B,u}^t)}{p_B(\tau_{B,u}^t)}\right) du, \tag{44}$$

and

$$\ln p_A(t) = \ln G_A^{-1}(\underline{m}_A(t)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} \ln\left(\frac{p_A(\tau_{A,u}^t)}{p_B(\tau_{A,u}^t)}\right) du, \tag{45}$$

for $t \geq 0$.

In Theorem 2 of the subsection that follows, it is stated that (44)-(45) represents the system equations for the price processes of a solution to the SDE (31)-(32). In the present subsection, we simply present the following result.

Proposition 2. *If $p_B(0) < p_A(0)$, then P-a.s. there exists a unique solution (p_B, p_A) to (44)-(45). Moreover, $p_B(t) < p_A(t)$ for $t \geq 0$.*

We now point out that the price processes p_B and p_A defined by (44)-(45) have the following property. For any two points in time $0 \leq t_1 < t_2$, let $[t_1, t_2]$ be an up excursion interval of the process $\tilde{S}_A$ if and only if $\tilde{S}_A(t_1) = \tilde{S}_A(t_2)$ and $\tilde{S}_A(t) \geq \tilde{S}_A(t_1)$ for all $t_1 < t < t_2$. We remark that our definition of an excursion interval is slightly different than the terminology of "excursion" commonly used in the literature. In our definition, we only require that $\tilde{S}_A(t) \geq \tilde{S}_A(t_1)$ for all $t_1 < t < t_2$ instead of the commonly used assumption that requires $\tilde{S}_A$ to be strictly greater than $\tilde{S}_A(t_1)$ in the interval (t_1, t_2). It is clear that if $[t_1, t_2]$ is an up excursion interval of $\tilde{S}_A$, then $\underline{m}_A(t_1) = \underline{m}_A(t_2)$. Also, from the definition of $\tau_{A,x}^t$, one can easily show that $\tau_{A,x}^{t_1} = \tau_{A,x}^{t_2}$ for any $x < \tilde{S}_A(t_1)$. Hence, it follows from (45) that

$$\begin{aligned}
\ln p_A(t_2) &= \ln G_A^{-1}(\underline{m}_A(t_2)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t_2)}^{\tilde{S}_A(t_2)} \ln\left(\frac{p_A(\tau_{A,u}^{t_2})}{p_B(\tau_{A,u}^{t_2})}\right) du \\
&= \ln G_A^{-1}(\underline{m}_A(t_1)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t_1)}^{\tilde{S}_A(t_1)} \ln\left(\frac{p_A(\tau_{A,u}^{t_1})}{p_B(\tau_{A,u}^{t_1})}\right) du \\
&= \ln p_A(t_1).
\end{aligned} \tag{46}$$

Equation (46) implies that the price p_A is the same at the two endpoints for every up excursion interval $[t_1, t_2]$. This is an important property of the price process defined in (44)-(45) which we refer to as the "excursion property". In a similar manner, one can derive the excursion property for p_B.

6.2 The Order Book Process

Before presenting our main result of this section, the following notation is needed. Given the unique pair of price processes (p_B, p_A) solving (44)-(45), for each $t \geq 0$ let

$$\underline{p}_B(t) = \inf_{0 \leq u \leq t} p_B(u) \text{ and } \bar{p}_A(t) = \sup_{0 \leq u \leq t} p_A(u)$$

be the minimum bid price and maximum ask price, respectively, up until time t. Next, for each $\underline{p}_B(t) \leq x < p_B(t)$ set

$$\tau^t_{pB,x} \quad = \quad \sup\{0 \leq u \leq t : p_B(u) = x\} \tag{47}$$

to be the last visit time of p_B to the price x before time t. Similarly, for each $p_A(t) < x \leq \overline{p}_A(t)$ set

$$\tau^t_{pA,x} \quad = \quad \sup\{0 \leq u \leq t : p_A(u) = x\} \tag{48}$$

to be the last visit time of p_A to the price x before time t.

The following is our main result of this section.

Theorem 2. *Suppose that* $\mathbb{E}(\ln X^B), \mathbb{E}(\ln X^A) < 0$. *If* $\tilde{\mu}_B(0)$ *and* $\tilde{\mu}_A(0)$ *are* P*-a.s. absolutely continuous, then a solution* $(\tilde{\mu}_B, \tilde{\mu}_A)$ *to the SDE* (31)-(32) *is such that for each* $t \geq 0$, $\tilde{\mu}_B(t)$ *is absolutely continuous with density*

$$\frac{d\tilde{\mu}_B(t)}{dx} = \begin{cases} d\tilde{\mu}_B(0)/dx & \text{if } x < \underline{p}_B(t), \\ \left[\mathbb{E}(\ln X^B) x \ln\left(\frac{x}{p_A(\tau^t_{pB,x})}\right)\right]^{-1} & \text{if } \underline{p}_B(t) \leq x < p_B(t), \\ 0 & \text{if } x \geq p_B(t), \end{cases} \tag{49}$$

and $\tilde{\mu}_A(t)$ *is absolutely continuous with density*

$$\frac{d\tilde{\mu}_A(t)}{dx} = \begin{cases} 0 & \text{if } x \leq p_A(t), \\ \left[\mathbb{E}(\ln X^A) x \ln\left(\frac{p_B(\tau^t_{pA,x})}{x}\right)\right]^{-1} & \text{if } p_A(t) < x \leq \overline{p}_A(t), \\ d\tilde{\mu}_A(0)/dx & \text{if } x > \overline{p}_A(t), \end{cases} \tag{50}$$

where (p_B, p_A) *is the unique solution to* (44)-(45).

We now provide some interpretation of Theorem 2. Let $t \geq 0$ and consider the density $d\tilde{\mu}_A(t)/dx$ of the ask side of the book as given by (50). There are 3 cases to discuss. The first case of $x \leq p_A(t)$ is straightforward since by definition the best ask is the lowest price on the ask side of the book.

Next, consider the case of $p_A(t) < x \leq \overline{p}_A(t)$. This is the most interesting one. First note that by the continuity of p_A, it follows that $p_A(\tau^t_{pA,x}) = x$. The expression for the density on the righthand side of (50) can then be written as

$$\left[\mathbb{E}(\ln X^A) p_A(\tau^t_{pA,x}) \ln\left(\frac{p_B(\tau^t_{pA,x})}{p_A(\tau^t_{pA,x})}\right)\right]^{-1}. \tag{51}$$

Notice now the similarity between this expression and (17) of Section 3. One may then loosely interpret the expression inside the parenthesis above as being proportional to the average distance from the best ask price that limit sell orders were placed at the last time that the best ask price was at the level x. The reciprocal of this quantity naturally corresponds to the density of orders. We therefore see that the density of the order book provides a history of its previous states.

The final case in (50) of $x > \overline{p}_A(t)$ corresponds to price levels that the best ask has not yet reached. It is not surprising that for such x the density of the order book remains as it was at time 0. The analysis of the bid side of the book using (49) follows similarly.

We also note there exists a direct relationship between the pair of equations (44)-(45) for the limiting price processes (p_B, p_A) and the formulas (49)-(50) for the limiting measures $(\tilde{\mu}_B, \tilde{\mu}_A)$. Specifically, it turns out that at each point time $t \geq 0$ the integrands in (44)-(45) may be written in terms of the densities (49)-(50) of the current state of the book. In order to see that this is the case, first note from Proposition 6 in the appendix that for each $t \geq 0$ and $u \in [\underline{m}_A(t), \tilde{S}_A(t)]$,

$$\tau^t_{A,u} \quad = \quad \tau^t_{p_A, p_A(\tau^t_{A,u})}. \tag{52}$$

On the other hand, it can also be shown that given $\tilde{\mu}_A(t)$ one has

$$p_A(\tau^t_{A,u}) \quad = \quad \inf\{x \in \mathbb{R} : \tilde{\mu}_A(t)(x, \infty) < u\}. \tag{53}$$

We henceforth for convenience set $p_A(\tau^t_{A,x}) = p_A(t,x)$. It then follows from (50) and after some algebra that

$$\ln\left(\frac{p_A(\tau^t_{A,u})}{p_B(\tau^t_{A,u})}\right) = \left(\mathbb{E}(\ln X^A) p_A(t,u)\frac{d\tilde{\mu}_A(t)}{dx}(p_A(t,u))\right)^{-1}. \tag{54}$$

Note moreover that the lefthand side above is exactly the integrand in equation (45) for p_A.

In a similar manner, for each $t \geq 0$ and $u \in [\underline{m}_B(t), \tilde{S}_B(t)]$ letting

$$p_B(t,u) = \sup\{x \in \mathbb{R} : \tilde{\mu}_B(t)(-\infty, x) < u\}, \tag{55}$$

it may be shown that

$$\ln\left(\frac{p_A(\tau^t_{B,u})}{p_B(\tau^t_{B,u})}\right) = \left(\mathbb{E}(\ln X^B) p_B(t,u)\frac{d\tilde{\mu}_B(t)}{dx}(p_B(t,u))\right)^{-1}. \tag{56}$$

Note moreover that the lefthand side above is exactly the integrand in equation (44) for p_B. We therefore see that at each point in time $t \geq 0$ the integrands in (44)-(45) for the limiting price processes (p_B, p_A) are functions of the current state $(\tilde{\mu}_B(t), \tilde{\mu}_A(t))$ of the order book. The additional quantities $(\tilde{S}_B(t), \tilde{S}_A(t))$ and $(\underline{m}_B(t), \underline{m}_A(t))$ appearing on the righthand sides of (44)-(45) are easily seen to be functions of $(\tilde{\mu}_B(t), \tilde{\mu}_A(t))$ too.

6.3 Price process approximation

In the proof of Proposition 4 of the appendix, it is shown that the pair $(\ln p_B, \ln p_A)$ of log price processes solving (44)-(45) is the unique fixed point of a contraction mapping. This fact may be used to construct a sequence of approximations to the solution to (44)-(45). We proceed as follows.

For the sake of simplicity, assume that $\tilde{S}_A(0) = \tilde{a}_A$ and $\tilde{S}_B(0) = \tilde{a}_B$. The more general case can be handled too, the only difference being the expressions below become more involved. Next, fix a $T \geq 0$ and define the linear operators A_B and A_A by setting for each $x \in C([0,T], \mathbb{R})$ and $0 \leq t \leq T$,

$$A_B(x)(t) = \mathbb{E}(\ln X^B) \int_{\tilde{a}_B}^{\tilde{S}_B(t)} x(\tau^t_{B,u})du \tag{57}$$

and

$$A_A(x)(t) = \mathbb{E}(\ln X^A) \int_{\tilde{a}_A}^{\tilde{S}_A(t)} x(\tau^t_{A,u})du. \tag{58}$$

In the proof of Proposition 4, it is shown that $A_B, A_A : C([0,T],\mathbb{R}) \mapsto C([0,T],\mathbb{R})$. Moreover, selecting T sufficiently small A_B and A_A become contraction mappings with respect to the uniform norm. The choice of T does however depend on $\omega \in \Omega$.

Equations (44)-(45) may now be written for $0 \leq t \leq T$ as

$$\ln p_B(t) \quad = \ln p_B(0) + A_B(\ln p_B - \ln p_A)(t) \tag{59}$$

and

$$\ln p_A(t) \quad = \ln p_A(0) + A_A(\ln p_A - \ln p_B)(t). \tag{60}$$

Moreover, denote by $A_A + A_B$ the summation of the operators A_A and A_B. That is,

$$(A_A + A_B)(x) \quad = \quad A_A(x) + A_B(x) \quad \text{for} \quad x \in C([0,T],\mathbb{R}). \tag{61}$$

Then, subtracting (59) from (60) and iteratively applying $A_A + A_B$ to both sides of the resulting equation, we obtain using the contraction mapping property of A_A and A_B that

$$\ln p_A - \ln p_B \quad = \quad \sum_{n=0}^{\infty} (A_A + A_B)^n (\ln p_A(0) - \ln p_B(0)), \tag{62}$$

where $(A_A + A_B)^n$ denotes the composition of $A_A + A_B$ with itself n times.

Now substituting (62) into (59)-(60), it follows that for $0 \leq t \leq T$ the solution to (44)-(45) can be represented as the infinite series

$$\ln p_B \quad = \quad \ln p_B(0) + A_B \left(\sum_{n=0}^{\infty} (A_A + A_B)^n (\ln p_B(0) - \ln p_A(0)) \right) \tag{63}$$

and

$$\ln p_A \quad = \quad \ln p_A(0) + A_A \left(\sum_{n=0}^{\infty} (A_A + A_B)^n (\ln p_A(0) - \ln p_B(0)) \right). \tag{64}$$

We may therefore approximate $(\ln p_B, \ln p_A)$ by truncating the infinite series (63)-(64). For instance, truncating at $n = 1$ yields the first-order approximation for $(\ln p_B, \ln p_A)$ given by

$$\begin{aligned}
\ln p_B^{(1)}(t) \quad &= \quad \ln p_B(0) + \mathbb{E}(\ln X^B)(\ln p_B(0) - \ln p_A(0))(\tilde{S}_B(t) - \tilde{a}_B), \\
\ln p_A^{(1)}(t) \quad &= \quad \ln p_A(0) + \mathbb{E}(\ln X^A)(\ln p_A(0) - \ln p_B(0))(\tilde{S}_A(t) - \tilde{a}_A),
\end{aligned}$$

for $0 \leq t \leq T$. Truncating at $n = 2$ leads to the second-order approximation given by

$$
\begin{aligned}
\ln p_B^{(2)}(t) &= \ln p_B(0) + \mathbb{E}(\ln X^B)(\ln p_B(0) - \ln p_A(0)) \Bigg[\mathbb{E}(\ln X^A) \int_{\tilde{a}_B}^{\tilde{S}_B(t)} \left(\tilde{S}_A(\tau_{B,u}^t) - \tilde{a}_A \right) du \\
&\quad + \left((\tilde{S}_B(t) - \tilde{a}_B) + \frac{1}{2}\mathbb{E}(\ln X^B)(\tilde{S}_B(t) - \tilde{a}_B)^2 \right) \Bigg], \\
\ln p_A^{(2)}(t) &= \ln p_A(0) + \mathbb{E}(\ln X^A)(\ln p_A(0) - \ln p_B(0)) \Bigg[\mathbb{E}(\ln X^B) \int_{\tilde{a}_A}^{\tilde{S}_A(t)} \left(\tilde{S}_B(\tau_{A,u}^t) - \tilde{a}_B \right) du \\
&\quad + \left((\tilde{S}_A(t) - \tilde{a}_A) + \frac{1}{2}\mathbb{E}(\ln X^A)(\tilde{S}_A(t) - \tilde{a}_A)^2 \right) \Bigg],
\end{aligned}
$$

for $0 \leq t \leq T$.

In Figure 6 below, we plot a simulated sample path of $(\ln p_B, \ln p_A)$ and compare it to the first and second order sample paths given by the approximations above. In Figure 7 below, we plot a Monte Carlo estimation of the marginal distribution of $\ln p_A(1)$ and compare it to the first and second order Monte Carlo estimations using the approximations above.

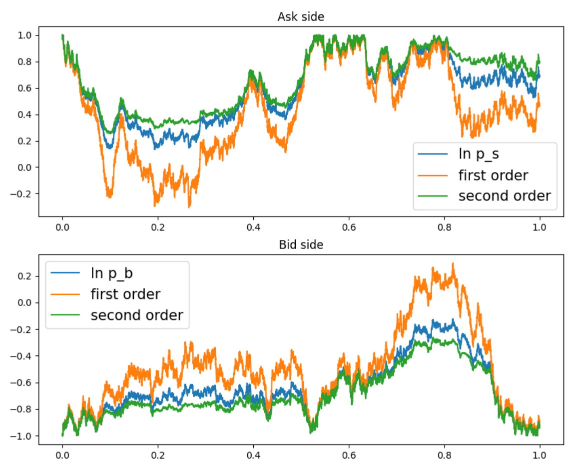


Figure 6: Sample paths of $\ln p_A$ and $\ln p_B$ together with their first and second order approximations.

6.4 Examples

The equations (44)-(45) for $(\ln p_B, \ln p_A)$ cannot in general be solved in closed form. There are however a couple of special cases where an exact solution may be found. We next discuss these two special cases.

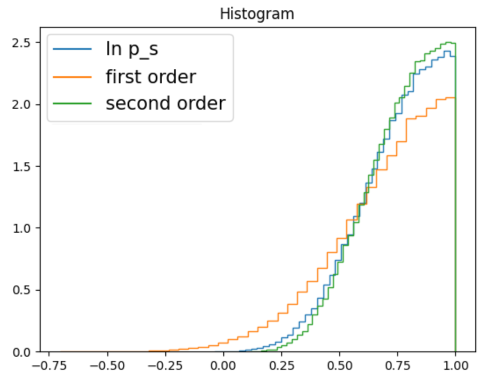


Figure 7: The marginal distribution of $\ln p_A(1)$ based on Monte Carlo simulation and its first and second order approximations.

6.4.1 The case where either $\mathbb{E}(\ln X^B)$ or $\mathbb{E}(\ln X^A)$ is zero

Theorem 2 also holds when either $\mathbb{E}(\ln X^B)$ or $\mathbb{E}(\ln X^A)$ is zero with the other expectation still being negative. Consider for instance the case where $\mathbb{E}(\ln X^B) = 0$ and $\mathbb{E}(\ln X^A) < 0$. This loosely corresponds to the average price of incoming limit buy orders being equal to the best bid, and the average price of incoming limit sell orders being less than the best ask. It turns out that in this case the integral term in (44) vanishes and so we obtain a explicit expression for the best bid. The dynamics of the best ask given by (45) are more complicated. However, assuming that $\tilde{S}_B(0) = \tilde{a}_B$ and $\tilde{S}_A(0) = \tilde{a}_A$, we can explicitly solve for p_A as well. We proceed as follows.

First note that if both $\mathbb{E}(\ln X^B) = 0$ and $\tilde{S}_B(0) = \tilde{a}_B$, then by (44) it follows that $p_B(t) = p_B(0)$ for $t \geq 0$. Next, if $\tilde{S}_A(0) = \tilde{a}_A$, then $G_A^{-1}(\underline{m}_A(t)) = p_A(0)$ for $t \geq 0$. Now let $x(t) = \ln p_A(t) - \ln p_B(0)$. It then follows subtracting (44) from (45) that

$$x(t) \quad = \quad x(0) + \mathbb{E}(\ln X^A) \int_{\tilde{a}_A}^{\tilde{S}_A(t)} x(\tau^t_{A,u}))du \tag{65}$$

for $t \geq 0$. Next, applying the change-of-variables $s = \tau^t_{A,u}$ to the integral in (65), we obtain

$$x(t) \quad = \quad x(0) + \mathbb{E}(\ln X^A) \int_0^t x(s) dT^t_A(s), \tag{66}$$

where T^t_A is given by (115) in the appendix. It then follows by Lemma 2 in the appendix that

$$\int_0^t x(s)d\tilde{S}_A(s) - \int_0^t x(s)dT_A^t(s) \quad = \quad -\lambda_A \mathbb{E}(\ln X^A) \int_0^t x(s)ds. \tag{67}$$

On the other hand, recall by (66) that

$$\int_0^t x(s)dT_A^t(s) \quad = \quad \frac{x(t) - x(0)}{E(\ln X^A)}. \tag{68}$$

Substituting (68) into (67), we then obtain that

$$x(t) = x(0) + \mathbb{E}(\ln X^A) \int_0^t x(s)d\tilde{S}_A(s) + \lambda_A \mathbb{E}^2(\ln X^A) \int_0^t x(s)ds. \tag{69}$$

The above is the SDE for geometric Brownian motion where the Brownian motion has been replaced by a reflected Brownian motion. Nevertheless, after some algebra its solution is given by

$$x(t) = x(0) \exp\left(\mathbb{E}(\ln X^A)(\tilde{S}_A(t) - \tilde{a}_A)\right).$$

Recalling now that $x(t) = \ln p_A(t) - \ln p_B(0)$, we arrive at

$$\ln p_A(t) = \ln p_B(0) + (\ln p_A(0) - \ln p_B(0)) \exp\left(\mathbb{E}(\ln X^A)(\tilde{S}_A(t) - \tilde{a}_A)\right).$$

Moreover, by (50) the density function of $\tilde{\mu}_A(t)$ is given by

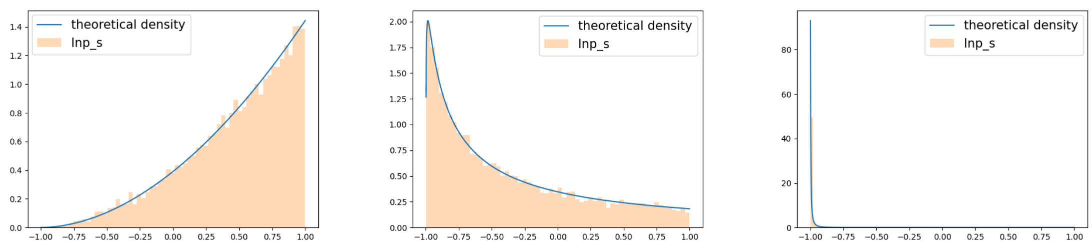
$$f_A(x) = -\left[\mathbb{E}(\ln X^A)x(\ln x - \ln p_B(0))\right]^{-1} \text{ for } p_A(t) \leq x \leq p_A(0). \tag{70}$$

Note that f_A as given above is continuous and differentiable in the interval $[p_A(t), p_A(0)]$. Moreover, the support of $\tilde{\mu}_A(t)$ depends on the time t but its actual density $f_A(x)$ does not as long as x is within the support $[p_A(t), p_A(0)]$.

Now recall that $\tilde{S}_A(t) - \tilde{a}_A$ is a Brownian motion reflected at 0 with constant drift θ_A and infinitesimal variance $2\lambda_A$. Therefore, using the transient distribution of reflected Brownian motion, the distribution of $p_A(t)$ is given by

$$\begin{aligned} &P\Big(\ln p_A(t) \leq \ln p_A(0) + (\ln p_A(0) - \ln p_B(0))z\Big) \\ =&1 - \Phi\left(\frac{\ln z/\mathbb{E}(\ln X^A) - \theta_A t}{\sqrt{2\lambda_A t}}\right) + \exp\left(\frac{\theta_A}{\lambda_A}\frac{\ln z}{\mathbb{E}(\ln X^A)}\right)\Phi\left(\frac{-\ln z/\mathbb{E}(\ln X^A) - \theta_A t}{\sqrt{2\lambda_A t}}\right), \end{aligned} \tag{71}$$

where Φ is the c.d.f. of a standard normal distribution. In Figure 8 below, we use Monte Carlo simulation to generate a histogram of the best ask price at time $t = 1$, and compare it with the theoretical density function in (71). We consider three cases where the drift of the reflected Brownian motion $\tilde{S}_A$ is negative, zero and positive. From Figure 8 we can clearly see that the density function (71) closely matches the histogram generated from the Monte Carlo simulation.



(a) $\theta_s = -1$ (b) $\theta_s = 0$ (c) $\theta_s = 1$

Figure 8: Histogram based on Monte Carlo simulation and the theoretical density function of $p_A(1)$ in the case that $\mathbb{E}(\ln X^B) = 0$.

6.4.2 The case where $\tilde{S}_B(t) = \tilde{S}_A(t)$

We next consider the case in which $\tilde{S}_B(t) = \tilde{S}_A(t)$ for $t \geq 0$. For convenience, let $\tilde{S} = \tilde{S}_B = \tilde{S}_A$ and set

$$T^t(u) \quad = \quad T^t_B(u) = T^t_A(u) \text{ for } 0 \leq u \leq t, \tag{72}$$

and $x(t) = \ln p_A(t) - \ln p_B(t)$. Recall that the definitions of T^t_A and T^t_B are given in (115) and (116) of the appendix. Subtracting (44) from (45) and using the definitions of T^t_B and T^t_A together with a change-of-variables, yields after a little bit of algebra that

$$x(t) = x(0) + (\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B)) \int_0^t x(u) dT^t(u)$$

for $t \geq 0$. Applying Lemma 2 in the appendix and setting $\lambda = \lambda_A = \lambda_B$, we then arrive at

$$x(t) \quad = \quad x(0) + \big(\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B)\big) \left(\int_0^t x(u) d\tilde{S}(u) + \lambda(\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B)) \int_0^t x(u) du \right),$$

which is equivalent to

$$\frac{dx(t)}{x(t)} = (\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B))d\tilde{S}(t) + \lambda(\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B))^2 dt.$$

The above is the SDE for a geometric Brownian motion where the Brownian motion has been replaced by a reflecing Brownian motion. Its solution is given by

$$x(t) = x(0)\exp\Big((\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B))(\tilde{S}(t) - \tilde{a})\Big) \tag{73}$$

for $t \geq 0$, where $\tilde{a} = \tilde{a}_S = \tilde{a}_B$.

Now note that multiplying (44) by $E(\ln X^A)$, and (45) by $E(\ln X^B)$, and then adding the resulting two expressions together, we obtain that

$$E(\ln X^B)\ln p_A(t) + E(\ln X^A)\ln p_B(t) = E(\ln X^B)\ln p_A(0) + E(\ln X^A)\ln p_B(0). \tag{74}$$

(74) implies that the appropriately weighted average of the log of the two price processes remains constant. In particular, the mid-log price does not change if $E(\ln X^B) = E(\ln X^A)$. Combining (73) with (74) now yields explicit expressions for $p_A(t)$ and $p_B(t)$ given by

$$\ln p_A(t) = \frac{E(\ln X^B)\ln p_A(0) + E(\ln X^A)\ln p_B(0) + E(\ln X^A)x(t)}{E(\ln X^A) + E(\ln X^B)} \tag{75}$$

and

$$\ln p_B(t) \quad = \quad \frac{E(\ln X^B)\ln p_A(0) + E(\ln X^A)\ln p_B(0) - E(\ln X^B)x(t)}{E(\ln X^A) + E(\ln X^B)} \tag{76}$$

for $t \geq 0$.

It also follows that substituting (76) into (50) we obtain the density function of $\tilde{\mu}_A(t)$ for $x \in [p_A(t), p_A(0)]$ given by

$$f_A(x) = -\Big[x\Big((E(\ln X^A) + E(\ln X^B))\ln x - E(\ln X^B)\ln p_A(0) - E(\ln X^A)\ln p_B(0)\Big)\Big]^{-1}. \tag{77}$$

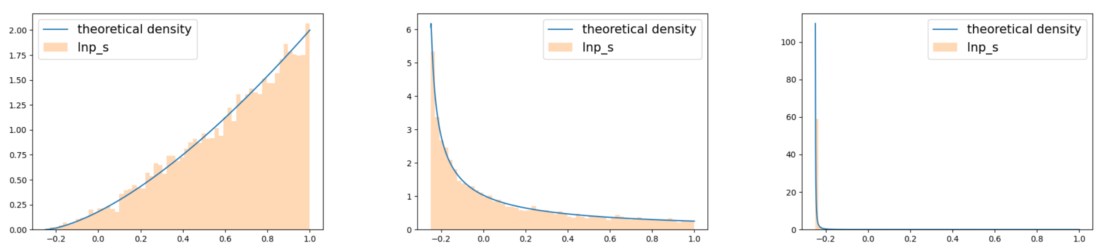
Note in particular that the density function of $\tilde{\mu}_A(t)$ is continuous and differentiable in the interval $[p_A(t), p_A(0)]$. Moreover, the time t only affects the support of $\tilde{\mu}_A(t)$, while the density $f_A(x)$ does not depend on t as long as x is in the support $[p_A(t), p_A(0)]$. In a similar manner, it may be shown that the density function of $\tilde{\mu}_B(t)$ for $x \in [p_B(0), p_B(t)]$ is given by

$$f_B(x) = \Big[x\Big((E(\ln X^A) + E(\ln X^B))\ln x - E(\ln X^B)\ln p_A(0) - E(\ln X^A)\ln p_B(0)\Big)\Big]^{-1}. \tag{78}$$

Now recall that $\tilde{S}_A(t) - \tilde{a}_A$ is a Brownian motion reflected at 0 with a constant drift θ_A and infinitesimal variance $2\lambda_A$. Using (75) and the transient distribution of reflected Brownian motion, the distribution of $p_A(t)$ is then given by

$$P\left(\ln p_A(t) \le \frac{E(\ln X^B)\ln p_A(0) + E(\ln X^A)\ln p_B(0) + E(\ln X^A)(\ln p_A(0) - \ln p_B(0))z}{E(\ln X^A) + E(\ln X^B)}\right)$$
$$= \Phi\left(\frac{\frac{\ln z}{\mathbb{E}(\ln X^A)+\mathbb{E}(\ln X^B)} - \theta_A t}{\sqrt{2\lambda_A t}}\right) - \exp\left(\frac{\theta_A \ln z}{\lambda_A(\mathbb{E}(\ln X^A) + \mathbb{E}(\ln X^B))}\right)\Phi\left(\frac{-\frac{\ln z}{\mathbb{E}(\ln X^A)+\mathbb{E}(\ln X^B)} - \theta_A t}{\sqrt{2\lambda_A t}}\right) \tag{79}$$

for $z \in \mathbb{R}$, where Φ denotes the c.d.f. of a standard normal distribution. In Figure 9 below, we use Monte Carlo simulation to generate a histogram of the best ask price at time $t = 1$, and compare it with the theoretical density function in (79) in the case where the drift of the reflected Brownian motion $\tilde{S}_A$ is negative, zero and positive. The results in Figure 9 show a clear match between the density function (79) and the histogram generated from the Monte Carlo simulation.



(a) $\theta_A = -1$ (b) $\theta_A = 0$ (c) $\theta_A = 1$

Figure 9: Histogram based on Monte Carlo simulation and the theoretical density function of $p_A(1)$ in the case that $\tilde{S}_B(t) = \tilde{S}_A(t)$.

Finally, we remark that although the results in this section are based on the assumption that $\tilde{S}_B(t) = \tilde{S}_A(t)$, their derivation can also be extended to cover the case where $\tilde{S}_B(t) - \tilde{a}_B = c(\tilde{S}_A(t) - \tilde{a}_A)$ for arbitrary $c > 0$.

7 The case of $E[\ln X^B]$ and $E[\ln X^A]$ having differing signs

In this section, we discuss the case where $E[\ln X^B]$ and $E[\ln X^A]$ have differing signs. The solution to the SDE (31)-(32) can then be characterized as a mixture of the two cases discussed previously. Specifically, in the case where $\mathbb{E}(\ln X^B) \geq 0$ and $\mathbb{E}(\ln X^A) < 0$, consider the system of equations

$$\ln p_B(t) = \ln p_B(0) - \lambda_B \mathbb{E}(\ln X^B) \int_0^t \ln\left(\frac{p_A(s)}{p_B(s)}\right) \frac{ds}{\alpha_B(s)}, \tag{80}$$

$$\ln p_A(t) = \ln G_A^{-1}(\underline{m}_A(t)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} \ln\left(\frac{p_A(\tau_{A,u}^t)}{p_B(\tau_{A,u}^t)}\right) du, \tag{81}$$

for $t \geq 0$. We then have the following result.

Theorem 3. *If $\mathbb{E}(\ln X^B) \geq 0$ and $\mathbb{E}(\ln X^A) < 0$, then a solution $(\tilde{\mu}_B, \tilde{\mu}_A)$ to the SDE* (31)-(32) *is such that for each $t \geq 0$, $\tilde{\mu}_B(t)$ is given by*

$$\tilde{\mu}_B(t)([0,x]) = \begin{cases} \tilde{\mu}_B(0)([0,x]) & \text{if } 0 \leq x < p_B(t), \\ \tilde{S}_B(t) & \text{if } x \geq p_B(t), \end{cases} \tag{82}$$

and $\tilde{\mu}_A(t)$ is absolutely continuous with density

$$f_{t,x}^A = \begin{cases} 0 & \text{if } x \leq p_A(t), \\ \left[\mathbb{E}(\ln X^A) x \ln\left(\frac{p_B(\tau_{t,x}^{p_A})}{x}\right)\right]^{-1} & \text{if } p_A(t) < x \leq \overline{p}_A(t), \\ f_{0,x}^A & \text{if } x > \overline{p}_A(t), \end{cases} \tag{83}$$

where (p_B, p_A) is the unique solution to the system of equations (80)-(81).

Proof. By Proposition 7 in the appendix, it follows that $p_B(t) < p_A(t)$ for $t \geq 0$. The result then follows as in the proofs of Theorems 1 and 2. □

A Proofs of Main Results

In the appendix, we provide the proofs of our main results.

A.1 Proof of Theorem 1

Before providing the proof of Theorem 1, we first have the following result.

Proposition 3. *If $p_B(0) < p_A(0)$, then P-a.s. there exists a unique solution to* (35)-(36).

Proof. First let $\bar{x} = (x_1, x_2)$ and consider the solution to the system of equations

$$\bar{x}(t) \;=\; \bar{x}(0) + \int_0^t g(s, \bar{x}(s))ds \text{ for } t \geq 0, \tag{84}$$

where $g : (\mathbb{R}_+, \mathbb{R}^2) \mapsto \mathbb{R}^2$ is given by

$$g(t, \bar{x}) \;=\; \left(-c_1 \frac{x_2(t) - x_1(t)}{h_1(t, x_1(t))}, c_2 \frac{x_2(t) - x_1(t)}{h_2(t, x_2(t))} \right), \tag{85}$$

with $c_1, c_2 > 0$, and where

$$h_1(t, x) \;=\; \tilde{S}_B(t) - \tilde{\mu}_B(0)(0, \exp(x)) \tag{86}$$

and

$$h_2(t, x) \;=\; \tilde{S}_A(t) - \tilde{\mu}_A(0)(\exp(x), \infty). \tag{87}$$

The result of the proposition then follows if there exists a unique solution to (84). It therefore suffices to show that for each $t \geq 0$, there exists a constant κ_t such that for each $0 \leq s \leq t$, $g(s, \cdot)$ is Lipschitz continuous with constant κ_t. We proceed as follows.

First note by the assumptions of the theorem there exists a constant $c > 0$ such that $h_1(t, x), h_2(t, x) > c$. It then follows by the fundamental theorem of calculus that for $i = 1, 2$,

$$\left| \frac{1}{h_i(t, x)} - \frac{1}{h_i(t, x)} \right| \;\leq\; \frac{1}{c^2} \cdot |h_i(t, x) - h_i(t, y)| \text{ for } x, y \in \mathbb{R}. \tag{88}$$

Moreover, by Gronwall's inequality, for each $t \geq 0$ there exists a constant $\bar{c}_t$ such that if $\bar{x}$ is a solution to (84), then $\|\bar{x}_s\| \leq \|x_0\| \bar{c}_t$ for $0 \leq s \leq t$. Thus, let $\bar{x}, \bar{y} \in \mathbb{R}^2$ be such that $\|\bar{x}\|, \|\bar{y}\| \leq \|x_0\| \bar{c}_t$. It then follows after some simple algebra that for $0 \leq s \leq t$,

$$\|g(s, \bar{x}) - g(s, \bar{y})\| \;\leq\; \frac{c_1 + c_2}{c} \|\bar{x} - \bar{y}\| \tag{89}$$

$$+ \frac{\bar{c}_t}{c^2} \left(c_1 |h_1(s, \bar{x}_1) - h_1(s, \bar{y}_1)| + c_2 |h_2(s, \bar{x}_2) - h_2(s, \bar{y}_2)| \right). \tag{90}$$

However, by the assumptions that $\tilde{\mu}_B(0)(0, x)$ and $\tilde{\mu}_A(0)(x, \infty)$ are Lipschitz continuous, together with the fact that $\exp(x)$ is Lipsschitz continuous on compact sets, it follows there exists a constant $\bar{\nu}_t$ such that for $0 \leq s \leq t$,

$$|h_1(s, \bar{x}_1) - h_1(s, \bar{y}_1)| \;\leq\; \kappa_t \|\bar{x} - \bar{y}\|. \tag{91}$$

(89) and (91) now complete the proof. □

The proof of Theorem 1 is now as follows.

Proof of Theorem 1. By Proposition 3, P-a.s. there exists a unique solution to (35)-(36). Thus, it suffices to prove that $(\tilde{\mu}_B, \tilde{\mu}_A)$ given by (33)-(37) is a solution to (31)-(32). We prove that (31) is satisfied. The proof that (32) is satisfied follows similarly and hence it not shown.

First note that by (33) and (37), it follows that

$$\langle \tilde{\mu}_B(t), \phi \rangle = \int_{(0,p_B(t))} \phi(u) d\tilde{\mu}_B(0)(u) + \alpha_B(t)\phi(p_B(t)), \tag{92}$$

for each $t \geq 0$ and $\phi \in C_b^2(\mathbb{R})$. Hence, in order to verify that $(\tilde{\mu}_B, \tilde{\mu}_A)$ satisfies (31), it is sufficient to show that

$$\begin{aligned} \alpha_B(t)\phi(p_B(t)) &= \langle \tilde{\mu}_B(0), \phi \rangle - \int_{(0,p_B(t))} \phi(u) d\tilde{\mu}_B(0)(u) \\ &\quad + \int_0^t \phi(p_B(s)) d\tilde{S}_B(s) \\ &\quad - \lambda_B \mathbb{E}(\ln X^B) \int_0^t \phi'(p_B(s)) p_B(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right) ds. \end{aligned} \tag{93}$$

First note by (35) that p_B is monotonically decreasing and so also using the fact that $\alpha_B(t) = \tilde{S}_B(t) - \tilde{\mu}_B(0)((0, p_B(t)))$ for $t \geq 0$, we may integrate-by-parts to obtain that

$$\alpha_B(t)\phi(p_B(t)) = \alpha_B(0)\phi(p_B(0)) + \int_0^t \alpha_B(s) d\phi(p_B(s)) + \int_0^t \phi(p_B(s)) d\alpha_B(s). \tag{94}$$

Regarding the middle term on the righthand side above, note by (35) it follows that

$$\begin{aligned} \int_0^t \alpha_B(s) d\phi(p_B(s)) &= \int_0^t \alpha_B(s)\phi'(p_B(s)) dp_B(s) \\ &= -\lambda_B \mathbb{E}(\ln X^B) \int_0^t \phi'(p_B(s)) p_B(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right) ds, \end{aligned} \tag{95}$$

and so substituting (95) into (94) we obtain that

$$\begin{aligned} \alpha_B(t)\phi(p_B(t)) &= \alpha_B(0)\phi(p_B(0)) + \int_0^t \phi(p_B(s)) d\alpha_B(s) \\ &\quad - \lambda_B \mathbb{E}(\ln X^B) \int_0^t \phi'(p_B(s)) p_B(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right) ds. \end{aligned} \tag{96}$$

Next, note that since $\alpha_B(t) = \tilde{S}_B(t) - \tilde{\mu}_B(0)((0, p_B(t)))$, it follows after some algebra that

$$\begin{aligned} & \alpha_B(0)\phi(p_B(0)) + \int_0^t \phi(p_B(s))d\alpha_B(s) \qquad (97) \\ = \quad & \alpha_B(0)\phi(p_B(0)) - \int_0^t \phi(p_B(s))d\tilde{\mu}_B(0)((0, p_B(s))) + \int_0^t \phi(p_B(s))d\tilde{S}_B(s). \end{aligned}$$

Substituting (97) into (96), in order to complete the proof it now suffices after some algebra to show that

$$\begin{aligned} & \int_{(0,p_B(t))} \phi(u)d\tilde{\mu}_B(0)(u) - \int_0^t \phi(p_B(s))d\tilde{\mu}_B(0)((0, p_B(s))) + \alpha_B(0)\phi(p_B(0)) \qquad (98) \\ = \quad & \langle \tilde{\mu}_B(0), \phi \rangle. \end{aligned}$$

However, since $\mathbb{E}(\ln X^B) \geq 0$, it follows by (35) that p_B is non-increasing, and so by a change-of-variables we obtain that

$$\int_0^t \phi(p_B(s))d\tilde{\mu}_B(0)((0, p_B(s))) \quad = \quad - \int_{[p_B(t), p_B(0))} \phi(u)d\tilde{\mu}_B(0)(u).$$

The equality (98) now follows by (92). $\square$

A.2 Proof of Theorem 2

In this section, we provide the proof of Theorem 2. We begin by proving Proposition 2. Then, we proceed to the proof of Theorem 2.

A.2.1 Proof of Proposition 2

Let $b_B = \ln p_B(0), b_A = \ln p_A(0)$, and for each $T \geq 0$ denote by $C_T(b_B, b_A) = \{f \in C([0,T], \mathbb{R}^2) : f(0) = (b_B, b_A)\}$ the metric space with the L_∞ norm induced metric. Next, define the mapping $\mathcal{T}$ as follows. For each $x = (x_1, x_2) \in C_T(b_B, b_A)$, let

$$\begin{aligned} \mathcal{T}(x)(t) \quad = \quad \Bigg(& \ln \bar{F}_{B,0}^{-1}(\underline{m}_B(t)) + c_2 \int_{\underline{m}_B(t)}^{\tilde{S}_B(t)} \left(x_2(\tau_{B,u}^t) - x_1(\tau_{B,u}^t) \right) du, \\ & \ln \bar{F}_{A,0}^{-1}(\underline{m}_A(t)) + c_1 \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} \left(x_1(\tau_{A,u}^t) - x_2(\tau_{A,u}^t) \right) du \Bigg) \qquad (99) \end{aligned}$$

for $0 \leq t \leq T$, where $b_B, b_A, c_1, c_2 \in \mathbb{R}$ and $b_B < b_A$. The following lemma implies that P-a.s. $\mathcal{T} : C_T(b_B, b_A) \mapsto C_T(b_B, b_A)$.

Lemma 1. *Let $f : [0,\infty) \to \mathbb{R}$ be a measurable function that is bounded on compact sets. Then, the mappings $F_B : [0,\infty) \to \mathbb{R}$ and $F_A : [0,\infty) \to \mathbb{R}$ defined for $t \geq 0$ by*

$$F_B(t) = \int_{\underline{m}_B(t)}^{\tilde{S}_B(t)} f(\tau^t_{B,u})du \quad \text{and} \quad F_A(t) = \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} f(\tau^t_{A,u})du$$

are P-a.s. continuous.

Proof. We prove the result for F_B, the proof for F_A follows similarly. Let $0 \leq x \leq y \leq t$. Then, by the triangle inequality

$$\begin{aligned}
&\left| \int_{\underline{m}_B(x)}^{\tilde{S}_B(x)} f(\tau^x_{B,u})du - \int_{\underline{m}_B(y)}^{\tilde{S}_B(y)} f(\tau^y_{B,u})du \right| \\
\leq &\left| \int_{\underline{m}_B(x)}^{\tilde{S}_B(x)} f(\tau^x_{B,u})du - \int_{\underline{m}_B(x)}^{\tilde{S}_B(x)} f(\tau^y_{B,u})du \right| + \left| \int_{\underline{m}_B(x)}^{\tilde{S}_B(x)} f(\tau^y_{B,u})du - \int_{\underline{m}_B(y)}^{\tilde{S}_B(x)} f(\tau^y_{B,u})du \right| \\
&+ \left| \int_{\underline{m}_B(y)}^{\tilde{S}_B(x)} f(\tau^y_{B,u})du - \int_{\underline{m}_B(y)}^{\tilde{S}_B(y)} f(\tau^y_{B,u})du \right| \\
\leq &\int_{\inf_{x\leq v\leq y} \tilde{S}_B(v)}^{\sup_{x\leq v\leq y} \tilde{S}_B(v)} \left| f(\tau^x_{B,u}) - f(\tau^y_{B,u}) \right| du + |\underline{m}_B(x) - \underline{m}_B(y)| \sup_{0\leq s\leq y} |f(s)| \\
&+ \left| \tilde{S}_B(x) - \tilde{S}_B(y) \right| \sup_{0\leq s\leq y} |f(s)| \\
\leq &2 \left| \sup_{x\leq v\leq y} \tilde{S}_B(v) - \inf_{x\leq v\leq y} \tilde{S}_B(v) \right| \sup_{0\leq s\leq y} |f(s)| + |\underline{m}_B(x) - \underline{m}_B(y)| \sup_{0\leq s\leq y} |f(s)| \qquad (100) \\
&+ \left| \tilde{S}_B(x) - \tilde{S}_B(y) \right| \sup_{0\leq s\leq y} |f(s)|.
\end{aligned}$$

In the above, we use the fact that

$$\tau^x_{B,u} = \tau^y_{B,u} \text{ for } u \notin \left[\inf_{x\leq v\leq y} \tilde{S}_B(v), \sup_{x\leq v\leq y} \tilde{S}_B(v) \right].$$

Now since $\tilde{S}_B$ is P-a.s. continuous, it is clear that P-a.s. (100) goes to zero as $|x-y| \to 0$. $\square$

We are now ready to prove the existence and uniqueness portion of Proposition 2.

Proposition 4. *If $p_B(0) < p_A(0)$, then P-a.s. there exists a unique solution (p_B, p_A) to* (44)-(45).

Proof of Proposition 4. For each $i = 1, 2, ...$, let

$$\begin{aligned}
\lambda_i \quad = \quad \min\Bigg(&\inf\left\{ s > \lambda_{i-1} : |\tilde{S}_B(s) - \inf\{\tilde{S}_B(x) : \lambda_{i-1} \leq x \leq s\}| \geq \frac{1}{4|c_2|} \right\}, \qquad (101) \\
&\inf\left\{ s > \lambda_{i-1} : |\tilde{S}_A(s) - \inf\{\tilde{S}_A(x) : \lambda_{i-1} \leq x \leq s\}| \geq \frac{1}{4|c_1|} \right\} \Bigg),
\end{aligned}$$

where c_1, c_2 are as in (99) and $\lambda_0 = 0$. Note that P-a.s. $\lambda_i \to \infty$ as $i \to \infty$ since otherwise the increasing sequence $\{\lambda_i, i = 0, 1, 2, ...\}$ converges to a finite value, which violates the P-a.s. continuity of $\tilde{S}_B$ and $\tilde{S}_A$. Hence, the set of $\omega \in \Omega$ such that $\tilde{S}_B(\omega)$ and $\tilde{S}_A(\omega)$ are continuous and $\lambda_i(\omega) \to \infty$ has probability 1. We now complete the proof by showing that for all such ω, there exists a unique solution $(p_B(\omega), p_A(\omega))$ to (44)-(45) on $[0, \lambda_i(\omega)]$ for $i \geq 0$. We proceed by induction on i.

Let $\omega \in \Omega$ be as described above and note that in the base case of $i = 0$, we have that $\lambda_0(\omega) = 0$ and so the unique solution $(p_B(\omega), p_A(\omega))$ to (44)-(45) on $[0, \lambda_0]$ is trivially given by $(p_B(\omega, 0), p_A(\omega, 0))$. Next, suppose that there exists a unique solution (p_B, p_A) to (44)-(45) on $[0, \lambda_{i-1}]$. In order to complete the proof, it suffices to show that there exists a unique solution to (44)-(45) on $[0, \lambda_i]$. We proceed as follows.

Consider the metric space $C_i(p_B, p_A) = \{x \in C([0, \lambda_i], \mathbb{R}^2) : x(s) = (p_B(s), p_A(s)), 0 \leq s \leq \lambda_{i-1}\}$ with the L_∞ norm induced metric, and for $x = (x_1, x_2) \in C_i(p_B, p_A)$ define the mapping $\mathcal{T}_i$ by setting $\mathcal{T}_i(x)(s)$ equal to

$$\begin{aligned} \Bigg(&\ln \bar{F}_{B,0}^{-1}(\underline{m}_B(t)) + c_2 \int_{\underline{m}_B(t)}^{\tilde{S}_B(s)} \left(x_2(\tau_{B,u}^s) - x_1(\tau_{B,u}^s) \right) du, \\ &\ln \bar{F}_{A,0}^{-1}(\underline{m}_A(t)) + c_1 \int_{\underline{m}_A(t)}^{\tilde{S}_A(s)} \left(x_1(\tau_{A,u}^s) - x_2(\tau_{A,u}^s) \right) du \Bigg), \end{aligned} \tag{102}$$

for $0 \leq s \leq \lambda_i$. By Lemma 1 above and the induction hypothesis, it follows that $\mathcal{T}_i : C_i(p_B, p_A) \mapsto C_i(p_B, p_A)$. Moreover, by the induction hypothesis and (102), any solution to (44)-(45) on $[0, \lambda_i]$ must lie in $C_i(p_B, p_A)$ and be a fixed point of $\mathcal{T}_i$. Hence, in order to complete the proof it suffices to show that there exists a unique fixed point to $\mathcal{T}_i$.

By the Banach fixed-point theorem [Granas et al., 2003], it suffices to show that $\mathcal{T}_i : C_i(p_B, p_A) \mapsto C_i(p_B, p_A)$ is a contraction mapping. Let $x, y \in C_i(p_B, p_A)$. Then, since $x(s) = y(s) = (p_B(s), p_A(s))$ for $0 \leq s \leq \lambda_{i-1}$, it follows that $\mathcal{T}_i(x)(s) = \mathcal{T}_i(y)(s)$ for $0 \leq s \leq \lambda_{i-1}$. Next, let

$$m_B^i = \inf_{\lambda_{i-1} \leq s \leq \lambda_i} \tilde{S}_B(s) \quad \text{and} \quad m_A^i = \inf_{\lambda_{i-1} \leq s \leq \lambda_i} \tilde{S}_A(s) \tag{103}$$

and note that $0 \leq \tau_{B,u}^s \leq \lambda_{i-1}$ for $\lambda_{i-1} \leq s \leq \lambda_i$ and $0 \leq u \leq m_B^i$. Similarly, $0 \leq \tau_{A,u}^s \leq \lambda_{i-1}$ for $0 \leq u \leq m_A^i$ and $\lambda_{i-1} \leq s \leq \lambda_i$. Thus, for each $x, y \in C_i(p_B, p_A)$ we have that

$\mathcal{T}_i(x)(s) - \mathcal{T}_i(y)(s) = 0$ for $0 \le s \le \lambda_{i-1}$ and

$$\mathcal{T}_i(x)(s) - \mathcal{T}_i(y)(s) = \Bigg(c_2 \int_{m_B^i}^{\tilde{S}_B(s)} \big(x_2(\tau_{B,u}^s) - y_2(\tau_{B,u}^s) + y_1(\tau_{B,u}^s) - x_1(\tau_{B,u}^s)\big)\, du, \quad (104)$$

$$c_1 \int_{m_A^i}^{\tilde{S}_A(s)} \big(x_1(\tau_{A,u}^s) - y_1(\tau_{A,u}^s) + y_2(\tau_{A,u}^s) - x_2(\tau_{A,u}^s)\big)\, du \Bigg) \quad (105)$$

for $\lambda_{i-1} \le s \le \lambda_i$. It now follows from (101) that

$$\begin{aligned} & \|\mathcal{T}_i(x) - \mathcal{T}_i(y)\|_\infty \\ \le\ & 2\|x-y\|_\infty \max\left\{ |c_2| \sup_{\lambda_{i-1}\le s\le\lambda_i} |\tilde{S}_B(s) - m_B^i|, |c_1| \sup_{\lambda_{i-1}\le s\le\lambda_i} |\tilde{S}_A(s) - m_A^i| \right\} \\ =\ & \frac{1}{2}\|x-y\|_\infty. \end{aligned}$$

This implies that $\mathcal{T}_i$ is a contraction mapping, which completes the proof. □

Finally, we prove the positive spread part of Proposition 2.

Proposition 5. *If $p_B(0) < p_A(0)$, then the unique solution to* (44)-(45) *is such that $p_B(t) < p_A(t)$ for $t \ge 0$.*

Proof. Let $x(t) = \ln p_A(t) - \ln p_B(t)$ and note that subtracting (44) from (45) we obtain that

$$x(t) = d(t) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} x(\tau_{A,u}^t) du + \mathbb{E}(\ln X^B) \int_{\underline{m}_B(t)}^{\tilde{S}_B(t)} x(\tau_{B,u}^t) du \quad (106)$$

for $t \ge 0$, where $d(t) = \ln \bar{F}_{A,0}^{-1}(\underline{m}_A(t)) - \ln \bar{F}_{B,0}^{-1}(\underline{m}_B(t))$. Now suppose that the result does not hold and let $t^* = \inf\{t \ge 0 : p_B(t) = p_A(t)\}$. Then, by the continuity of p_B, p_A together with the continuity of $\tilde{S}_B, \tilde{S}_A$, and the fact that $x(t) > 0$ for $t \in [0, t^*)$, there exists a $\bar{t} \in [0, t^*)$ such that $x(t) \le x(\bar{t})$ for $t \in [\bar{t}, t^*]$ and

$$-3\mathbb{E}(\ln X^A)\left(\tilde{S}_A(t^*) - \inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)\right) \le 1 \quad \text{and} \quad -3\mathbb{E}(\ln X^B)\left(\tilde{S}_B(t^*) - \inf_{\bar{t}\le t\le t^*} \tilde{S}_B(t)\right) \le 1.$$

Next note that d is a non-decreasing function, and so it follows that

$$\begin{aligned} x(\bar{t}) = x(\bar{t}) - x(t^*) \le\ & \mathbb{E}(\ln X^A)\left(\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau_{A,u}^{\bar{t}}) du - \int_{\underline{m}_A(t^*)}^{\tilde{S}_A(t^*)} x(\tau_{A,u}^{t^*}) du \right) \quad (107) \\ & + \mathbb{E}(\ln X^B)\left(\int_{\underline{m}_B(\bar{t})}^{\tilde{S}_B(\bar{t})} x(\tau_{B,u}^{\bar{t}}) du - \int_{\underline{m}_B(t^*)}^{\tilde{S}_B(t^*)} x(\tau_{B,u}^{t^*}) du \right). \end{aligned}$$

Regarding the first term on the righthand side above, if

$$\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t) < \underline{m}_A(\bar{t}), \quad \text{then} \quad \inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t) = \underline{m}_A(t^*) \tag{108}$$

and so it follows that

$$\begin{aligned}
\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u})du - \int_{\underline{m}_A(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du &\ge -\int_{\underline{m}_A(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du &(109)\\
&= -\int_{\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du. &(110)
\end{aligned}$$

On the other hand, if

$$\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t) \ge \underline{m}_A(\bar{t}), \quad \text{then} \quad \underline{m}_A(\bar{t}) = \underline{m}_A(t^*) \tag{111}$$

and so using the fact that $\tau^{t^*}_{A,u} = \tau^{\bar{t}}_{A,u}$ for $u \in \left[\underline{m}_A(\bar{t}), \inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)\right)$, it follows that

$$\begin{aligned}
\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u})du - \int_{\underline{m}_A(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du &= \int_{\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u})du - \int_{\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du\\
&\ge -\int_{\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du. &(112)
\end{aligned}$$

Note that in both cases above, we obtain the same inequality. Moreover, since $\tau^{t^*}_{A,u} \in [\bar{t}, t^*]$ for $u \in \left[\inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t), \tilde{S}_A(t^*)\right]$ and using the fact that $x(t) \le x(\bar{t})$ for $t \in [\bar{t}, t^*]$, it follows from the above and since $E(\ln X^A) < 0$ that

$$\begin{aligned}
&\mathbb{E}(\ln X^A)\left(\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u})du - \int_{\underline{m}_A(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u})du\right) &(113)\\
\le\ & -\mathbb{E}(\ln X^A)x(\bar{t})\left(\tilde{S}_A(t^*) - \inf_{\bar{t}\le t\le t^*} \tilde{S}_A(t)\right).\\
\le\ & \frac{x(\bar{t})}{3}.
\end{aligned}$$

Similarly, it may be shown that

$$\mathbb{E}(\ln X^B)\left(\int_{\underline{m}_B(\bar{t})}^{\tilde{S}_B(\bar{t})} x(\tau^{\bar{t}}_{B,u})du - \int_{\underline{m}_B(t^*)}^{\tilde{S}_B(t^*)} x(\tau^{t^*}_{B,u})du\right) \le \frac{x(\bar{t})}{3}. \tag{114}$$

Combining (112) with (113)-(114) yields that

$$x(\bar{t}) \le \frac{2}{3}x(\bar{t}),$$

which leads to a contradiction since $x(\bar{t}) > 0$. □

A.2.2 Proof of Theorem 2

In this section, we provide the proof of Theorem 2. In preparation for our first result, we must setup some notation. For each $t \geq 0$, let T_A^t be defined by setting

$$T_A^t(s) = \inf_{s \leq u \leq t} \tilde{S}_A(u) \text{ for } 0 \leq s \leq t. \tag{115}$$

Note that since $\tilde{S}_A$ is continuous, by definition T_A^t is a continuous, non-decreasing function with $T_A^t(0) = \tilde{S}_A(0)$ and $T_A^t(t) = \tilde{S}_A(t)$. Similarly, we define the function T_B^t by setting

$$T_B^t(s) = \inf_{s \leq u \leq t} \tilde{S}_B(u) \text{ for } 0 \leq s \leq t. \tag{116}$$

We then have the following lemma. An analogous result holds replacing T_A^t by T_B^t and $\tilde{S}_A$ by $\tilde{S}_B$.

Lemma 2. *If $q = \{q(t), t \in [0, \infty)\}$ is an adapted, continuous process, and*

$$h(t) \;=\; h(0) + \int_0^t q(s) dT_A^t(s) \tag{117}$$

for $t \geq 0$, then

$$\int_0^t h(s) d\tilde{S}_A(s) - \int_0^t h(s) dT_A^t(s) = -\lambda_A \int_0^t q(s) ds. \tag{118}$$

Proof. First we shall prove (118) for the case of "single jump processes"

$$q(s) = Q 1_{[u,\infty)}(s), \; h(0) = 0,$$

where $0 \leq u < \infty$ is a constant, and Q is an $\mathcal{F}_u$-measurable random variable. The identity in Lemma 1 is obviously true if $0 \leq t \leq u$, because then both sides are zero. Next we prove the identity for $t \geq u$. We have

$$h(s) = Q(\tilde{S}_A(s) - T_A^s(u))$$

for all $s \geq u$, hence

$$\int_0^t h(s) d(\tilde{S}_A(s) - T_A^t(s)) = Q \int_u^t (\tilde{S}_A(s) - T_A^s(u)) d\tilde{S}_A(s) - Q \int_u^t (\tilde{S}_A(s) - T_A^s(u)) dT_A^t(s). \tag{119}$$

We calculate the two integrals on the right-hand side separately. The first will be

$$\int_u^t (\tilde{S}_A(s) - T_A^s(u)) d\tilde{S}_A(s) = \int_u^t \tilde{S}_A(s) d\tilde{S}_A(s) - \int_u^t T_A^s(u) d\tilde{S}_A(s) =$$

$$\frac{1}{2}((\tilde{S}_A(t))^2 - (\tilde{S}_A(u))^2) - \frac{1}{2}(2\lambda_A)(t-u) - \int_u^t T_A^s(u)d\tilde{S}_A(s). \tag{120}$$

Using Ito's rule, the stochastic integral on the right-hand side can be written in the following way:

$$\int_u^t T_A^s(u)d\tilde{S}_A(s) = T_A^t(u)\tilde{S}_A(t) - (\tilde{S}_A(u))^2 - \int_u^t \tilde{S}_A(s)dT_A^s(u). \tag{121}$$

The function $p \mapsto T_A^p(u)$ with domain $[u,\infty)$ is decreasing at a point $s \geq u$ only if $\tilde{S}_A(\cdot)$ achieves its minimum on $[u,s]$ at the point s. Hence $p \mapsto T_A^p(u)$ decreases at the point s only if $T_A^s(u) = \tilde{S}_A(s)$. Thus we can cast the right-hand side (121) in the form

$$T_A^t(u)\tilde{S}_A(t) - (\tilde{S}_A(u))^2 - \int_u^t T_A^s(u)dT_A^s(u) =$$

$$T_A^t(u)\tilde{S}_A(t) - (\tilde{S}_A(u))^2 - \frac{1}{2}((T_A^t(u))^2 - (\tilde{S}_A(u))^2) = T_A^t(u)\tilde{S}_A(t) - \frac{1}{2}(\tilde{S}_A(u))^2 - \frac{1}{2}(T_A^t(u))^2.$$

We substitute the stochastic integral on the right-hand side of (120) with the last expression in the above identity, then after some algebra we get that the first integral on the right-hand side of (119) can be written as

$$\int_u^t (\tilde{S}_A(s) - T_A^s(u))d\tilde{S}_A(s) = \frac{1}{2}\left((\tilde{S}_A(t))^2 + (T_A^t(u))^2\right) - T_A^t(u)\tilde{S}_A(t) - \lambda_A(t-u). \tag{122}$$

Next we deal with the second integral on the right-hand side of (119). Notice that the function $T_A^t(\cdot)$ with domain $[0,t]$ is increasing at a point $s \in [0,t]$ only if $\tilde{S}_A(\cdot)$ reaches its minimum on $[s,t]$ at the point s. It follows that we we have $\tilde{S}_A(s) = T_A^t(s)$ whenever $s \in [0,t]$ is a point of increase of $T_A^t(\cdot)$. Therefore the second integral on the right-hand side of (119) can be written as

$$\int_u^t (\tilde{S}_A(s) - T_A^s(u))dT_A^t(s) = \int_u^t (T_A^t(s) - T_A^s(u))dT_A^t(s). \tag{123}$$

Similarly as before, one can see that if $s \in [0,t]$ is a point of increase of $T_A^t(\cdot)$, then $T_A^s(u) = T_A^t(u)$ for $u \leq s \leq t$. This follows again from the fact that the function $T_A^t(\cdot)$ is increasing at a point s only if $\tilde{S}_A(\cdot)$ reaches its minimum on $[s,t]$ at the point s. Thus we can cast (123) in the form

$$\int_u^t (\tilde{S}_A(s) - T_A^s(u))dT_A^t(s) = \int_u^t (T_A^t(s) - T_A^t(u))dT_A^t(s)$$

$$= \int_u^t T_A^t(s)dT_A^t(s) - T_A^t(u)(\tilde{S}_A(t) - T_A^t(u)) =$$

$$\frac{1}{2}\left((\tilde{S}_A(t))^2 - (T_A^t(u))^2\right) - T_A^t(u)\left(\tilde{S}_A(t) - T_A^t(u)\right) =$$
$$\frac{1}{2}\left((T_A^t(u))^2 + (\tilde{S}_A(t))^2\right) - T_A^t(u)\tilde{S}_A(t). \tag{124}$$

Equations (119), (122) and (124) imply

$$\int_0^t h(s)d(\tilde{S}_A(s) - T_A^t(s)) = -\lambda_A Q(t-u),$$

exactly what we wanted to prove.

In the next step we shall extend the previous result to the case in which q is an adapted, bounded, continuous process, and $h(0) = 0$. Fix a time $t \geq 0$. From the previous part follows that (118) is true if it is a process of the form

$$q(t) = \sum_{j=0}^{n-1} Q_j 1_{[t(j),t(j+1))}(t), \tag{125}$$

where $0 = t(0) < t(1) < \cdots < t(n) = t$ is a partition of the interval $[0,t]$, and Q_j is an $\mathcal{F}_{t(j)}$-measurable random variable for $j = 0, 1, , \ldots, n-1$. Indeed, this kind of process can be written as a linear combination of single jump processes, and we already know that equation (118) is true when q is a single jump process. We shall call processes that have the form given in (125) elementary processes. By Oksendal, Section 3.1 there exists a sequence of adapted elementary processes $\{q_n, n \geq 1\}$ such that each $|q_n| < C$ for some constant C, and

$$\lim_{n\to\infty} q_n(s) = q(s), \text{ a.s.} \quad s \leq t. \tag{126}$$

Let

$$h_n(s) = \int_0^s q_n(u)dT_A^s(u), \quad s \leq t, \tag{127}$$

and h be as in (117). By the Bounded Convergence Theorem we have

$$\lim_{n\to\infty} \int_0^t q_n(s)ds = \int_0^t q(s)ds, \text{ a.s.} \tag{128}$$

and

$$\lim_{n\to\infty} h_n(s) = h(s), \text{ a.s.} \quad s \leq t.$$

In addition, by the Dominated Convergence Theorem we also have

$$\lim_{n\to\infty} \int_0^t (h_n(s) - h(s))^2 ds = 0, \text{ a.s.}, \tag{129}$$

and

$$\lim_{n\to\infty}\int_0^t h_n(s)dT_A^t(s) = \int_0^t h(s)dT_A^t(s), \text{ a.s.}, \tag{130}$$

since we have the bound

$$|h_n(s)| \le C(S(s) - \tilde{a}_S). \tag{131}$$

From (129) follows that

$$\lim_{n\to\infty}\int_0^t h_n(s)d\tilde{S}_A(s) = \int_0^t h(s)d\tilde{S}_A(s), \tag{132}$$

in probability. We know that (118) holds if we replace q and h by q_n and h_n, respectively, so by taking limits and using (128), (130), and (132), we conclude that (118) indeed holds.

Next we shall assume that q is an adapted, continuous process, and $h(0) = 0$. Let

$$T_n = \inf\{t \ge 0 : |q(t)| \ge n\},$$

$$q_n(t) = q(t \wedge T_n),$$

and

$$h_n(t) = \int_0^t q_n(s)dT_A^t(s).$$

From the previous step we know that

$$\int_0^t h_n(s)d\tilde{S}_A(s) = \int_0^t h_n(s)dT_A^t(s) - \lambda_A\int_0^t q(s)ds, \tag{133}$$

thus substituting t by $t \wedge T_n$ and using the fact that for $s \le T_n$ we have $q_n(s) = q(s)$ and $h_n(s) = h(s)$, we get

$$\int_0^{t\wedge T_n} h(s)d\tilde{S}_A(s) = \int_0^{t\wedge T_n} h(s)dT_A^{t\wedge T_n}(s) - \lambda_A\int_0^{t\wedge T_n} q(s)ds. \tag{134}$$

Taking limits as $n \to \infty$ gives the desired result. Extending the result to cases when $h(0)$ is an arbitrary constant is obvious. □

Next, we have the following.

Proposition 6. *For each $t > 0$ and $x \in [\underline{m}_A(t), \tilde{S}_A(t)]$,*

$$\tau_{A,x}^t = \tau_{p_A, p_A(\tau_{A,x}^t)}^t, \tag{135}$$

and for each $t > 0$ and $x \in [\underline{m}_B(t), \tilde{S}_B(t)]$,

$$\tau_{B,x}^t = \tau_{p_B, p_B(\tau_{B,x}^t)}^t. \tag{136}$$

Proof of Proposition 6. We prove (135), the proof of (136) follows similarly. Let $t > 0$ and $x \in [\underline{m}_A(t), \tilde{S}_A(t)]$. Clearly

$$\tau^t_{A,x} \leq \tau^t_{p_A, p_A(\tau^t_{A,x})}.$$

Hence, in order to complete the proof it suffices to show that $p_A(u) < p_A(\tau^t_{A,x})$ for $u \in (\tau^t_{A,x}, t]$.

First note by the continuity of $\tilde{S}_A$ that $\tilde{S}_A(u) > x$ for $u \in (\tau^t_{A,x}, t]$. This implies that if $u \in (\tau^t_{A,x}, t]$, then $\underline{m}_A(u) = \underline{m}_A(t)$ and $\tau^u_{A,y} = \tau^t_{A,y}$ for $y \in [\underline{m}_A(t), x]$. Hence, for each $u \in (\tau^t_{A,x}, t]$,

$$\begin{aligned}
\ln p_A(u) &= \ln \bar{F}^{-1}_{A,0}(\underline{m}_A(u)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(u)}^{\tilde{S}_A(u)} \ln\left(\frac{p_A(\tau^u_{A,v})}{p_B(\tau^u_{A,v})}\right) dv \\
&= \ln \bar{F}^{-1}_{A,0}(\underline{m}_A(t)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t)}^{x} \ln\left(\frac{p_A(\tau^t_{A,v})}{p_B(\tau^t_{A,v})}\right) dv \\
&\quad + \mathbb{E}(\ln X^A) \int_{x}^{\tilde{S}_A(u)} \ln\left(\frac{p_A(\tau^u_{A,v})}{p_B(\tau^u_{A,v})}\right) dv \\
&= \ln p(\tau^t_{A,x}) + \mathbb{E}(\ln X^A) \int_{x}^{\tilde{S}_A(u)} \ln\left(\frac{p_A(\tau^u_{A,v})}{p_B(\tau^u_{A,v})}\right) dv \\
&< \ln p(\tau^t_{A,x}).
\end{aligned}$$

□

We now are in a position to prove Theorem 2.

Proof of Theorem 2. Recall by Proposition 2 that if $p_B(0) < p_A(0)$, then P-a.s. there exists a unique solution (p_B, p_A) to (44)-(45). It remains to show that $(\tilde{\mu}_B, \tilde{\mu}_A)$ given by (49)-(50) is a solution to the SDE (31)-(32). We prove that $\tilde{\mu}_A$ satisfies (32), the proof that $\tilde{\mu}_B$ satisfies (31) follows similarly.

First note that if $\tilde{\mu}_A$ is given by (50), then for each $\phi \in C_b(\mathbb{R})$,

$$\langle \tilde{\mu}_A(t), \phi \rangle \tag{137}$$

$$= \int_{p_A(t)}^{\overline{p}_A(t)} \phi(u) \left[\mathbb{E}(\ln X^A) u \ln\left(\frac{p_B(\tau^t_{pA,u})}{u}\right)\right]^{-1} du + \int_{\overline{p}_A(t)}^{\infty} \phi(u) d\tilde{\mu}_A(0,u) \tag{138}$$

for $t \geq 0$. Substituting (137) into the SDE (32), we then obtain that

$$\int_{p_A(t)}^{\bar{p}_A(t)} \phi(u) \left[\mathbb{E}(\ln X^A) u \ln\left(\frac{p_B(\tau^t_{pA,u})}{u}\right)\right]^{-1} du \tag{139}$$

$$= \int_0^{\bar{p}_A(t)} \phi(u) d\tilde{\mu}_A(0,u) + \lambda_A \mathbb{E}[\ln(X^A)] \int_0^t \left[\phi'(p_A(s)) p_A(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right)\right] ds \tag{140}$$

$$+ \int_0^t \phi(p_A(s)) d\tilde{S}_A(s)$$

for $t \geq 0$. In order to complete the proof, it suffices to show that the above equality holds.

First note for a fixed $t \geq 0$, it follows by the continuity of $\tilde{S}_A$ that $\tilde{S}_A(\tau^t_{A,x}) = x$ for $x \in [\underline{m}_A(t), \tilde{S}_A(t)]$. It then follows by a change-of-variables $v = \tau^t_{A,u}$ in (45) that

$$\ln p_A(t) = \ln G_A^{-1}(\underline{m}_A(t)) + \mathbb{E}(\ln X^A) \int_0^t \ln\left(\frac{p_A(v)}{p_B(v)}\right) dT^t_A(v). \tag{141}$$

Next, for a fixed $t > 0$ let

$$l(s) = \ln G_A^{-1}(\underline{m}_A(s)) + \mathbb{E}(\ln X^A) \int_0^s \ln\left(\frac{p_A(v)}{p_B(v)}\right) dT^t_A(v) \tag{142}$$

for $0 \leq s \leq t$ and note that $l(t) = \ln p_A(t)$ and $l(0) = \ln p_A(0)$. Moreover,

$$dl(s) = d \ln G_A^{-1}(\underline{m}_A(s)) + \mathbb{E}(\ln X^A) \ln\left(\frac{p_A(s)}{p_B(s)}\right) dT^t_A(s). \tag{143}$$

Hence, for $\phi \in C_b(\mathbb{R})$ that are differentiable

$$\phi(p_A(t)) = \phi(p_A(0)) + \int_0^t \phi'(p_A(s)) p_A(s) d \ln G_A^{-1}(\underline{m}_A(s)) \tag{144}$$

$$+\mathbb{E}(\ln X^A) \int_0^t \phi'(p_A(s)) p_A(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right) dT^t_A(s). \tag{145}$$

Now note since $G_A^{-1}(\underline{m}_A(s)) = \bar{p}_A(s)$ for $s \geq 0$, it follows after some algebra that

$$\int_0^t \phi'(p_A(s)) p_A(s) d \ln G_A^{-1}(\underline{m}_A(s)) = \phi(\bar{p}_A(t)) - \phi(p_A(0)) \tag{146}$$

for $t \geq 0$. Substituting the above into (144), we then obtain that

$$\phi(p_A(t)) - \phi(\bar{p}_A(t)) = \mathbb{E}(\ln X^A) \int_0^t \phi'(p_A(s)) p_A(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right) dT^t_A(s). \tag{147}$$

Now applying Lemma 2 to the above, it follows after some algebra that

$$\lambda_A \mathbb{E}(\ln X^A) \int_0^t \phi'(p_A(s)) p_A(s) \ln\left(\frac{p_A(s)}{p_B(s)}\right) ds + \int_0^t \phi(p_A(s)) d\tilde{S}_A(s) \tag{148}$$

$$= \int_0^t \phi(\bar{p}_A(s)) d(\tilde{S}_A(s) - T^t_A(s)) + \int_0^t \phi(p_A(s)) dT^t_A(s). \tag{149}$$

Now substituting the above into (139), it follows that in order to complete the proof it suffices to show that for each $\phi \in C_b(\mathbb{R})$,

$$\int_{p_A(t)}^{\bar{p}_A(t)} \phi(u) \left[\mathbb{E}(\ln X^A) u \ln \left(\frac{p_B(\tau^t_{pA,u})}{u} \right) \right]^{-1} du - \int_0^t \phi(p_A(s)) dT^t_A(s) \tag{150}$$

$$= \int_0^{\bar{p}_A(t)} \phi(u) d\tilde{\mu}_A(0,u) + \int_0^t \phi(\bar{p}_A(s)) d(\tilde{S}_A(s) - T^t_A(s)) \tag{151}$$

for $t \geq 0$. We proceed by showing both sides of the above are equal to 0.

Consider first the lefthand side of (150). Similar to the reasoning of (141), it follows that

$$\int_0^t \phi(p_A(s)) dT^t_A(s) = \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} \phi(p_A(\tau^t_{A,v})) dv \tag{152}$$

for $t \geq 0$. To show that the lefthand side of (150) vanishes, it then suffices to show that

$$\int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} \phi(p_A(\tau^t_{A,v})) dv = \int_{p_A(t)}^{\bar{p}_A(t)} \phi(u) \left[\mathbb{E}(\ln X^A) u \ln \left(\frac{p_B(\tau^t_{pA,u})}{u} \right) \right]^{-1} du. \tag{153}$$

Now fix $t \geq 0$ and note by (45) that for each $x \in [\underline{m}_A(t), \tilde{S}_A(t)]$,

$$\ln p_A(\tau^t_{A,x}) = \ln G_A^{-1}(\underline{m}_A(t)) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t)}^x \ln \left(\frac{p_A(\tau^t_{A,u})}{p_B(\tau^t_{A,u})} \right) du. \tag{154}$$

Next, (154) together with Proposition 5 implies that $p_A(\tau^t_{A,x})$ is continuous and strictly decreasing for $x \in [\underline{m}_A(t), \tilde{S}_A(t)]$. Differentiating (154) with respect to x then yields

$$\frac{dp_A(\tau^t_{A,x})}{dx} = \mathbb{E}(\ln X^A) p_A(\tau^t_{A,x}) \ln \left(\frac{p_A(\tau^t_{A,x})}{p_B(\tau^t_{A,x})} \right). \tag{155}$$

Now let $p_A^{-1}(\tau^t_{A,x}) : [p_A(t), \bar{p}_A(t)] \mapsto [\underline{m}_A(t), \tilde{S}_A(t)]$ denote the inverse function of $p_A(\tau^t_{A,x})$. By the inverse function theorem, it follows that

$$\frac{dp_A^{-1}(\tau^t_{A,x})(z)}{dz} = \frac{1}{\mathbb{E}(\ln X^A)} \frac{1}{p_A(\tau^t_{A,p_A^{-1}(\tau^t_{A,x})(z)})} \left[\ln \left(\frac{p_A(\tau^t_{A,p_A^{-1}(\tau^t_{A,x})(z)})}{p_B(\tau^t_{A,p_A^{-1}(\tau^t_{A,x})(z)})} \right) \right]^{-1} \tag{156}$$

for each $z \in [p_A(t), \bar{p}_A(t)]$. Moreover, recall by Proposition 6 that for each $z \in [p_A(t), \bar{p}_A(t)]$,

$$\tau^t_{A,p_A^{-1}(\tau^t_{A,x})(z)} = \tau^t_{pA,z}.$$

It therefore follows that

$$\frac{dp_A^{-1}(\tau^t_{A,x})(z)}{dz} = \frac{1}{\mathbb{E}(\ln X^A)} \cdot \frac{1}{z} \cdot \left[\ln \left(\frac{z}{p_B(\tau^t_{pA,z})} \right) \right]^{-1}. \tag{157}$$

Hence, using the change-of-variable $z = p^A(\tau^t_{A,v})$ in the lefthand side of (153) we obtain the desired result.

Finally, we show that for each $t \geq 0$,

$$-\int_0^{\bar{p}_A(t)} \phi(u)d\tilde{\mu}_A(0,u) \;\; = \;\; \int_0^t \phi(\bar{p}_A(s))d(\tilde{S}_A(s) - T^t_A(s)), \tag{158}$$

which implies that the righthand side of (150) is zero and completes the proof. First note for fixed $t \geq 0$ that by definition (115) we have the equality $T^t_A(s) = \underline{m}_A(t)$ for $0 \leq s \leq s_{max}$, where $s_{max} = \sup\{s \geq 0 : \tilde{S}_A(s) = \underline{m}_A(t)\}$. Moreover, note that $\bar{p}_A(s) = \bar{p}_A(t)$ for $s \geq s_{max}$. We then for each $\phi \in C_b(\mathbb{R})$ obtain the identity

$$\int_0^t \phi(\bar{p}_A(s))dT^t_A(s) \;\; = \;\; \phi(\bar{p}_A(t))(\tilde{S}_A(t) - \underline{m}_A(t)). \tag{159}$$

Next, since $\bar{p}_A$ is non-decreasing it follows integrating-by-parts that for each $\phi \in C_b(\mathbb{R})$,

$$\int_0^t \phi(\bar{p}_A(s))d\tilde{S}_A(s) \;\; = \;\; \tilde{S}_A(t)\phi(\bar{p}_A(t)) - \tilde{S}_A(0)\phi(\bar{p}_A(0)) - \int_0^t \tilde{S}_A(s)d\phi(\bar{p}_A(s)) \tag{160}$$

for $t \geq 0$. Moreover, since $\bar{p}_A$ only increases at times when $\tilde{S}_A$ achieves a new minimum and $\tilde{S}_A(s) = G_A(s)$ at each such point in time s, it follows using the change-of-variables $u = \bar{p}_A(s)$ that

$$\int_0^t \tilde{S}_A(s)d\phi(\bar{p}_A(s)) \;\; = \;\; \int_{p_A(0)}^{\bar{p}_A(t)} G_A(u)d\phi(u). \tag{161}$$

Finally, note that integrating-by-parts and using the identity $G_A(\bar{p}_A(t)) = \underline{m}_A(t)$ for $t \geq 0$, it follows that

$$-\int_{\bar{p}_A(0)}^{\bar{p}_A(t)} \phi(u)d\tilde{\mu}_A(0,u) \;\; = \;\; \phi(\bar{p}_A(t))\underline{m}_A(t) - \phi(\bar{p}_A(0)\tilde{S}_A(0) - \int_{\bar{p}_A(0}^{\bar{p}_A(t)} G_A(u)d\phi(u). \tag{162}$$

Combining (159)-(162) now yields the desired identity (158). $\square$

A.3 Proof of Theorem 3

In order to prove Theorem 3, it is sufficient to prove the following.

Proposition 7. *If* (p_B, p_A) *is a solution to* (80)-(81) *with* $p_B(0) < p_A(0)$*, then* $p_B(t) < p_A(t)$ *for* $t \geq 0$.

Proof. Set $x(t) = \ln p_A(t) - \ln p_B(t)$ for $t \geq 0$. Then, subtracting (80) from (81) it follows that

$$x(t) = d(t) + \mathbb{E}(\ln X^A) \int_{\underline{m}_A(t)}^{\tilde{S}_A(t)} x(\tau^t_{A,u}) du + \lambda_B \mathbb{E}(\ln X^B) \int_0^t \frac{x(s)}{\alpha_B(s)} ds, \tag{163}$$

where $d(t) = \ln G_A^{-1}(\underline{m}_A(t)) - \ln p_B(0)$. Suppose now that the result does not hold and denote by $t^* = \inf\{t \geq 0 : p_B(t) = p_A(t)\}$ the first time at which the best prices on both sides of the book are equal. By the continuity of $\tilde{S}_A$ and the fact that $x(t) > 0$ for $t \in [0, t^*)$, there then exists a $\bar{t} \in [0, t^*)$ such that for $t \in [\bar{t}, t^*]$,

$$x(t) \leq x(\bar{t}) \quad \text{and} \quad -2\mathbb{E}(\ln X^A) \left(\tilde{S}_A(t^*) - \inf_{\bar{t} \leq t \leq t^*} \tilde{S}_A(t) \right) \leq 1. \tag{164}$$

Now note that d is non-decreasing and so it follows by (163) that

$$\begin{aligned} x(\bar{t}) = x(\bar{t}) - x(t^*) &= d(t) - d(t^*) && (165)\\ &\quad + \mathbb{E}(\ln X^A) \left(\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u}) du - \int_{\underline{m}(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u}) du \right) && (166)\\ &\quad - \lambda_B \mathbb{E}(\ln X^B) \int_{\bar{t}}^{t^*} \frac{x(s)}{\alpha_B(s)} ds && (167)\\ &\leq \mathbb{E}(\ln X^A) \left(\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u}) du - \int_{\underline{m}(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u}) du \right). && (168) \end{aligned}$$

Then proceeding in the same manner as in the proof of Proposition 5, we arrive at

$$\mathbb{E}(\ln X^A) \left(\int_{\underline{m}_A(\bar{t})}^{\tilde{S}_A(\bar{t})} x(\tau^{\bar{t}}_{A,u}) du - \int_{\underline{m}(t^*)}^{\tilde{S}_A(t^*)} x(\tau^{t^*}_{A,u}) du \right) \leq \frac{x(\bar{t})}{2}. \tag{169}$$

By (164), this implies that $x(\bar{t}) < x(\bar{t})/2$, which is a contradiction since $x(\bar{t}) > 0$. □

References

Per Bak, Maya Paczuski, and Martin Shubik. Price variations in a stock market with many agents. *Physica A: Statistical Mechanics and its Applications*, 246(3-4):430–453, 1997.

Damien Challet and Robin Stinchcombe. Analyzing and modeling 1+ 1d markets. *Physica A: Statistical Mechanics and its Applications*, 300(1-2):285–299, 2001.

Samuel N Cohen and Lukasz Szpruch. A limit order book model for latency arbitrage. *Mathematics and Financial Economics*, 6(3):211–227, 2012.

Rama Cont and Jean-Philipe Bouchaud. Herd behavior and aggregate fluctuations in financial markets. *Macroeconomic dynamics*, 4(2):170–196, 2000.

Rama Cont and Adrien De Larrard. Price dynamics in a Markovian limit order market. *SIAM Journal on Financial Mathematics*, 4(1):1–25, 2013.

Rama Cont, Sasha Stoikov, and Rishi Talreja. A stochastic model for order book dynamics. *Operations research*, 58(3):549–563, 2010.

Rama Cont, Arseniy Kukanov, and Sasha Stoikov. The price impact of order book events. *Journal of financial econometrics*, 12(1):47–88, 2014.

Marcus G Daniels, J Doyne Farmer, László Gillemot, Giulia Iori, and Eric Smith. Quantitative model of price diffusion and market friction based on trading as a mechanistic random process. *Physical review letters*, 90(10):108102, 2003.

David Eliezer and Ian I Kogan. Scaling laws for the market microstructure of the interdealer broker markets. *arXiv preprint cond-mat/9808240*, 1998.

Stewart N Ethier and Thomas G Kurtz. *Markov processes: characterization and convergence.* John Wiley & Sons, 2009.

J Doyne Farmer, Paolo Patelli, and Ilija I Zovko. The predictive power of zero intelligence in financial markets. *Proceedings of the national academy of sciences of the united states of america*, 102(6):2254–2259, 2005.

Ronald Goettler, Christine Parlour, Uday Rajan, et al. Microstructure effects and asset pricing. *Preprint, available at http://en. scientificcommons. org/33345856*, 2006.

Andrzej Granas, James Dugundji, et al. *Fixed point theory*, volume 14. Springer, 2003.

Gao-Feng Gu and Wei-Xing Zhou. Emergence of long memory in stock volatility from a modified mike-farmer model. *EPL (Europhysics Letters)*, 86(4):48002, 2009.

J Michael Harrison. Stochastic networks and activity analysis. *American Mathematical Society Translations*, pages 53–76, 2002.

Burton Hollifield, Robert A Miller, and Patrik Sandås. Empirical analysis of limit order markets. *The Review of Economic Studies*, 71(4):1027–1063, 2004.

Peter Lakner, Josh Reed, and Sasha Stoikov. High frequency asymptotics for the limit order book. *Market Microstructure and Liquidity*, 2(01):1650004, 2016.

Sergei Maslov. Simple model of a limit order-driven market. *Physica A: Statistical Mechanics and its Applications*, 278(3-4):571–578, 2000.

Szabolcs Mike and J Doyne Farmer. An empirical behavioral model of liquidity and volatility. *Journal of Economic Dynamics and Control*, 32(1):200–234, 2008.

Christine A Parlour. Price dynamics in limit order markets. *The Review of Financial Studies*, 11(4):789–816, 1998.

Ioanid Roşu. A dynamic model of the limit order book. *The Review of Financial Studies*, 22(11):4601–4641, 2009.

František Slanina. Mean-field approximation for a limit order driven market model. *Physical review e*, 64(5):056136, 2001.

Eric Smith, J Doyne Farmer, L szl Gillemot, Supriya Krishnamurthy, et al. Statistical theory of the continuous double auction. *Quantitative finance*, 3(6):481–514, 2003.

Lei-Han Tang and Guang-Shan Tian. Reaction–diffusion–branching models of stock price fluctuations. *Physica A: Statistical Mechanics and its Applications*, 264(3-4):543–550, 1999.

Ioane Muni Toke. Market making in an order book model and its impact on the spread. In *Econophysics of order-driven markets*, pages 49–64. Springer, 2011.

Linqiao Zhao. *A model of limit-order book dynamics and a consistent estimation procedure.* PhD thesis, Carnegie Mellon University, 2010.