The Diminishing Value of Bundling Under Inventory Scarcity

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We study the impact of inventory constraints on bundling in a dynamic pricing setting, challenging the classical view that bundling consistently enhances revenue. Traditional bundling theory, which usually assumes abundant inventory and static pricing settings, typically asserts that bundling generates higher revenue than selling products individually. However, we show that limited inventory, in fact, distorts the revenue extraction capability of bundling in favor of selling the products or services separately. We study the optimal dynamic mixed bundling strategy in a large market regime where the market size grows relative to limited inventory. Leveraging this framework, we derive optimal dynamic pricing policies and value functions for commonly used bundling strategies, including mixed bundling, pure bundling, bundle-size pricing, and component pricing. Our analysis reveals that as inventory becomes more constrained relative to market size, the most general dynamic mixed bundling strategy converges to a dynamic component pricing strategy, outperforming both dynamic bundle-size pricing and dynamic pure bundling. Moreover, the performance gap between these strategies increases with the number of items, a factor typically viewed as favoring bundling when inventory is abundant. Notably, our numerical experiments suggest that these insights also extend to the fluid-regime policies, revealing that as the market size increases relative to inventory, the dynamic mixed bundling strategy derived from the fluid regime mimics a dynamic component pricing strategy.

1. Introduction

Bundling, the practice of offering multiple products together as a single package, is employed across industries by firms seeking to extract more value from heterogeneous consumer preferences. Whether in the form of subscription services, promotional packages, or mixed bundling offers, the practice allows firms to price discriminate, reduce transaction complexity, and enhance perceived consumer value. The conceptual foundation for bundling is rooted in the seminal works of Stigler (1963) and Adams and Yellen (1976), who demonstrated its effectiveness as a price discrimination tool and, hence, the ability to extract larger consumer surplus rather than selling the items or services separately. Subsequent research has explored bundling in various contexts, highlighting the aggregation benefits of bundling. However, most of these studies focus on static environments with non-perishable and unlimited inventory or capacity, which may be more representative of digital goods than physical ones. There is a consensus in the literature about the benefits of bundling in terms of extracting consumer surplus, best summarized by the following quote from Tjan (2010):

"*There is a simple and pretty consistent rule of thumb on the question [of bundling]. Here it is: unbundling or a la carte pricing benefits the buyer and packaged or bundled deals give the advantage to the seller.*"

There are mainly three types of bundling strategies in addition to selling the products separately that a firm may use:

(i) *Component Pricing (Additive Bundle Pricing)* (CP): The firm sets individual prices for each product separately. If a customer purchases multiple products (a bundle), they pay the sum of these individual prices—hence the term "additive bundling."

(ii) *Pure Bundling* (PB): The firm offers only a comprehensive bundle that includes all its products or services. Customers either choose to buy the entire bundle or nothing since purchasing items separately is not an option.

(iii) *Bundle Size Pricing* (BSP): The firm sets the bundles' prices based solely on the number of products they contain, regardless of which specific products or services are included.

(iv) *Mixed Bundling* (MB): The firm sets prices for each product or service, along with every possible combination of these as distinct bundles.

Mixed bundling represents the most general strategy, subsuming the other strategies as specific instances. Meanwhile, bundle size pricing subsumes pure bundling and certain instances of component pricing, such as a uniform price for all products. We illustrate the relationship among the four strategies in Figure 1. Considering this hierarchy, the literature primarily studies the value of bundles

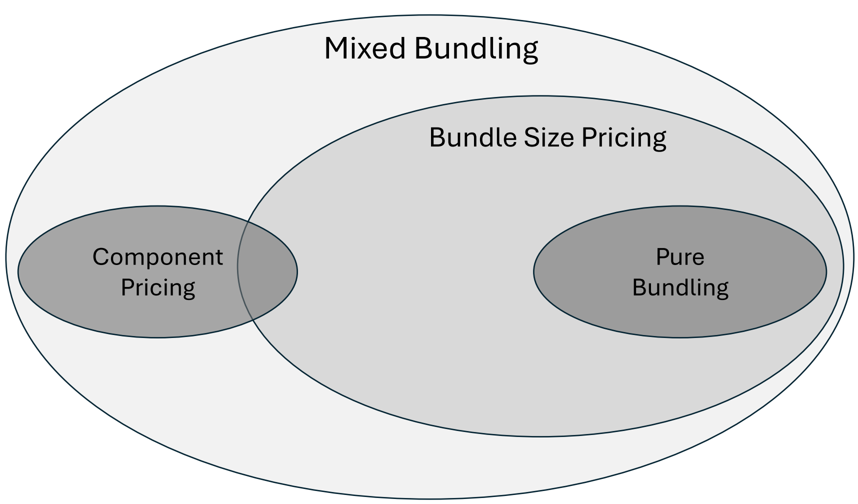


Figure 1 Relationship between the different bundling strategies

in static environments with unlimited inventory. In other words, the inventory significantly exceeds the market size, which refers to the potential demand. This situation typically involves a finite number of items, denoted as N (usually N=2), along with specific assumptions regarding the distribution of valuations, often for items that have zero marginal costs. Within this setting, the prevailing consensus on the surplus extraction capability of different bundling schemes can be summarized by the following ordering:

$$\text{MB} \succsim \text{BSP} \succ \text{PB} \succ \text{CP} \qquad [\text{Inv.} >> \text{Market Size} \mid N < \infty]$$

where Mixed Bundling (MB) is generally the most effective at extracting consumer surplus, whereas Component Pricing (CP) is the least. While mixed bundling is the most general strategy, it is also the most complicated. In particular, letting N represent the number of items, mixed bundling requires setting up to $2^N - 1$ prices, compared to N prices for BSP and CP, and a single price for PB. It is worth noting that despite BSP requiring only N prices, Chu et al. (2011) demonstrated that BSP closely approximates the revenue extraction of MB. Nevertheless, finding the optimal BSP and MB is generally intractable. For this reason, much of the bundling literature focuses on comparing CP versus PB as simpler strategies to characterize.

Another relevant line of research is the large-scale bundling literature, which analyzes the asymptotic performance of pricing policies as the number of items becomes large ($N \to \infty$) (Bakos and Brynjolfsson 1999, Abdallah et al. 2021). This literature provides valuable insights into the relative effectiveness of different pricing schemes in extracting surplus under general valuation distributions. In particular, it highlights the effectiveness of simple bundling strategies, demonstrating that even these simple pricing strategy can very well approximate the performance of the more general MB streategies. In particular, under assumptions of zero marginal costs and unlimited inventory for large N, the literature identifies the following ordering in terms of surplus extraction ability:

$$\text{MB} \sim \text{BSP} \sim \text{PB} \succ\succ \text{CP} \qquad [\text{Inv.} >> \text{Market Size} \mid N \to \infty]$$

In this asymptotic regime, simpler strategies, such as Bundle-Size Pricing (BSP) and Pure Bundling (PB), significantly outperform Component Pricing (CP) and achieve performance nearly comparable to that of mixed bundling (MB). These robust theoretical findings may help explain the success of digital platforms such as Netflix, Spotify, and Amazon Prime, which effectively employ pure bundling strategies by aggregating various goods and services into single comprehensive packages. These platforms notably benefit from near-zero marginal costs and virtually unlimited inventory, enabling them to leverage bundling as an effective mechanism for revenue optimization.

Inspired by these successes in digital products and services, bundling strategies have been increasingly adopted in industries that deal with physical goods, such as retail, hospitality, and transportation. Companies like Dollar Shave Club, Stitch Fix, and Sephora Play! utilize bundling as a key aspect of their business models, often incorporating it into subscription services or promotional campaigns. However, bundling in industries involving physical goods, where finite inventory and operational constraints prevail, introduces complexities that traditional models typically overlook. For example, retailers, airlines, and hotels frequently operate under conditions of scarce inventory, uncertain demand, and time-sensitive selling opportunities. The existing literature on bundling often abstracts away these operational realities, focusing instead on stylized scenarios without inventory or time constraints. Conversely, classical literature on multi-product dynamic pricing explicitly models inventory and time constraints but typically overlooks bundling effects by treating each bundle as an independent product with distinct demand (Gallego and Van Ryzin 1997).

In this paper, we bridge this gap by examining how operational factors, such as limited inventory levels and finite time horizons, influence the dynamics of bundling and its effectiveness in enhancing surplus extraction and price discrimination benefits. Furthermore, we address the effective design and implementation of dynamic bundling strategies when inventory is scarce relative to the potential market size. Specifically, we study the dynamic mixed bundling and pricing problem under the large market regime recently proposed by Abdallah and Reed (2025b,a), for the single-item setting where the market size (expected arrivals over the time horizon) is scaled while the inventory is held fixed. The large market regime contrasts with the fluid regime, in which both market size and inventory are both scaled proportionally (Gallego and Van Ryzin 1997). The large market regime provides clearer analytical insights compared to the fluid regime, where both market size and inventory scale together, resulting in a deterministic limiting problem that is complicated by non-convex inventory constraints, which are challenging to analyze.

Leveraging this large market regime framework, we characterize the convergence of optimal policies for dynamic mixed bundling, bundle-size pricing, pure bundling, and component pricing, along with their respective expected revenues under general valuation distributions and any number of items $N < \infty$. Importantly, our analysis reveals that in settings where inventory is scarce relative to the market size, the ordering of policies by their effectiveness in extracting consumer surplus reverses,

$$\text{MB} \sim \text{CP} \succ \text{BSP} \succ\succ \text{PB} \qquad [\text{Inv.} << \text{Market Size} \mid N < \infty]$$

where component pricing (CP) now significantly outperforms bundle size pricing (BSP) and pure bundling, achieving performance comparable to mixed bundling (MB).

This reversal highlights that the advantages of bundling diminish significantly under conditions of scarce inventory, causing optimal dynamic mixed bundling strategies to closely resemble dynamic component pricing (additive bundling). Furthermore, we demonstrate that the relative performance of pure bundling (PB) and bundle size pricing (BSP) decreases as the number of items N increases, reversing the insights from existing literature on large-scale bundling with abundant inventory, which suggest that increased number of items generally enhances the effectiveness of PB and BSP. Our contributions are summarized next.

1.1. Contribution

- We extend the large market regime introduced by Abdallah and Reed (2025b,a) from a single-item setting to a multi-item setting where firms must dynamically optimize their prices for individual products and bundles with limited inventory. Unlike prior work on bundling that ignores inventory effects, this paper develops a multi-dimensional framework that captures the complex interactions between dynamic bundling decisions and inventory constraints.
- We characterize the value functions and optimal dynamic pricing policies for the four pricing strategies, MB, BSP, PB, and CP, in the large market regime. Unlike prior work that often relies on specific distributional assumptions or asymptotic approximations, our results hold for any number of items $N < \infty$ and fairly general distributional assumptions.
- Our characterization of the value functions and optimal pricing policies for the four strategies allows for a systematic comparison of their performance under general valuation distributions and finite N. First, we show that as the market size increases relative to inventory, the revenue difference between mixed bundling (MB) and component pricing (CP) vanishes. That is, firms cannot leverage bundling strategies to increase revenue beyond what optimal component pricing achieves. This result fundamentally revises prior assumptions about the superiority of bundling, showing that in an inventory-scarce environment, the advantage of bundling diminishes. Second, we find that in the large market regime, pure bundling (PB) and bundle size pricing (BSP) become less effective as N increases, contrary to classical bundling theory, which suggests that bundling improves with more items.
- We demonstrate numerically that our theoretical insights hold beyond the large market regime assumptions, particularly for moderate inventory-to-market-size ratios (IMR). We observe that the revenue advantage of pure bundling (PB) over component pricing (CP) holds only for a large inventory-to-market-size ratio, but as the market size increases, CP catches up and eventually surpasses PB. The impact of the number of items N is also reversed as the market size grows, where larger N favors PB over CP for small IMR. However, as the market size increases, particularly beyond IMR thresholds between 22% and 42%, CP begins to dominate PB, with the gap widening for larger N.

- Our numerical results demonstrate that our insights regarding the diminishing value of bundling also apply to the policies of the fluid regime, where inventory and market size grow proportionally. In particular, we demonstrate that the optimal dynamic mixed bundling (MB) solution derived from the limiting fluid problem closely resembles dynamic component pricing (CP).

2. Related Literature

Our results are closely related to the literature on monopolistic bundling and network revenue management. Certain forms of opaque pricing strategies are also closely related to our work (see, for example, Briest and Röglin (2010) and Elmachtoub and Hamilton (2021)). We focus now on recent advances in these fields.

Bundling. There is extant literature on the static monopolistic bundling problem with unlimited inventory using a stylized two-item model that dates back to the work of Stigler (1963) and Adams and Yellen (1976). This literature emphasizes the benefits of bundling as a price discrimination tool. Interested readers are referred to the review by Venkatesh and Mahajan (2009).

An active research area (mainly in computer science) focuses on providing conditions under which simple selling mechanisms are optimal or nearly optimal in a setting with no inventory constraints and an arbitrary number of items. Hart and Nisan (2017) show that component pricing and pure bundling guarantee a fraction of the optimal mechanism revenue in this setting. This fraction, however, shrinks to zero as the number of items grows large. Babaioff et al. (2014) show that the maximum revenue of component pricing and pure bundling is a constant-factor approximation to the optimal revenue. Fang and Norman (2006) also compares component pricing and pure bundling and provides distribution-dependent conditions for when one strategy is better. Ma and Simchi-Levi (2015) study a new bundling strategy, called bundling with disposal, under a general cost structure and show that it can well approximate the optimal revenue mechanism. They also show that this strategy is asymptotically optimal as the number of items grows large.

Bakos and Brynjolfsson (1999) elegantly highlight the power of pure bundling for a large number of items with zero marginal costs and unlimited inventory. They show that as the number of items increases, a simple pure bundling strategy essentially extracts all of the consumer surplus. This is mainly due to the heterogeneity-reduction property of bundles, where for a large number of items, the valuation of a bundle becomes concentrated around its mean. Abdallah (2019) shows the limitations of pure bundling in the presence of positive marginal cost by providing distribution-free bounds on its asymptotic profit in this setting. Abdallah et al. (2021) consider a simple bundling policy called bundle size pricing. They show that bundle size pricing is more effective at extracting consumer surplus than pure bundling in the presence of marginal costs. They also provide a closed-form solution for the asymptotically optimal bundle size pricing policy.

An emerging research area (mainly in operations management) focuses on the optimization problem arising from the static pricing problem for a given bundling strategy without inventory constraints. Honhon and Pan (2017) examine bundling strategies in the context of vertically differentiated products, demonstrating that offering bundles can increase profits even in the absence of consumption complementarity. Their analysis explicitly considers positive variable costs, a departure from the typical assumption of negligible costs in bundling literature. Chen et al. (2017) study distribution-free pricing problems and show that they can be used efficiently for pure bundling for any number of products. Wu et al. (2008) study the bundle size pricing problem with deterministic valuations and propose a Lagrangian-based heuristic to solve it. Using clever reformulations Wu et al. (2018) show that bundle size pricing problems with deterministic valuations that satisfy the single-crossing property can be solved exactly using linear or dynamic programming approaches. Li et al. (2022) have shown that bundle size pricing problems with random valuations can be approximated by a convex optimization problem using a family of semi-parametric choice models. Recently, Sun et al. (2025) study the design and pricing of a single optimal bundle alongside individual sales of remaining products, demonstrating that this strategy can yield higher profitability and social welfare relative to pure bundling or separate component pricing.

There is limited research on the effects of inventory constraints in bundling problems, even in a static setting. Ernst and Kouvelis (1999) show that in a newsvendor problem with two items ignoring substitution effects in the presence of bundles and stockouts is suboptimal. Cao et al. (2015) study the bundling problem of two items in a newsvendor setting when the supply of the attractive product is limited. They show that, in this case, bundling extracts a larger surplus. Song and Xue (2021) study a multi-period joint replenishment and pricing problem for product assemblies where a product can be viewed as a bundle of sub-assemblies. They provide exact and heuristic methods for different assembly systems.

Multi-product Dynamic Pricing. Gallego and Van Ryzin (1997) is the canonical paper in the multi-product dynamic pricing literature. They study this problem in a fluid regime that scales both the inventory levels and the arrival rate. They propose asymptotically optimal heuristics based on the resultant limiting deterministic problem. Jasin (2014) and Chen et al. (2015) propose improved heuristics for the same problem requiring minimal real-time price adjustments. There have been several extensions to this canonical model to include dynamic pricing joinlty with other considerations such as learning (see Araman and Caldentey (2009), Besbes and Zeevi (2012), Keskin and Zeevi (2014), den Boer and Zwart (2015)), consumer behavior (Chen and Farias (2018), Liu and Cooper (2015), Najafi et al. (2024)), and competition (Martínez-de Albéniz and Talluri (2011), Adida and Perakis (2010), Gallego and Hu (2014)). Closely related to the multi-product dynamic pricing literature is the multi-product capacity control literature (see, for example, Talluri and Van Ryzin (1998), van Ryzin and Vulcano (2008), Topaloglu (2009)), Kunnumkal and Talluri (2015)). Interested readers are also referred to the books by Talluri and Van Ryzin (2006) and Gallego et al. (2018).

The paper most closely related to our work from a modeling perspective is Bulut et al. (2009). They numerically solve a single-period static bundling problem using heuristics with two items and limited inventory. They also extend their numerical results to a multi-period dynamic bundling problem, noting that the performance of bundling is highly sensitive to model parameters in terms of demand and inventory levels. Liao et al. (2017) study the capacity control version of two-item dynamic bundling in a multi-period discrete time setting. The components' and bundle prices are fixed, but the firm must decide which bids to accept or reject. They provide a comparative statics analysis and propose heuristics for cases involving more than two items.

Our model follows the canonical framework of Gallego and Van Ryzin (1997), where bundles are composed of multiple items assembled as products. We note, however, that Gallego and Van Ryzin (1997) assume that the demand for each product/bundle is exogenous and satisfies some concavity property and that demand for each product/bundle is independent. In our case, the demand for products/bundles is determined endogenously, as it depends on item valuations and set prices.

3. The Model

We consider a firm selling $N \geq 1$ different item types over a finite time horizon of length $t > 0$. The initial inventory of item n is $Q_{0,n}$ units for $n = 1, ..., N$. We set $Q_0 = (Q_{0,1}, Q_{0,2}, \ldots, Q_{0,N})$ to be the initial inventory position of the firm. We assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Customers arrive according to a Poisson process M with a rate $\lambda > 0$ known to the firm. Customer m ($m \geq 1$) arrives at time τ_m and has idiosyncratic nonnegative valuation $X_{m,n}$ for item n. We set $X_m = (X_{m,1}, \ldots, X_{m,N})$ to be the item valuation vector for customer m. We also assume that the $X_{m,n}$ are independent and identically distributed (i.i.d.) across customers and item types, and we denote their common distribution function by F, which we assume to be absolutely continuous.

Given N item types, the firm may offer $B = 2^N - 1$ unique bundles to price accordingly. We index these bundles by $b = 1, ..., B$. Let H be an $N \times B$ matrix where the (n, b)th entry of H equals 1 if bundle b contains item $n = 1, \ldots, N$ and 0 otherwise. We refer to H as the bundle design matrix. Consistent with the bundling literature, we assume that a customer's valuation for a bundle is additive in nature. Specifically, the bundle valuations for a customer with items valuation vector $X \in \mathbb{R}_+^N$ are given by $V = H'X \in \mathbb{R}_+^B$, where H' denotes the transpose of the bundle design matrix. We note that the non-negative valuation can be relaxed if we allow for the free disposal of items with negative values.

At each point in time $0 \leq s \leq t$, the firm sets a price $p_s \in \mathbb{R}_+^B$ for the bundles. Given the additive valuation, we make two assumptions on p_s without loss of optimality. Our first assumption is the monotonicity of p with respect to the items in a bundle. Specifically, for each bundle type $b \in \{1,2,...,B\}$, denote by $\psi_b \subseteq \{1,2,..,N\}$ the set of items contained in bundle b. We assume that if $b,b^{'} \in \{1,2,...,B\}$ are such that $\psi_b \subseteq \psi_{b'}$, then $p_b \leq p_{b'}$. In other words, this assumption states that adding an item to a bundle will not decrease its price. Our second assumption on p_s is that it is subadditive as a function on the set of items $\{1,2,...,N\}$. Technically speaking, this assumption may be written as $p_b \leq p_{b'} + p_{b''}$ for $b,b^{'},b^{''} \in \{1,2,...,B\}$ such that $\psi_b = \psi_{b'} \cup \psi_{b''}$. The primary implication of subadditivity is that purchasing a set of items as a single bundle is never more expensive than purchasing them as two or more smaller bundles. Moreover, if some item $n = 1,\ldots,N$ is stocked out, we force the price of any bundle b that contains item n to ∞. Therefore, given $q \in \mathbb{R}^N_+$, we denote the feasible pricing vectors by $\mathcal{P}(q)$.

An arriving customer will determine which bundles to purchase by maximizing the surplus of their bundle valuation vector V relative to the firm's current pricing vector. We assume that customers have a unit demand for each item type. Therefore, using the monotonicity assumption of the pricing policy, it is straightforward to show that a customer will never purchase bundles with overlapping items. Also, note that the firm's inventory position constrains the bundles available for customers to purchase, which is reflected in the pricing vector $p \in \mathcal{P}(q)$. Given the unit demand assumption, let $y \in \{0,1\}^B$ denote the bundle purchasing decision of a customer where $y_b = 1$ if the customer purchases bundle $b = 1,\ldots,B$ and is zero otherwise.

Given a pricing vector p and inventory position q, the optimal bundle purchasing decision of a customer with bundle valuation vector V is then given by a solution to the optimization problem

$$\max_{y \in \mathcal{B}(q)} (V-p)'y \qquad (1)$$
$$\text{s.t. } H \cdot y \leq e,$$

where e is the all-ones vector of length N and $\mathcal{B}(q)$ is the set of feasible binary bundle purchase vectors for bundles that are not stocked out. More specifically, $\mathcal{B}(q) \subseteq \{0,1\}^B$ is the set of all binary vectors such that $y_b = 0$ for $b = 1\ldots,B$ if there exists $n \in \psi_b$ such that $q_n = 0$. That is, $\mathcal{B}(q)$ is the feasible set of decisions that take into account the firm's inventory position and restricts the customers' choices only to bundles that do not include a stocked-out item. Note that there may exist more than one optimal solution to (1); however, from a technical perspective, this can be ignored since we assume that the valuation distribution F is absolutely continuous. We now denote by $y^\star(X,q,p)$ the unique choice P-a.s.

The firm may dynamically adjust the price of each bundle in response to the previous history of customer purchases and its current inventory position. We denote the set of admissible pricing policies by $\mathcal{U}$. To be admissible, a pricing policy $p = \{p_s, 0 \leq s \leq t\}$ in an $\mathbb{R}^B_+$-dimensional space, where $p_s = p(Q_s, t-s) \in \mathcal{P}(Q_s)$, must fulfill two conditions. The first is that p must P-a.s. be right-continuous with left limits. Second, for each $b \in \{1, ..., B\}$, we require that the 1-dimensional bundle b pricing process $p(b) = \{p_s(b), 0 \leq s \leq t\}$ be measurable and adapted to the filtration $\{\mathcal{F}_s^D, s \geq 0\}$, where

$$D_s(p) = \sum_{m=1}^{M_t} y^\star(Q_{\tau_m -}, X_m, p_{\tau_m -}) \tag{2}$$

is a B-dimensional vector representing the cumulative amount of each type of bundle purchased by time s, and

$$Q_s(p) = Q_0 - HD_s(p) \tag{3}$$

is the time s inventory position of the firm. For any admissible pricing policy p, it can be shown that there exists a unique solution to (2) and (3). Also, since by (1) customers take into account the inventory position of the firm when making their purchasing decision, it follows that for any $p \in \mathcal{U}$, $Q_s(p) \geq 0$ for $0 \leq s \leq t$.

Consistent with the revenue management literature, we assume all items have zero marginal cost. The firm is, therefore, interested in maximizing its expected revenue. Now note that for any admissible pricing policy $p \in \mathcal{U}$, the realized revenue of the firm over the selling horizon $[0, t]$ may be expressed as

$$\sum_{m=1}^{M_s} p_{\tau_m -} \cdot y^\star(Q_{\tau_m -}, X_m, p_{\tau_m -}) = \int_0^t p'_{s-} dD_s(p).$$

Given an admissible policy $p \in \mathcal{U}$ and initial inventory position $q \in \mathbb{N}^N$, we denote the expected revenue of the firm by

$$J_p(\lambda; q, t) = E\left[\int_0^t p(Q_s, t-s)' dD_s(p)\right]. \tag{4}$$

Thus, the optimization problem of the firm may be written as

$$J^\star(\lambda; q, t) = \sup_{p \in \mathcal{U}} J_p(\lambda; q, t). \tag{5}$$

Next, we show that the firm's optimization problem may be written as a stochastic intensity control problem and provide some results on its corresponding HJB equations and optimality conditions.

3.1. Optimality Conditions

First note that by assumption on the tie-breaking rule at most 1 bundle will be purchased by a customer. Hence, for any pricing policy $p \in \mathcal{U}$, it follows by the results in II.1 of Brémaud (1981) that $D(p)$ is a B-dimensional point process. Moreover, it can be shown that for each $b \in \{1, 2, ..., B\}$, the process D_b is a 1-dimensional point process with stochastic intensity function $\lambda_b(s) = \lambda_b(Q_{s-}, p_{s-})$ for $0 \le s \le t$, where $\lambda_b : \mathbb{N}_+^N \times \mathbb{R}_+^B \mapsto \mathbb{R}_+$ is given by

$$\lambda_b(q,p) = \lambda \mathbb{E}[y_b^\star(q,X,p)] \text{ for } (q,p) \in \mathbb{N}_+^N \times \mathbb{R}_+^B. \tag{6}$$

It then follows by D7 of II.3 of Brémaud (1981) that

$$\mathbb{E}\left[\int_0^t p'(Q_s, t-s) dD_s(p)\right] = \mathbb{E}\left[\int_0^t p'(Q_s, t-s)\lambda(Q_s, p(Q_s, t-s))ds\right],$$

where $\lambda(q,p) = (\lambda_1(q,p), \lambda_2(q,p), ..., \lambda_B(q,p)) \in \mathbb{R}_+^B$, and so we may write

$$J^\star(\lambda; q, t) = \sup_{p \in \mathcal{U}} \mathbb{E}\left[\int_0^t p'(Q_s, t-s)\lambda(Q_s, p(Q_s, t-s))ds\right].$$

We now arrive at the following result by applying C2 and T3 of VII.2 of Brémaud (1981).

Theorem 1. *The family of functions* $\{J^\star(\lambda; q, \cdot), q \in \mathbb{N}^N\}$ *are a solution to the system of equations*

$$\frac{\partial J^\star(\lambda; q, t)}{\partial t} = \sup_{p \in \mathcal{P}(q)} \left\{ \sum_{b=1}^B \lambda_b(q,p)\left[p_b - \Delta J^\star(\lambda; q, b, t)\right] \right\}, \quad \forall t \ge 0, \;\; q \in \mathbb{N}^N, \tag{7}$$

$$J^\star(q, 0) = 0, \quad q \in \mathbb{N}^N, \tag{8}$$

$$J^\star(0, \lambda t) = 0, \quad t \ge 0, \tag{9}$$

where $\Delta J^\star(\lambda; q, b, t) = J^\star(\lambda; q, t) - J^\star(\lambda; q - H_b, t)$ *and* H_b *denotes the* b*th column of* H *for* $b = 1, 2, ..., B$. *Moreover, there exists an optimal solution* $p^\star$ *to* (5) *such that* $p_t^\star(\omega) = p^\star(Q_t, t)$, *where the family of functions* $\{p^\star(\lambda; q, \cdot), q \in \mathbb{N}^N\}$ *are measurable and such that*

$$p^\star(\lambda; q, t) = \arg \sup_{p \in \mathbb{R}_+^B} \left\{ \sum_{b=1}^B \lambda_b(q,p)\left[p_b - \Delta J^\star(\lambda; q, b, t)\right] \right\}, \;\; t \ge 0. \tag{10}$$

Closed-form solutions for the above system of nonlinear differential equations are not known except in a few instances for the single-item setting (see Gallego et al. (2018)). Note, however, that the HJB equations (7)-(9) are similar in structure to those of the classical network revenue management problem studied by Gallego and Van Ryzin (1997), who examine their multi-item problem under an asymptotic fluid regime that scales both the market size λt and initial inventory levels Q_0 to ∞. In the following subsection, we present a simple example illustrating why the fluid regime may be ill-suited for analyzing dynamic mixed-bundle pricing problems. We then study the problem using the large-market regime recently introduced by Abdallah et al. Abdallah and Reed (2025b,a) in the single-item setting, where the initial inventory remains fixed while the market size λt approaches infinity..

3.2. Challenges of the Fluid Regime: A Simple Example

The fluid regime is defined by the following scaling, given $k>0$ $q=k\lambda t$ and $\lambda t\to\infty$. That is, both the inventory and market size are scaled proportionally to infinity. One of the main advantages of the fluid regime is that it leverages the law of large numbers to show that the "stochastic" value function $J^\star(q,t)$ can be well approximated with a "deterministic" value function, which is the solution to a simpler optimization problem. Furthermore, one can leverage the solutions to the deterministic problem to obtain policies that perform well for the original stochastic problem. In our setting, the fluid deterministic problem can be written as

$$J^d(\lambda;q,t)=(\lambda t)\sup_{p\in\mathcal{P}(q)}\sum_{b=1}^{B}\mathbb{E}[y_b^\star(q,X,p)]p_b \tag{11}$$

$$\text{subject to}\quad \sum_{b:n\in\psi_b}\mathbb{E}[y_b^\star(q,X,p)]\le\frac{q_n}{\lambda t}\quad\text{for } n=1,\dots,N$$

It is worth noting that one of the key assumptions for the convergence in the fluid regime to hold is that the revenue function $\sum_{b=1}^{B}\mathbb{E}[y_b^\star(q,X,p)]p_b$ needs to be concave and the inventory-related constraints should be convex, which is not the case even for the simplest mixed bundling problem. Nonetheless, even if we assume that the convergence holds for the dynamic bundling problem, the purchase probabilities under mixed bundling $\mathbb{E}[y_b^\star(q,X,p)]$ do not admit closed-form expressions in general. To the best of our knowledge, the one exception is the uniform distribution with only two items. In particular, letting $N=2$ and denoting by $b=1,2,3$ the bundles containing item 1 only, item 2 only, and the bundle, respectively, the deterministic problem under i.i.d. standard uniform valuations is given by

$$\begin{aligned}J^d(\lambda;q,t)=(\lambda t)\sup_{p}\quad & p_1(1-p_1)(p_3-p_1)+p_2(1-p_2)(p_3-p_2)\\ & +p_3\left((1-p_3+p_1)(1-p_3+p_2)-\frac{1}{2}(p_1+p_2-p_3)^2\right)\end{aligned} \tag{12}$$

$$\text{subject to}\quad (1-p_1)(p_3-p_1)+(1-p_3+p_1)(1-p_3+p_2)-\frac{1}{2}(p_1+p_2-p_3)^2\le\frac{q_1}{\lambda t} \tag{13}$$

$$(1-p_2)(p_3-p_2)+(1-p_3+p_1)(1-p_3+p_2)-\frac{1}{2}(p_1+p_2-p_3)^2\le\frac{q_2}{\lambda t} \tag{14}$$

$$0\le p_1\le 1, 0\le p_2\le 1, 0\le p_3\le p_1+p_2$$

If we ignore the non-convex inventory constraints (13)-(14), a closed-form solution exists for this simple case without inventory constraints (see, for example, Eckalbar (2010)). When the non-convex inventory constraints are present, the problem becomes more challenging, even for this simple example. Additionally, as the number of items N increases modestly, the problem becomes even more challenging due to the combinatorial number of decisions and constraints involved and the fact that it becomes very difficult to explicitly capture the interdependencies between the bundles. This renders the fluid regime not suited for solving the dynamic bundle pricing problem. Instead, we next study the dynamic bundle pricing problem in the recently introduced large market regime.

4. The Large Market Regime

In the large market regime, we consider the stochastic control problem (5) of the firm when the market size $\lambda t \to \infty$ while the inventory is kept fixed. Observe that straightforward change of variables in the Hamilton–Jacobi–Bellman equation (7) yields $J^\star(\lambda; q, t) = J^\star(1; q, \lambda t) = J^\star(\lambda t; q, 1)$ implying that the optimal value function depends only on the initial inventory q and the composite quantity λt, rather than on λ, q, and t separately. We therefore adopt the simplified two-parameter notation $J^\star(q, \lambda t)$, from which the full three-parameter form may be recovered via the change of variables.

In this regime, the pressure to liquidate inventory quickly to avoid unsold inventory at the end of the selling horizon is significantly reduced. However, as shown by Abdallah and Reed (2025b,a), this pressure still exists because the uncertainty is not resolved as λt becomes very large due to the structure of the problem, which is fundamentally governed by extreme value behavior.

Indeed, extreme value theory provides the appropriate lens through which the large market regime should be analyzed. Without this perspective, one might incorrectly conclude that the problem becomes trivial in the limit and that a static pricing policy, where the firm simply sets a high price and waits, is the optimal policy. As demonstrated by Abdallah and Reed (2025b), such static "price-high-and-wait" policies perform poorly in this regime. A key insight arises from the Fisher–Tippett theorem, the central result in extreme value theory, which characterizes the limiting distribution of the maximum of a sequence of random variables. Notably, under this limiting behavior, the maximum customer valuation remains stochastic even as λt becomes large. Consequently, demand uncertainty persists, and the firm can continue to benefit from intertemporal price discrimination, whether through dynamic pricing or, in our multidimensional setting, dynamic bundling strategies.

In particular, let $M_N = \max\{X_1, \dots, X_N\}$ be the maximum valuation out N random valuations. The Fisher-Tippett theorem, stated next, characterizes the convergence in distribution of an appropriately scaled M_N.

THEOREM 2 **(Fisher-Tippett Theorem)**. *If there exists norming constants $b_N \in \mathbb{R}$ and $a_N > 0$ for $N \geq 1$ and some non-degenerate distribution G such that*

$$\frac{M_N - b_N}{a_N} \xrightarrow{d} G \text{ as } N \to \infty,$$

then G belongs to one of the following three extreme value distributions:

$$\begin{aligned}
\textit{Type I (Gumbel):} \quad \Lambda(x) &= \quad \exp(-\exp(-x)), x \in \mathbb{R}, \\
\textit{Type II (Frechet):} \quad \Phi_\alpha(x) &= \begin{cases} 0, & x \leq 0, \\ \exp(-x^{-\alpha}), & x > 0, \end{cases} \\
\textit{Type III (Weibull):} \quad \Psi_\alpha(x) &= \begin{cases} \exp(-(-x)^\alpha), & x < 0, \\ 1, & x \geq 0, \end{cases}
\end{aligned}$$

where $\alpha > 0$ for either Type II or Type III.

In addition, the tail characteristics of the distribution F of X determine the resulting distribution as N approaches infinity. We note that a_N and b_N can be adapted to constants with continuous indexes $a(t)$ and $b(t)$. In addition, the Fisher-Tippett theorem continues to hold if the index N is replaced by a Poisson random variable $N(t)$ with rate t (see Chapter 4 in (Embrechts et al. 2013)).

We say that the distribution F belongs to the domain of attraction of G if there exist sequences $a_N > 0$ and $b_N \in \mathbb{R}$ such that $P(M_N \le b_N + a_N x) \to G(x)$ as $N \to \infty$. In this paper, we focus on distributions in the Weibull and Fréchet domains of attraction. However, we show numerically in Section 8 that the key insights derived under these domains extend to the Gumbel domain as well.

Even though the Fisher-Tippett theorem is not used directly in our proofs, we present an equivalent statement to the Fisher-Tippett in terms of the purchasing probability $1-F$, which plays a central role in our proofs (see Proposition 3.1.1 in Embrechts et al. (2013)).

Proposition 1. *Given $a_N \ge 0$ and $b_N \in \mathbb{R}$, the following are equivalent*

$$\frac{M_N - b_N}{a_N} \xrightarrow{d} G \quad \text{as } N \to \infty$$
$$N\left(1 - F(b_N + a_N x)\right) \to -\log G(x) \quad \text{as } N \to \infty.$$

for each x such that $G(x) > 0$.

5. A Generalized System of Equations

Before presenting our main results on the optimal dynamic bundling policies in the large market regime, we first recall a system of equations introduced by Abdallah and Reed (2025a) in the single-item setting. We also present its multidimensional generalization, which enables us to characterize various optimal dynamic bundle pricing policies in the large market regime.

In the single-item setting and for $\kappa > 0$, Abdallah and Reed (2025a) introduced the following system of equations

$$v_q^{\kappa-1} = \frac{1}{\kappa}\left(v_q^{\kappa} - v_{q-1}^{\kappa}\right) \quad \text{for } q \in \mathbb{N}_+, \tag{15}$$

where $v_0 = 0$. The value of κ depends on the tail properties of the distribution and corresponds to its domain of attraction. In particular, $\kappa > 1$, $\kappa = 1$, and $\kappa < 1$ correspond to the Weibull, Gumbel, and Frechet domains of attraction, respectively. The system of equations (15) has a unique positive solution that increases with κ, satisfying $v_q(\kappa) = q$ when $\kappa = 1$. Therefore, $v_q(\kappa) > q$ for $\kappa < 1$ and $v_q(\kappa) < q$ for $\kappa > 1$. Figure 2 illustrates the relationship between $v_q(\kappa)$ and q for different values of κ.

Furthermore, $v_q(\kappa)$ admits the following asymptotic expansion (Abdallah and Reed 2025a)

$$v_q(\kappa) = q + \frac{\kappa - 1}{2}\ln q + o(\ln q) \text{ as } q \to \infty. \tag{16}$$

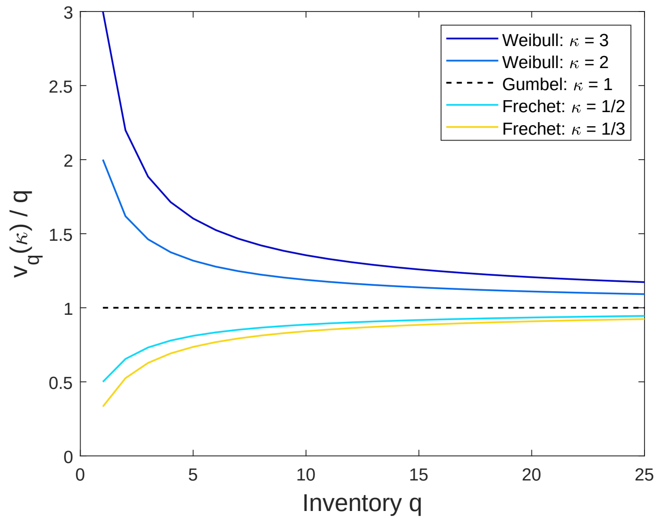


Figure 2 **The values of** $\{v_q(\kappa)/q, q \geq 1\}$ **for different values of** κ**.**

The same system of equations (15) helps characterize the optimal dynamic mixed and pure bundling policies in the large market regime. However, this system of equations is not suited to characterize the optimal dynamic bundle-size pricing policies. Next, we introduce a new system of equations that allows us to characterize the optimal dynamic bundle-size pricing in large markets.

Given $q \in \mathbb{N}^N$, let $\mathcal{N}(q) = \{n \in \{1,..,N\} : q_n \geq 1\}$ be the set of items that are not stocked out and denote by $|\mathcal{N}(q)|$ its cardinality. Now for any $\kappa > 0$ and q such that $|\mathcal{N}(q)| \geq 1$, consider the system of equations

$$\tilde{v}_q^{\kappa-1} = \frac{|\mathcal{N}(q)|^{(\kappa-1)/\kappa}}{\kappa}\left(\tilde{v}_q^{\kappa} - \frac{\sum_{n\in N(q)} \tilde{v}_{q-e_n}^{\kappa}}{|\mathcal{N}(q)|}\right) \tag{17}$$

where $\tilde{v} : \mathbb{N}^N \mapsto \mathbb{R}_+$, $\tilde{v}_0 = 0$, and e_n is a single-entry vector with 1 in the n^{th} entry. Note that for $\kappa > 0$ and $|\mathcal{N}(q)| = 1$, then $\tilde{v}_q(\kappa) = v_{q_n}(\kappa)$ for $n \in \mathcal{N}(q)$. Therefore, the system of equations (17) can be viewed as a multi-dimension generalization of (15). We summarize the properties of the solution to (17) in the following proposition.

Proposition 2. *The solution to the system of equations (17), $\tilde{v}_q(\kappa)$, satisfies the following properties*

i. For every $\kappa > 0$, there is a unique positive solution $\{\tilde{v}_q(\kappa), q \in \mathbb{N}^N : |\mathcal{N}(q)| \geq 1\}$.

ii. For each $q \in \mathbb{N}^N$ with $|\mathcal{N}(q)| \geq 1$, $\tilde{v}_q(\kappa)$ is continuous and strictly increasing in $\kappa > 0$,

iii. For each $q \in \mathbb{N}^N$ and $\kappa > 0$, $\tilde{v}_q(\kappa)$ is strictly increasing in q, where $\tilde{v}_{q+e_n}(\kappa) > \tilde{v}_q(\kappa)$ for every $n = 1, \ldots, N$.

iv. For each $q \in \mathbb{N}^N$ and $\kappa > 0$, $\tilde{v}_q(\kappa)$ is permutation invariant with respect to q. That is, letting $\sigma(q)$ be a permutation of q, then $\tilde{v}_{\sigma(q)}(\kappa) = \tilde{v}_q(\kappa)$.

v. *For each $q \in \mathbb{N}$ such that $|\mathcal{N}(q)| \geq 1$, the relationship between the multidimensional $\tilde{v}_q$ and v_{q_n} satisfies*

$$\tilde{v}_q^\kappa(\kappa) - \sum_{n=1} v_{q_n}^\kappa(\kappa) \begin{cases} \geq 0 & \text{for } \kappa > 1, \\ = 0 & \text{for } \kappa = 1, \\ \leq 0 & \text{for } 0 < \kappa < 1. \end{cases} \tag{18}$$

vi. *For each $q \in \mathbb{N}$ with $\mathcal{N}(q) \geq 1$ and for every $\kappa > 0$ where $\kappa \neq 1$, the relation $\tilde{v}_q^\kappa(\kappa) - \sum_{n=1} v_{q_n}^\kappa = 0$ holds if and only if $|\mathcal{N}(q)| = 1$ or $|\mathcal{N}(q)| > 1$ with $q_n = 1$ for each $n \in \mathcal{N}(q)$.*

6. Weibull Domain of Attraction

We begin our analysis of the optimal dynamic bundling problem in the large-market regime by focusing on distributions that belong to the Weibull domain of attraction. This class includes distributions with a finite right endpoint $x_U < \infty$, where $x_U = \sup\{x : F(x) < 1\}$, and whose upper tail, roughly speaking, exhibits polynomial decay near x_U. Examples include the uniform and beta distributions. The following result, due to Resnick (2013), characterizes all distributions that belong to the Weibull domain of attraction.

Definition 1 (Weibull Domain of Attraction). The item valuation distribution F is in the Weibull domain of attraction if and only if $x_U < \infty$ and for some index $\alpha > 0$,

$$F(x_U - 1/x) = 1 - x^{-\alpha} L(x) \text{ for } x > 0,$$

where $L : \mathbb{R}_+ \to (0, \infty)$ is a slowly varying function. That is, for any $x > 0$, $L(xt)/L(t) \to 0$ as $t \to \infty$. We focus on a fundamental class of distributions belonging to the Weibull domain of attraction that satisfies the so-called von Mises condition. (Resnick 2013).

Definition 2. A distribution function F with a finite right endpoint $x_U < \infty$ is said to satisfy the von Mises condition of the Weibull domain of attraction if F is absolutely continuous in a left neighborhood of x_U with positive density f and satisfies

$$\lim_{x \uparrow x_U} (x_U - x) \frac{f(x)}{1 - F(x)} = \alpha > 0. \tag{19}$$

The von Mises conditions are classical sufficiency conditions in extreme value theory, with each domain of attraction having its own set of these conditions. The Weibull domain of attraction includes distributions that satisfy the von Mises condition or are tail equivalent to a distribution that satisfies the von Mises condition. Almost all continuous distributions satisfy their respective von Mises conditions (see, for example, Chapter 8 of Arnold et al. (2008)). It can be verified that both the Uniform and Beta distributions satisfy the von Mises condition (19). Moreover, if F is in the Weibull domain of attraction with a non-increasing probability density function f, then (19) hold (Resnick 2013).

The norming and centering functions in the Fisher-Tippett Theorem 2 for distributions that belong to the Weibull domain of attraction are given by

$$b(t) \;=\; x_U \quad \text{and} \quad a(t) \;=\; x_U - F^{-1}(1-t^{-1}). \tag{20}$$

where

$$a(t) = t^{-1/\alpha} L_1(t), \tag{21}$$

for some L_1 slowly varying function.

We are now ready to state our main result for the optimal dynamic mixed bundling in the large market regime for valuation distributions that belong to the Weibull attraction domain.

THEOREM 3. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$, the optimal value function is given by*

$$J^\star(q,\lambda t) = \sum_{n\in\mathcal{N}(q)} q_n F^{-1}\left(1-\left(\frac{w_{q_n}(\alpha)}{q_n}\right)^{\alpha}\frac{w_{q_n}(\alpha)}{\lambda t}\right) + o(a(\lambda t)). \tag{22}$$

Moreover, an optimal dynamic pricing policy for bundles $b \in \{1,2,...,B\}$, satisfies

$$p_b^\star(q,\lambda t) = \sum_{n\in\psi_b} F^{-1}\left(1-\frac{w_{q_n}(\alpha)}{\lambda t}\right) + o(a(\lambda t)). \tag{23}$$

and the purchasing probabilities under the optimal pricing policy are given by

$$\pi_b^\star(q,\lambda t) = \begin{cases} \frac{w_{q_n}(\alpha)}{\lambda t} + o(1/(\lambda t)) & \text{if } \psi_b = \{n\} \\ o(1/(\lambda t)) & \text{if } |\psi_b| \geq 2, \end{cases} \tag{24}$$

where $w_{q_n}(\alpha) = v_{q_n}((\alpha+1)/\alpha)$.

In the single-item setting, $N = 1$, Abdallah and Reed (2025a) show that the optimal pricing policy for distributions in the Weibull domain of attraction is given by $p^\star(q,\lambda t) = F^{-1}\left(1-\frac{w_q(\alpha)}{\lambda t}\right) + o(a(\lambda t))$. Building on this, Theorem 3 implies that the optimal mixed-bundle pricing policy asymptotically resembles additive (dynamic) pricing, i.e., dynamic component pricing. Moreover, the additional value gained from selling any bundle beyond its individual components is asymptotically negligible and of order $o(a(\lambda t))$. In the large market regime, equation (24) further implies that the probability of purchasing a bundle involving more than one item (i.e., $|\psi_b| \geq 2$) is negligible compared to the probability of purchasing single-item bundles.

Leveraging the results of Abdallah and Reed (2025a) for the single-item case, it is straightforward to establish that an additive dynamic bundle pricing policy (dynamic component pricing) is asymptotically optimal in the large market regime, which we state as a corollary.

COROLLARY 1. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$, an additive dynamic bundle pricing policy (i.e., component pricing) $p \in \mathcal{U}$ where for each bundle $b \in \{1, 2, ..., B\}$,*

$$p_b(q, \lambda t) = \sum_{n \in \psi_b} F^{-1}\left(1 - \frac{w_{q_n}(\alpha)}{\lambda t}\right) \tag{25}$$

is asymptotically optimal such that

$$\frac{J^\star(q, \lambda t) - J^p(q, \lambda t)}{a(\lambda t)} \to 1 \quad as\ \lambda t \to \infty. \tag{26}$$

Theorem 3 and Corollary 1 show that the optimal and asymptotical dynamic mixed bundling policies mimic a dynamic component pricing policy in which the value of selling any bundle diminishes in the large market regime. Next, we characterize the performance two important bundling strategies in the large market regime: 1) *pure bundling*, where all items are sold together only as one grand bundle, and 2) *bundle size pricing*, where the price is set based on the number of items in the bundle, regardless of which items are included in the bundle (Chu et al. (2011), Abdallah et al. (2021)).

6.1. Optimal Dynamic Pure Bundling

In this section, we restrict the space of policies to pure bundling. Under a pure bundling policy, customers cannot buy the items separately and can either buy all of the offered items or none. A pure bundling strategy is mathematically equivalent to a special mixed bundling strategy where all bundles are priced equally. In this case, given the non-negative (or with free disposal) additive valuation assumption for the bundles, customers will only consider the grand bundle or nothing. In particular, letting B denote the largest bundle that includes all the items, under a pure bundling pricing policy, the set of admissible pricing policies should satisfy $p_b(q, \lambda t) = p_B(q, \lambda t)$ for every $b = 1, \dots, B$. In this case,

$$V_B - p_B(q, \lambda t) = \max_{b=1,\dots,B}\{V_b - p_b(q, \lambda t)\}$$

Denoting by $F^{*,N}$ the N-fold convolution distribution of the sum of N random valuations, the valuation of the grand bundle $V_B = \sum_{i=1}^N X_i$ follows the distribution $F^{*,N}$. To keep the characterization simple, when considering pure bundling policies, we only consider settings when the starting inventories of all items are equal, that is, $q_1 = \dots = q_N \geq 1$. The result can be naturally extended to settings where the initial inventories are imbalanced.

Letting $J^{\star,\mathsf{PB}}$ represent the optimal value function under pure bundling and assuming equal initial inventories, the HJB conditions (7)-(9) now reduce to

$$\frac{\partial J^{\star,\mathsf{PB}}(q, \lambda t)}{\partial t} = \lambda \sup_{p_B \geq 0} \left\{1 - F^{*,N}(p_B)\left(p_B - \Delta J^{\star,\mathsf{PB}}(q, B, \lambda t)\right)\right\}, \quad \forall t \geq 0,\ q_1 = \dots = q_N \geq 1, \tag{27}$$

$$J^\star(q, 0) = 0, \quad \text{for } q_1 = \dots = q_N \geq 0, \tag{28}$$

$$J^\star(0, \lambda t) = 0, \quad \text{for } t \geq 0, \tag{29}$$

Observe that the structure of the dynamic pure bundling problem is identical to that of the single-item problem studied by Abdallah and Reed (2025a) but with the convolution distribution $F^{*,N}$ instead of F. Therefore, to be able to leverage the results of Abdallah and Reed (2025a), we state an important property related to the closure of convolutions of distributions within the Weibull domain of attraction.

Lemma 1 (**Maddipatla et al. (2011)**). *Consider independent random variables, $X_n \sim F_{X_n}$, $n=1,\dots,N$, such that F_{X_n} belongs to the Weibull domain of attraction with index $\alpha_n > 0$, then $F^{*,N}$ belongs to the Weibull domain of attraction with index $\sum_{n=1}^N \alpha_n$.*

Since $F^{*,N}$ is in the Weibull domain of attraction, then given the i.i.d. valuation assumption, we can choose the centering and norming constants for the dynamic pure bundling problem as

$$b^{*,N}(t) = Nx_U \quad \text{and } a^{*,N}(t) = \left(Nx_U - \overset{-1}{F^{*,N}}(1-1/t)\right) = t^{-1/(N\alpha)} L_1^{*,N}(t). \tag{30}$$

where $\overset{-1}{F^{*,N}}$ is the generalized inverse of $F^{*,N}$ and $L_1^{*,N}(t)$ is a slowly varying function.

We can now use the results of Abdallah and Reed (2025a) for the single-item setting to characterize the optimal dynamic pure bundling in the large market regime which is stated next as a Corollary.

Corollary 2. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and $F^{*,N}$ satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$ such that $q_1 = \ldots = q_N \geq 1$ as $\lambda t \to \infty$, the optimal value function under dynamic pure bundling is given by*

$$J^{\star,\mathsf{PB}}(q,\lambda t) = q_N \overset{-1}{F^{*,N}}\left(1 - \left(\frac{w_{q_N}(N\alpha)}{q_N}\right)^{N\alpha} \frac{w_{q_N}(N\alpha)}{\lambda t}\right) + o(a^{*,N}(\lambda t)) \tag{31}$$

and the optimal dynamic pure bundle price is given by

$$p_B^\star(q,\lambda t) = \overset{-1}{F^{*,N}}\left(1 - \frac{w_{q_N}(N\alpha)}{\lambda t}\right) + o(a^{*,N}(\lambda t)), \tag{32}$$

where $w_q(N\alpha) = v_q((N\alpha+1)/N\alpha)$.

The characterizations in Corollary 2 and Theorem 3 allow us to compare the performance of dynamic component pricing relative to pure bundling in the large market regime for distributions that belong to the Weibull domain of attraction. First, notice that since $\alpha > 0$ then $a(\lambda t) \to 0$ and $a^{*,N}(\lambda t) \to 0$ as $\lambda t \to \infty$. Therefore, Corollary 2 and Theorem 3 imply, that for $q_1 = \ldots = q_N$,

$$\lim_{\lambda t \to \infty} J^\star(q,\lambda t) = \lim_{\lambda t \to \infty} J^{\star,\mathsf{PB}}(q,\lambda t) = N q_N x_U.$$

Upon initial inspection, it may appear that dynamic component pricing and pure bundling yield similar results in the large market regime. However, this conclusion would be misleading, as it does not account for the appropriate extreme value scaling in the large market regime. In fact, it is possible to construct *static* component and pure bundle pricing policies that can achieve the same limit Nq_Nx_U. However, static policies tend to perform poorly in the large market regime, as shown by Abdallah and Reed (2025b).

To understand the difference in the behavior of these selling strategies, it is important to note that the optimal dynamic component pricing policy achieves the bound Nq_Nx_U at a rate $a(\lambda t) = (\lambda t)^{-1/\alpha}L_1(\lambda t)$, while the optimal dynamic component pricing policy achieves the bound at a rate $a^{*,N}(\lambda t) = (\lambda t)^{-1/N\alpha}L_1^{*,N}(\lambda t)$. Therefore, for $N \geq 2$, the dynamic component pricing policy achieves the bound faster. Our subsequent corollary makes this comparison precise at the appropriate extreme value scale.

Corollary 3. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and both F and $F^{*,N}$ satisfy the von-Mises condition (19), respectively, then for any $N \geq 2$ and $q \in \mathbb{N}_+^N$ such that $q_1 = \ldots = q_N \geq 1$,*

$$\frac{J^\star(q,\lambda t) - J^{\star,\mathsf{PB}}(q,\lambda t)}{a^{*,N}(\lambda t)} \to (w_{q_N}(N\alpha))^{(N\alpha+1)/N\alpha} \quad \text{as } \lambda t \to \infty, \tag{33}$$

and,

$$\frac{J^\star(q,\lambda t) - J^{\star,\mathsf{PB}}(q,\lambda t)}{a(\lambda t)} \to +\infty \quad \text{as } \lambda t \to \infty. \tag{34}$$

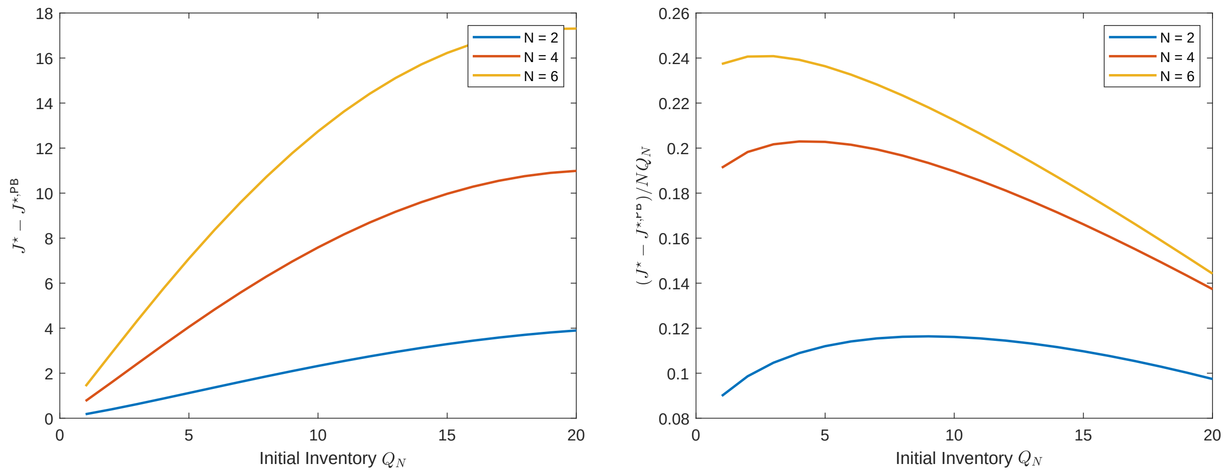
More generally, we have

$$J^\star(q,\lambda t) - J^{\star,\mathsf{PB}}(q,\lambda t) = \left(w_{q_N}(N\alpha)^{(N\alpha+1)/N\alpha}\right) a^{*,N}(\lambda t) - \left(N w_{q_N}(\alpha)^{(\alpha+1)/\alpha}\right) a(\lambda t) + o(a^{*,N}(\lambda t)). \tag{35}$$

Corollary 3 highlights the limitations of pure bundling in the large market regime for valuation distributions in the Weibull domain of attraction. To better visualize the difference in performance, we plot in Figure 3 the right-hand side of (35) (ignoring $o(a^{*,N}(\lambda t)$ term) valuations that follow a standard uniform distribution and for different values of equal initial inventory per item $q_1 = \ldots = q_N$.

Figure 3 shows that as the number of items N increases, the revenue loss from pure bundling becomes larger. However, the loss per unit sold eventually decreases as the initial inventory increases. This suggests that as the inventory becomes more abundant, the price discrimination advantage of bundles starts to counterbalance the limitations of the large market regime.

Next, we consider the performance of dynamic bundle size pricing, which is more general than pure bundling but simpler than mixed bundling.



(a) Expected revenue loss

(b) Expected revenue loss per unit sold

Figure 3 **Revenue loss from optimal dynamic pure bundling compared to optimal dynamic component pricing for market size $\lambda t = 100$ for valuations from a standard uniform distribution ($\alpha = 1$)**

6.2. Optimal Dynamic Bundle Size Pricing

We now restrict the space of policies to bundle size pricing (BSP), where the firm prices the bundles based on the number of items included in the selected bundle, i.e., the *bundle size*, regardless of the specific items included in the bundle. Therefore, given N items, the firm only sets the price of bundle sizes $k = 1, \ldots, N$ compared to the $2^N - 1$ prices of mixed bundling.

We denote by $\mathcal{B}_k = \{b : |\psi_b| = k\}$ the set of all bundles of size $k = 1, \ldots, N$. For $q \in \mathbb{N}^N$, let $\mathcal{B}_k(q) \subseteq \mathcal{B}_k$ be the set of all the bundles of size k that are not stocked out and recall that $|\mathcal{N}(q)|$ corresponds to the largest possible bundle size that can be purchased from items that are not stocked out. Now, given $q \in \mathbb{N}^N$ and remaining time $t > 0$, we denote a bundle size pricing policy by $\rho \in \mathcal{U}^{\mathsf{BSP}}$ where $\mathcal{U}^{\mathsf{BSP}}$ is the set of all admissible bundle size pricing policies. Note that $\mathcal{U}^{\mathsf{BSP}} \subset \mathcal{U}$ since any $\rho \in \mathcal{U}^{\mathsf{BSP}}$ is mathematically equivalent to special mixed bundling policies that set equal prices for the bundles of the same size $k = 1, \ldots, N$, i.e.,

$$p_b(q, \lambda t) = \rho_k(q, \lambda t) \quad \text{for } b \in \mathcal{B}_k.$$

It is worth noting that the size $k = N$ represents the grand bundle where all the items are included. Therefore, setting the prices of all bundle sizes $k = 1, \ldots, N-1$ greater than or equal to the price of the grand bundle with $k = N$, BSP reduces to pure bundling.

Letting $\pi_k(q, \lambda t) = \lambda_k(q, \rho)/\lambda$ be the probability that an arriving customer picks a bundle of size $k = 1, \ldots, K$, then conditional on the size of the chosen bundle and given the i.i.d. valuation assumption, the probability that a bundle $b \in \mathcal{B}_k(q)$ is chosen is $1/|\mathcal{B}_k(q)|$. It now follows that for $q \in \mathbb{N}^N$, the HJB equations for the optimal dynamic BSP policy can written as

$$\frac{\partial J^{\star,\mathsf{BSP}}(q,\lambda t)}{\partial t} = \lambda \sup_{\rho \in [0,\infty)^N} \left\{ \sum_{k=1}^{N} \pi_k(q,\lambda t) \left(\rho_k - \frac{1}{|\mathcal{B}_k(q)|} \sum_{b \in \mathcal{B}_k(q)} \Delta J^{\star,\mathsf{BSP}}(q,b,\lambda t) \right) \right\}, \ \forall t \geq 0, \ |\mathcal{N}(q)| \geq 1, \tag{36}$$

$$J^{\star,\mathsf{BSP}}(q,0) = 0, \quad q \in \mathbb{N}^N, \tag{37}$$

$$J^{\star,\mathsf{BSP}}(0,\lambda t) = 0, \quad t \geq 0. \tag{38}$$

Next, we characterize the optimal dynamic BSP policy and its value function in the large market regime for valuations that belong to the Weibull domain of attraction.

THEOREM 4. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$ with $|\mathcal{N}(q)| \geq 1$ and under the optimal dynamic bundle size pricing policy the optimal value function is given by*

$$J^{\star,\mathsf{BSP}}(q,\lambda t) = \sum_{n \in \mathcal{N}(q)} q_n F^{-1}\left(1 - \left(\frac{\tilde{w}_q(\alpha)}{|\mathcal{N}(q)| q_n}\right)^{\alpha} \frac{\tilde{w}_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)). \tag{39}$$

Moreover, only the price of size one bundles contributes meaningfully, and is given by

$$\rho_1^{\star}(q,\lambda t) = F^{-1}\left(1 - \left(\frac{1}{|\mathcal{N}(q)|^{\alpha/(\alpha+1)}}\right) \frac{\tilde{w}_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{40}$$

while the prices of bundles of size $k \geq 2$ are inconsequential. In particular, the purchasing probabilities for bundle sizes $k = 1, \dots, N$ are given by

$$\pi_k(q,\lambda t) = \begin{cases} \frac{1}{|\mathcal{N}(q)|^{\alpha/(\alpha+1)}} \frac{\tilde{w}_q(\alpha)}{\lambda t} + o(1/(\lambda t)) & \text{for } k = 1, \\ o(1/(\lambda t)) & \text{for } k = 2, \dots, N, \end{cases} \tag{41}$$

where $\tilde{w}_q(\alpha) = \tilde{v}_q((\alpha+1)/\alpha)$.

First, comparing the remainder terms (39) with (31) and (22), we observe that the remainder term from the optimal dynamic BSP policy matches that for the mixed bundling and is (asymptotically) smaller than pure bundling. Hence, roughly speaking, dynamic bundle size pricing strictly improves on dynamic pure bundling and may be comparable to component pricing on the same extreme value scale. However, this improvement stems from the fact that the optimal bundle size pricing policy addresses the diminishing value of bundles, and its optimal pricing policy mimics a BSP of size 1, which restricts customers to picking only one item. In fact, we next show that a dynamic BSP that allows only purchases of bundles of size 1 achieves the same value function of the optimal dynamic BSP (up to $o(a(\lambda t))$) in the large market regime.

Proposition 3. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$ such that $|\mathcal{N}(q)| \geq 1$, a single size bundle size pricing policy $\rho \in \mathcal{U}^{BSP}$ where*

$$\rho_k(q, \lambda t) = \begin{cases} F^{-1}\left(1 - \left(\frac{1}{N^{\alpha/(\alpha+1)}}\right)\frac{\tilde{w}_q(\alpha)}{\lambda t}\right) & \text{for } k = 1, \\ k x_U & \text{for } k = 2, \ldots, N, \end{cases} \tag{42}$$

is asymptotically optimal among BSP policies such that

$$\frac{J^{\star,\mathsf{BSP}}(q, \lambda t) - J^{\rho,\mathsf{BSP}}(q, \lambda t)}{a(\lambda t)} \to 1 \quad \text{as } \lambda t \to \infty. \tag{43}$$

Next, we compare the optimal dynamic BSP to the optimal dynamic mixed bundling.

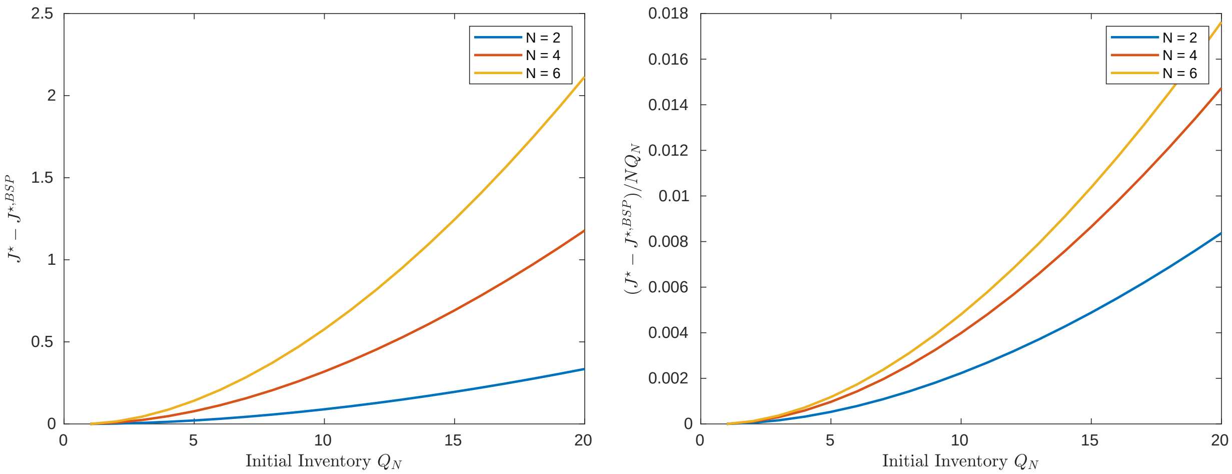
Corollary 4. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and F satisfies the von-Mises condition (19), then for $q \in \mathbb{N}^N$, we have*

$$\frac{J^{\star}(q, \lambda t) - J^{\star,\mathsf{BSP}}(q, \lambda t)}{a(\lambda t)} \to \tilde{w}_q^{(\alpha+1)/\alpha}(\alpha) - \sum_{n=1}^{N} w_{q_n}^{(\alpha+1)/\alpha}(\alpha) \geq 0 \quad \text{as } \lambda t \to \infty. \tag{44}$$

Focusing on (44) and invoking property (vi) from Proposition 2, it turns out that, except in the special cases $N = 1$ or $q_n = 1$ for every $n \in \mathcal{N}(q)$, the optimal dynamic BSP is strictly dominated by the optimal dynamic component pricing policy. This is perhaps surprising, especially since in the large market regime, the value of a bundle of two or more items is negligible (see Theorem 4), and hence it may not be immediately clear why BSP is inefficient.

In fact, the inefficiency of dynamic BSP in our setting is related to the inefficiencies of (static) BSP policies under heterogeneous costs highlighted by Abdallah et al. (2021). Note that, even though the items have zero marginal cost in our setting, the dynamics of the pricing problem effectively give rise to virtual marginal costs, which are determined by the marginal value of depleting an item. This can be observed in the optimization problem (36), where the virtual marginal cost is given by $\sum_{b \in \mathcal{B}_k(q)} \Delta J^{\star,\mathsf{BSP}}(q, b, \lambda t)/|\mathcal{B}_k(q)|$. A seller who adopts a BSP policy cannot control which items customers choose, so they must set prices based on the average marginal value of those items. This can lead to inefficiencies, as items with lower available inventory have a higher marginal value compared to items that are more abundant.

To better visualize the right-hand side of (44), we plot in Figure 4 the difference between $J^{\star}$ and $J^{\star,\mathsf{BSP}}$ for valuations that follow a standard uniform distribution and for different numbers of items with equal initial inventory per item $q_1 = \ldots = q_N$. First, compared to the optimal dynamic pure bundling in Figure 3, we observe that the revenue loss is smaller yet can still be significant, especially for larger values of N. In contrast to Figure 3, the revenue loss per unit sold appears to increase with the initial inventory. This is likely because, with a larger initial inventory, the possible imbalances in the inventory along the sample paths are more pronounced, which, as argued earlier, is the major source of the inefficiency of BSP.



(a) Expected revenue loss (b) Expected revenue loss per unit sold

Figure 4 **Revenue loss from optimal dynamic bundle size pricing compared to optimal dynamic component pricing for market size $\lambda t = 100$ for valuations from a standard uniform distribution ($\alpha = 1$)**

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7. Frechet Domain of Attraction

Next, we consider valuation distributions that belong to the Frechet domain of attraction. This domain of attraction corresponds to item valuation distributions with a "heavy" power-law tail and infinite support. The Pareto and Cauchy distributions are some examples of distributions in the Frechet domain of attraction. More generally, the following definition characterizes all distributions that belong to the Frechet domain of attraction (Resnick 2013).

Definition 3 (Frechet Domain of Attraction). The item valuation distribution F is in the Frechet domain of attraction if and only if $x_U = \infty$ and for some index $\alpha > 0$,

$$F(x) = 1 - L(x)x^{-\alpha} \text{ for } x > 0,$$

where $L : \mathbb{R}_+ \to (0, \infty)$ is a slowly varying function, that is, for any $x > 0$, $L(xt)/L(t) \to 0$ as $t \to \infty$.

We refer to α as the tail-index of the distribution. Note that the distribution has an infinite mean when $\alpha \leq 1$, which can cause our problem to be ill-defined in the limit. For this reason, we consider only $\alpha > 1$ in the Frechet domain of attraction.

Similar to the Weibull domain of attraction, the von-Mises condition of the Frechet domain of attraction provides sufficient conditions for distributions to belong to it, which we state next.

Definition 4. A distribution function F with an infinite right endpoint $x_U = \infty$ is said to satisfy the von-Mises condition of the Frechet domain of attraction if F is absolutely continuous with positive density f on (x_0, ∞) for some $x_0 > 0$ and satisfies

$$\lim_{x \to \infty} \frac{xf(x)}{1 - F(x)} = \alpha > 0. \tag{45}$$

For the rest of this section, we assume that the item valuation distributions satisfy the von Mises condition of the Frechet domain of attraction with $\alpha > 1$. In this case, the centering and norming functions of the Fisher-Tippett Theorem 2 are given by

$$a(t) \;=\; F^{-1}(1-t^{-1}) \;\text{ and }\; b(t)=0, \tag{46}$$

for $t>0$. Moreover, it may be shown that $a(t)=t^{1/\alpha}L_1(t)$, where L_1 is a slowly varying function.

Next, we characterize the optimal value function and the optimal dynamic mixed bundle pricing policy in the large market regime.

Theorem 5. *If F is in the Frechet domain of attraction with index $\alpha>1$ and satisfies the von Mises condition (45), then for each $q\in\mathbb{N}^N$ as $\lambda t\to\infty$,*

$$J^\star(q,\lambda t) = \sum_{n=1}^{N} q_n F^{-1}\left(1-\left(\frac{q_n}{\phi_{q_n}(\alpha)}\right)^{\alpha}\frac{\phi_{q_n}(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{47}$$

Moreover, an optimal dynamic pricing policy for bundles $b\in\{1,2,...,B\}$, satisfies

$$p_b^\star(q,\lambda t) = \sum_{n\in\psi_b} F^{-1}\left(1-\frac{\phi_{q_n}(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{48}$$

and the purchasing probabilities under the optimal pricing policy are given by

$$\pi_b^\star(q,\lambda t) = \begin{cases} \frac{\phi_{q_n}(\alpha)}{\lambda t} + o(1/(\lambda t)) & \text{if } \psi_b=\{n\} \\ o(1/(\lambda t)) & \text{if } |\psi_b|\ge 2, \end{cases} \tag{49}$$

where $\phi_{q_n}(\alpha)=v_{q_n}((\alpha-1)/\alpha)$ for $n=1,\dots,N$.

In the Frechet domain of attraction, similar to the Weibull domain, it turns out that in the large market regime, the optimal dynamic mixed bundle pricing policy also mimics a dynamic additive bundle pricing policy or, in other words, a dynamic component pricing policy. We can leverage the results of Abdallah and Reed (2025a) to show that an additive dynamic bundle pricing policy is asymptotically optimal for the Frechet domain of attraction.

Corollary 5. *If F is in the Frechet domain of attraction with index $\alpha>1$ and satisfies the von Mises condition (45), then for each $q\in\mathbb{N}^N$, an additive dynamic bundle pricing policy (i.e., component pricing) $p\in\mathcal{U}$ where for each bundle $b\in\{1,2,...,B\}$,*

$$p_b(q,\lambda t) = \sum_{n\in\psi_b} F^{-1}\left(1-\frac{\phi_{q_n}(\alpha)}{\lambda t}\right) \tag{50}$$

is asymptotically optimal where

$$\frac{J^\star(q,\lambda t)-J^p(q,\lambda t)}{a(\lambda t)} \to 1 \quad \text{as } \lambda t\to\infty. \tag{51}$$

7.1. Optimal Dynamic Pure Bundling

We now restrict the policies to dynamic pure bundling in order to characterize the performance of the optimal dynamic pure bundling policy. For the Fréchet domain of attraction, we next provide a stronger characterization for the optimal dynamic pure bundling than in the Weibull domain of attraction (Corollary 2) that does not depend on the convolution distribution $F^{*,N}$—the distribution of the bundle's valuation.

PROPOSITION 4. *If F is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}_+^N$ such that $q_1 = \ldots = q_N \geq 1$ as $\lambda t \to \infty$,*

$$J^{\star,\mathsf{PB}}(q,\lambda t) = q_N N^{1/\alpha} F^{-1}\left(1 - \left(\frac{q_N}{\phi_{q_N}(\alpha)}\right)^{\alpha} \frac{\phi_{q_N}(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{52}$$

and

$$p_B^{\star}(q,\lambda t) = N^{1/\alpha} F^{-1}\left(1 - \frac{\phi_{q_N}(\alpha)}{\lambda t}\right) + o(a(\lambda t)), \tag{53}$$

where $\phi_{q_N}(\alpha) = v_{q_N}((\alpha-1)/\alpha)$.

First observe that in contrast to the Weibull domain of attraction, the remainder terms in (52) and (53) are, loosely speaking, on the same order ($o(a(\lambda t))$), of those for the optimal mixed bundling policy (component pricing) (47) and (48), respectively. On the other hand, for $N \geq 2$, and since $\alpha > 1$, we observe that when the inventory is balanced the first order term in the optimal dynamic pure bundling price (53) is strictly lower than the sum of the component prices in the optimal mixed bundling policy. This lower price also reflects in lower revenues as captured in the first order term in (52). We make the value function comparison precise in the following corollary.

COROLLARY 6. *If F is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (45), then for any $N \geq 1$ and $q \in \mathbb{N}_+^N$ such that $q_1 = \ldots = q_N \geq 1$, we have*

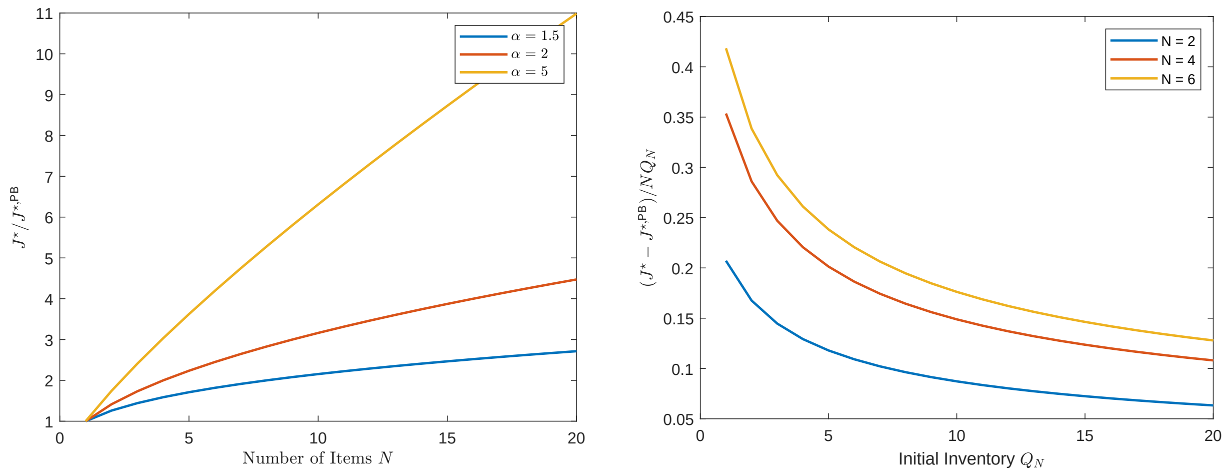
$$\frac{J^{\star}(q,\lambda t)}{J^{\star,\mathsf{PB}}(q,\lambda t)} \to N^{(\alpha-1)/\alpha} \quad \text{as } \lambda t \to \infty, \tag{54}$$

and

$$\frac{J^{\star}(q,\lambda t) - J^{\star,\mathsf{PB}}(q,\lambda t)}{N q_N a(\lambda t)} \to \left(\frac{\phi_{q_N}^{(\alpha-1)/\alpha}(\alpha)}{q_N}\right)(1 - N^{(1-\alpha)/\alpha}) \quad \text{as } \lambda t \to \infty, \tag{55}$$

To better visualize the impact of the α, N, and initial inventory Q_N, we plot the right-hand sides of (54) and (55) in Figure 5.

We observe that in the Frechet domain of attraction and similar to the Weibull domain of attraction, the optimal dynamic pure bundling is strictly dominated by the optimal dynamic additive bundle pricing (component pricing) for $N \geq 2$. Furthermore, the right-hand sides in (54) and (55) are



(a) Limiting Ratio of Revenues

(b) Limiting Revenue loss per unit sold ($\alpha = 2$)

Figure 5 **Limiting performance of the optimal dynamic pure bundling policy versus the optimal dynamic component pricing policy**

both strictly decreasing in N. This implies that as the number of items N grows large the performance of pure bundling deteriorates. This is in contrast to the celebrated result of Bakos and Brynjolfsson (1999) that states that when items have zero marginal costs and there is sufficient (unlimited) inventory, pure bundling can extract all the consumer surplus. Next, notice that the right-hand sides in (54) and (55) are both strictly increasing in $\alpha > 1$. This implies that the performance of pure bundling deteriorates as the tail gets lighter.

Finally, regarding the impact of inventory levels q_N, note that the right-hand side of the normalized revenue loss per unit sold (55) depends on $\phi_{q_N}^{(\alpha-1)/\alpha}(\alpha)/q_N$. However, from Abdallah and Reed (2025a) we have $\phi_{q_N}(\alpha) = q - (1/2\alpha)\ln q + o(\ln q)$ and hence

$$\frac{\phi_{q_N}^{(\alpha-1)/\alpha}(\alpha)}{q_N} = q^{-1/\alpha}\left(1 - \frac{1}{2\alpha}\frac{\ln q_N}{q_N}\right)^{(\alpha-1)/\alpha} + o\left((\ln q_N)^{(\alpha-1)/\alpha}/q_N\right)$$

Observe that up to the $o(\cdot)$ term, $\phi_{q_N}^{(\alpha-1)/\alpha}(\alpha)/q_N$ is strictly decreasing in q_N and approaches zero as $q_N \to \infty$. This again highlights that bundling performs poorly in the large market regime mainly due to the scarce inventory. As the inventory levels increase, the relative performance of the bundles improves.

7.2. Optimal Dynamic Bundle Size Pricing

Next, we restrict the space of policies to bundle size pricing and characterize the optimal dynamic BSP policy and its value function in the large market regime for valuations that belong to the Frechet domain of attraction. Recall that under a BSP, the firm sets the prices based on the number of items in the bundle, regardless of which items are included.

Theorem 6. *If F is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (45), then under the optimal dynamic bundle size pricing policy for each $q \in \mathbb{N}^N$ with $|\mathcal{N}(q)| \geq 1$, we have*

$$J^{\star,\mathsf{BSP}}(q,\lambda t) = \sum_{n\in\mathcal{N}(q)} q_n F^{-1}\left(1 - \left(\frac{|\mathcal{N}(q)|q_n}{\tilde{\phi}_q(\alpha)}\right)^{\alpha} \frac{\tilde{\phi}_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{56}$$

Moreover, only the price of size one bundles contributes meaningfully, and is given by

$$\rho_1^\star(q,\lambda t) = |\mathcal{N}(q)|^{1/(\alpha-1)} F^{-1}\left(1 - \frac{\tilde{\phi}_q(\alpha)}{\lambda t}\right) + o(a(\lambda t)) \tag{57}$$

while the prices of bundles of size $k \geq 2$ are inconsequential. In particular, the purchasing probabilities for bundle sizes $k = 1, \ldots, N$ are given by

$$\pi_k^\star(q,\lambda t) = \begin{cases} \frac{1}{|\mathcal{N}(q)|^{\alpha/(\alpha-1)}} \frac{\tilde{\phi}_q(\alpha)}{\lambda t} + o(1/(\lambda t)) & \text{for } k = 1, \\ o(1/(\lambda t)) & \text{for } k = 2, \ldots, N, \end{cases} \tag{58}$$

where $\tilde{\phi}_q(\alpha) = \tilde{v}_q((\alpha-1)/\alpha)$.

We next show that similar to the Weibull domain of attraction, a dynamic BSP that allows only purchases of bundles of size 1, achieves the same value function of the optimal dynamic BSP (up to $o(a(\lambda t))$) in the large market regime.

Proposition 5. *If F_X is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von-Mises condition (45), then for each $q \in \mathbb{N}^N$ with $\mathcal{N}(q) \geq 1$, a single-size bundle size pricing policy $\rho \in \mathcal{U}^{BSP}$ where*

$$\rho_k(q,\lambda t) = \begin{cases} |\mathcal{N}(q)|^{1/(\alpha-1)} F^{-1}\left(1 - \frac{\tilde{\phi}_q(\alpha)}{\lambda t}\right) & \text{for } k = 1, \\ \infty & \text{for } k = 2, \ldots, N, \end{cases} \tag{59}$$

is asymptotically optimal among BSP policies where

$$\frac{J^{\star,\mathsf{BSP}}(q,\lambda t) - J^{\rho,\mathsf{BSP}}(q,\lambda t)}{a(\lambda t)} \to 1 \quad \text{as } \lambda t \to \infty. \tag{60}$$

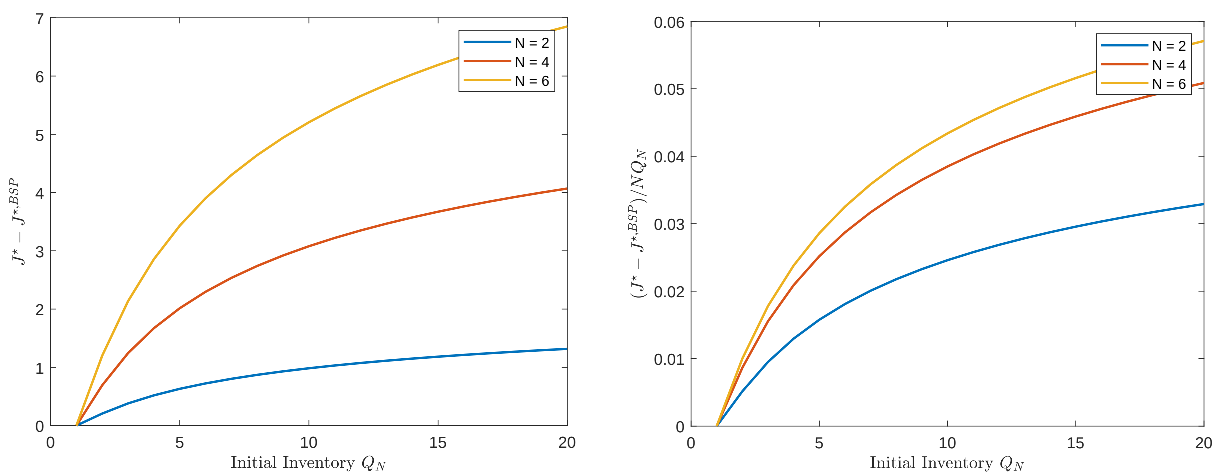
Next, we compare the optimal dynamic BSP to the optimal dynamic mixed bundling.

Corollary 7. *If F is in the Frechet domain of attraction with index $\alpha > 1$ and F satisfies the von-Mises condition (45), then for $q \in \mathbb{N}_+^N$, we have*

$$\frac{J^\star(q,\lambda t) - J^{\star,\mathsf{BSP}}(q,\lambda t)}{a(\lambda t)} \to \sum_{n=1}^{N} \phi_{q_n}^{(\alpha-1)/\alpha}(\alpha) - \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha) \geq 0 \quad \text{as } \lambda t \to \infty. \tag{61}$$

Note that by Property (vi) in Proposition 2, it turns out that, similar to the Weibull domain of attraction and in most of the cases, the optimal dynamic BSP is strictly dominated by the optimal dynamic component pricing policy. The only cases in which they are asymptotically equivalent are $N = 1$ or $q_n = 1$ for all $n \in \mathcal{N}(q)$. Again, the inefficiency of BSP policies stems from the fact that the problem dynamics yield (virtual) heterogeneous marginal costs represented by the marginal value of losing one unit of inventory, where items with lower available inventory have a larger marginal value than items with abundant inventories.

To better visualize the right-hand side of (61), we plot in Figure 6 the difference between $J^{\star}$ and $J^{\star,\mathsf{BSP}}$ and its normalized version per unit sold for valuations that follow a standard Pareto distribution ($\alpha = 2$) and for different numbers of items and assuming equal initial inventory per item $q_1 = \ldots = q_N$.



(a) Expected revenue loss (b) Expected revenue loss per unit sold

Figure 6 **Revenue loss from optimal dynamic bundle size pricing compared to optimal dynamic component pricing for market size $\lambda t = 100$ for valuations from a standard Pareto distribution ($\alpha = 2$)**

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First, compared to the optimal dynamic pure bundling in Figure 5, the revenue loss is significantly smaller under the optimal BSP, yet can be large, especially for larger N. Moreover, the revenue loss per unit sold appears to be increasing in the initial inventory and number of items, which is consistent with what was observed in the Weibull domain of attraction, Figure 4. Similarly, this is likely because, with larger initial inventories, the possible imbalances in the inventory along the sample paths are more pronounced, which, as argued earlier, is the major source of the inefficiency of BSP.

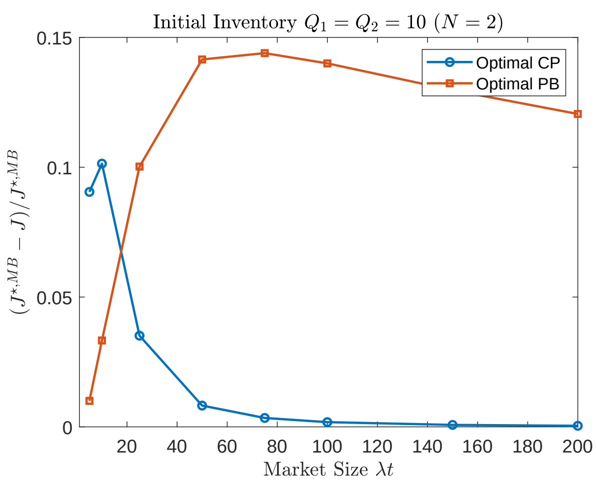
8. Numerics

In this section, we perform several numerical simulations to examine how inventory levels and market size influence the effectiveness of various bundle pricing strategies. The results are organized into three subsections, corresponding to the three domains of attraction of extreme value theory.

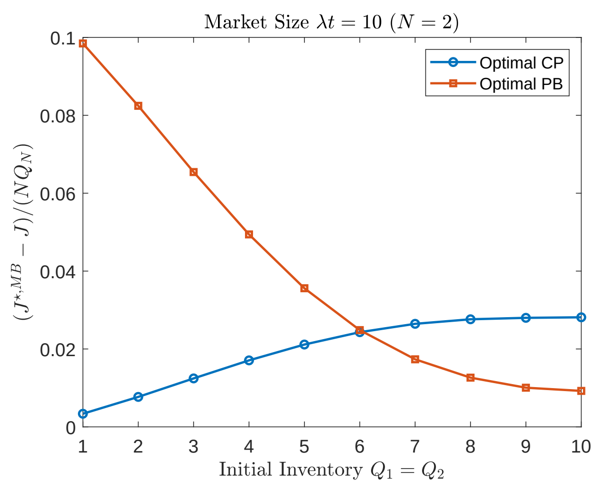
8.1. Uniform Distribution: Weibull Domain of Attraction

We revisit the example from Section 3.2, where we analyze a setting with two items, $N = 2$, and valuations that follow a standard uniform distribution that belongs to the Weibull domain of attraction. We focus on this setting since it allows us to derive closed-form expressions for the purchase probabilities associated with mixed bundling (MB). As a result, we can numerically determine the optimal dynamic mixed bundling policy and the dynamic fluid-optimal mixed bundling prices obtained using the fluid-resolving heuristic.

Figure 7 presents two plots that compare the effectiveness of the optimal dynamic component pricing (CP) and pure bundling (PB) strategies against the optimal dynamic mixed bundling policy. Figure 7a, we fix the initial inventory $Q_1 = Q_2$ and vary the market size, while in Figure 7b we fix the market size $\lambda t = 10$ and vary the initial inventory of both items while keeping them equal.



(a) Relative expected revenue loss



(b) Expected revenue loss per unit sold

Figure 7 **Expected revenue loss from the optimal dynamic component pricing and optimal dynamic pure bundle pricing relative to the optimal dynamic mixed bundling for $N = 2$ and valuations that follow a standard uniform distribution (Weibull domain of attraction)**

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We observe from Figure 7a that when the market size is small ($\lambda t \leq 20$), the insights from the classical bundling literature hold, where the optimal dynamic pure bundling outperforms optimal dynamic component pricing. However, as the market size increases, we observe that component pricing dominates pure bundling and eventually approximates optimal dynamic mixed bundling closely.

Figure 7b provides a complementary view of the initial inventory's impact on the performance of pricing strategies while keeping the market size fixed, $\lambda t = 10$. When inventory is scarce ($q_1 = q_2 \leq 6$), dynamic component pricing dominates dynamic pure bundle pricing. Yet, dynamic pure bundling dominates component pricing as inventory increases and becomes abundant ($q_1 = q_2 > 6$).

Next, we compare the performance of the asymptotically optimal (in their respective regimes) LM and fluid policies to the optimal dynamic mixed bundling policy and the component pricing policy in Figure 8. In particular, the asymptotically optimal Fluid policy in the fluid regime dynamically re-optimizes the deterministic problem (12)-(14).

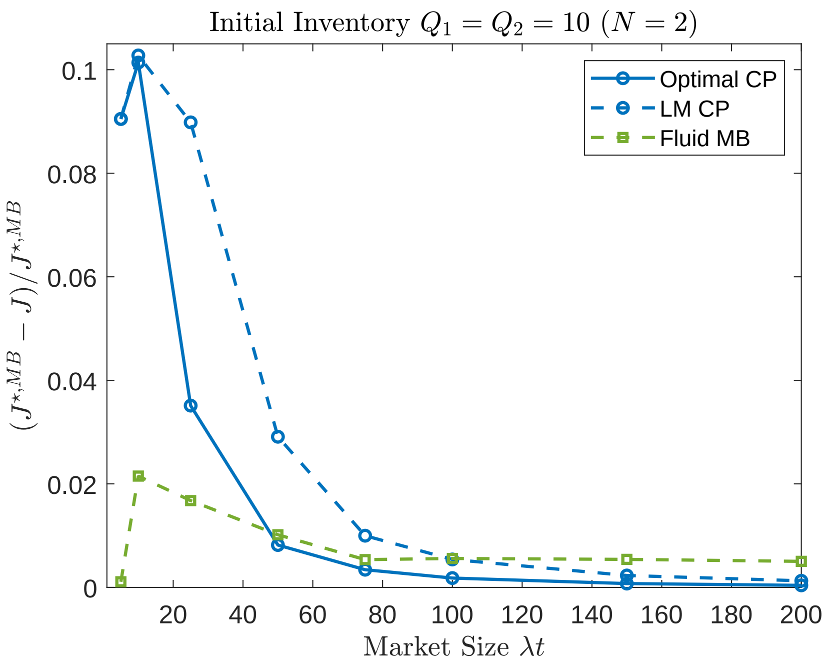

Figure 8 **Expected revenue loss of the LM dynamic component pricing policy and the Fluid dynamic mixed bundling relative to the optimal dynamic mixed bundling and dynamic component for $N = 2$ and valuations that follow a standard uniform distribution (Weibull domain of attraction with $\alpha = 1$)**

We observe from Figure 8 that the fluid dynamic mixed bundling policy dominates the optimal dynamic component pricing policy and the LM dynamic component pricing policy for small market sizes, while the reverse is true for large market sizes. It is worth noting that the mixed bundling policy is more complicated than component pricing as a selling strategy. Apart from the uniform distribution case with $N = 2$, it is not possible to write the deterministic problem explicitly; hence, one may need to resort to simulation optimization. On the other hand, even if there is no closed form for the optimal component pricing policy, it is easier to compute numerically. Meanwhile, the most straightforward policy to compute is the LM policy, whose performance is competitive with the optimal mixed bundling policy for moderate to large market sizes, $\lambda t \geq 75$.

The insights from Figures 7 and 8 align with the findings of Abdallah and Reed (2025a), which indicate that the inventory-to-market-size ratio (IMR) is the primary driver of policy performance in large market regimes. To explore this further, we conduct a numerical investigation of the IMR boundary to identify when one policy begins to outperform another. These boundaries are illustrated in Figure 9. Specifically, we focus on the boundary between optimal dynamic pure bundling and optimal dynamic component pricing in addition to the boundaries separating the fluid-optimal dynamic mixed bundling policy from the optimal dynamic component pricing policy and the LM-optimal dynamic component pricing policies.

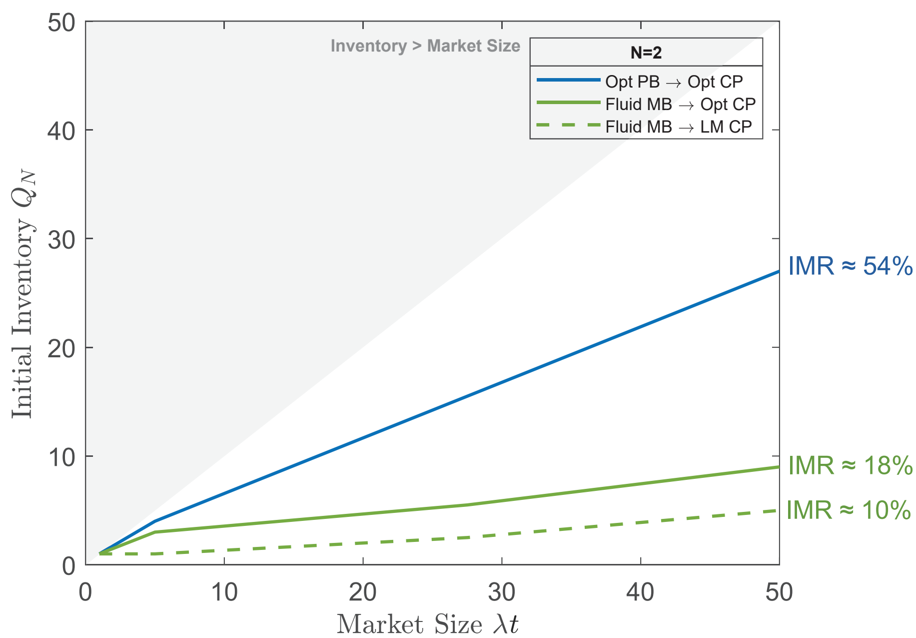

Figure 9 **Inventory-to-market-size ratio boundary under which dynamic component pricing policy starts dominating dynamic pure bundle pricing (Weibull domain of attraction)**

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We observe from Figure 9 that the optimal dynamic component pricing policy outperforms the optimal dynamic pure bundling policy (both determined numerically) when the IMR falls below 54%. Additionally, the optimal dynamic component pricing outperforms the fluid-optimal dynamic mixed bundling policy when the IMR is under 18%. Furthermore, when the IMR drops below 10%, even the LM-optimal dynamic component pricing policy surpasses the more complicated fluid-optimal dynamic mixed bundling policy.

Finally, we examine numerically how the IMR impacts the fluid-optimal dynamic component pricing, pure bundling, and mixed bundling relative to the optimal mixed bundling policy. For the fluid policies, we continuously reoptimize their corresponding deterministic problems, which are analogous to (12)-(14) depending on the policy type.

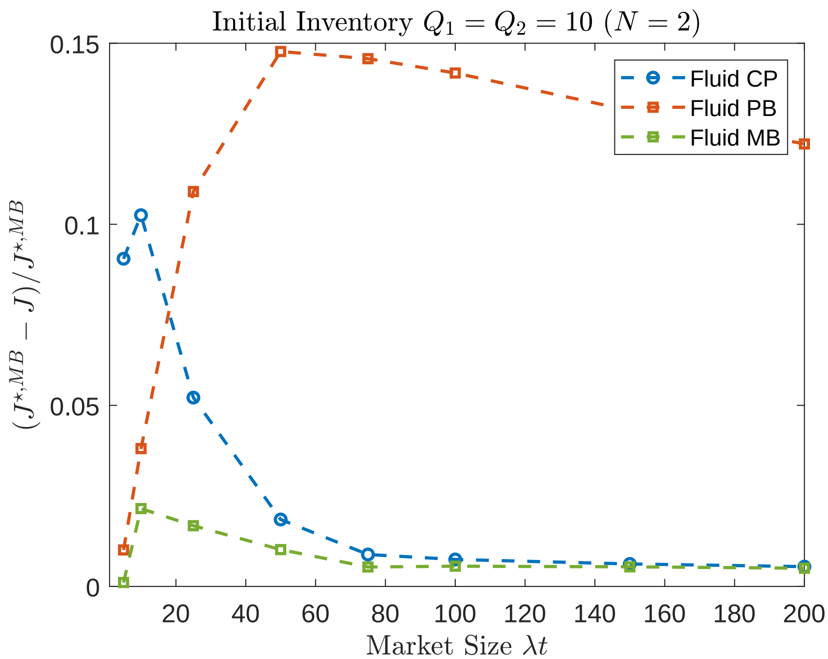

Figure 10 **Expected revenue loss of the Fluid-Optimal component, pure bundling, mixed bundling pricing relative to the optimal dynamic mixed bundling and dynamic component for $N = 2$ and valuations that follow a standard uniform distribution (Weibull domain of attraction with $\alpha = 1$)**

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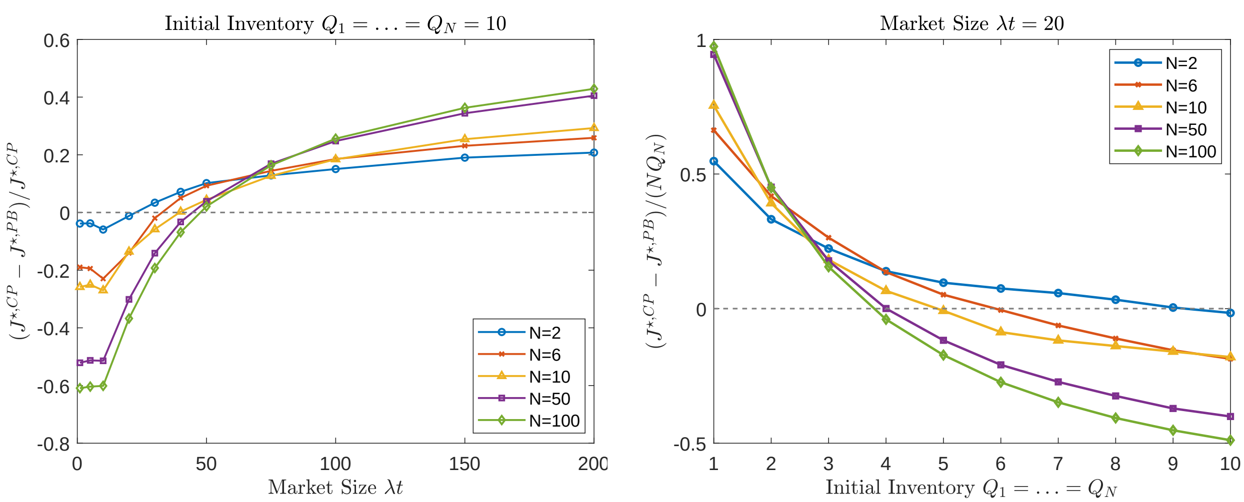
Interestingly, we observe in Figure 10 that the IMR effects on fluid-optimal policies align with those for large market regime policies. When inventory is below 50% of the market size, fluid-optimal dynamic component pricing starts dominating the fluid-optimal dynamic pure bundling policy. Furthermore, inventory becomes scarce; the fluid-optimal dynamic component pricing closely matches the fluid-optimal mixed bundling policy.

8.2. Pareto Distribution: Frechet Domain of Attraction

Next, we consider valuations distributed according to a standard Pareto distribution with $\alpha = 2$ that belongs to the Fréchet attraction domain. We note that for a Pareto distribution, the purchase probabilities under a mixed bundling policy do not have closed-form expressions. For this reason, we focus on comparing component pricing with pure bundling.

In Figure 11, we show two plots that compare the relative value function difference between the optimal dynamic component pricing (CP) and the optimal dynamic pure bundle pricing (PB) for different market sizes (λt), initial inventory levels, and the number of items N. We assume equal initial inventories for all item types in both plots, i.e., $Q_1 = ... = Q_N$.

Similar to the case of uniform distribution, we observe that the relative performance depends mainly on the IMR, where a lower IMR favors dynamic component pricing. In particular, Figure 11a illustrates that the principles of classical bundling theory regarding the profitability of pure bundling apply to small market sizes ($\lambda t < 25$). However, we observe an interesting effect of the number of items N on the relative performances of the policy. For small market sizes, we observe that larger N increases the dominance of pure bundling relative to component pricing. However, as the market size increases, component pricing catches up and surpasses pure bundling; however, the impact of N is reversed, where now a larger N favors component pricing more. Figure 11a shows a comparable trend when we keep the market size constant and adjust the initial inventory, where abundant inventory favors pure bundling, especially for larger N.



(a) Relative expected revenue difference

(b) Expected revenue difference per unit sold

Figure 11 **Relative expected revenue from the optimal dynamic component pricing compared to the optimal dynamic pure bundle pricing for valuations that follow a standard Pareto distribution (Frechet domain of attraction)**

In Figure 12, we analyze the inventory-to-market-size ratio boundary (IMR) boundary to determine when optimal dynamic component pricing becomes more effective than pure bundling while varying the number of items, N. The transition point where component pricing surpasses pure bundling appears to occur when the IMR is between 22% and 42%, depending on N. Notably, for smaller values of N, this transition occurs at a higher IMR threshold. As N increases, the boundary becomes more stable and less sensitive to changes in N.

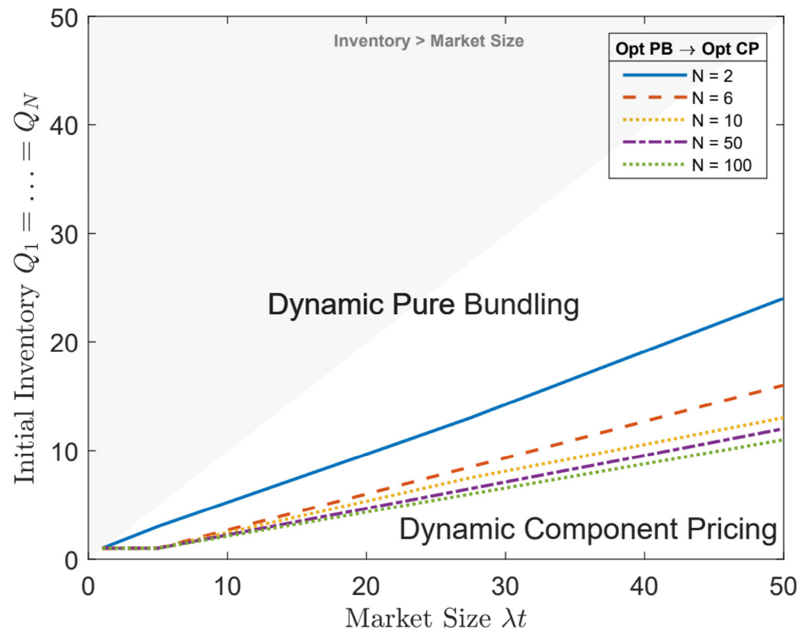
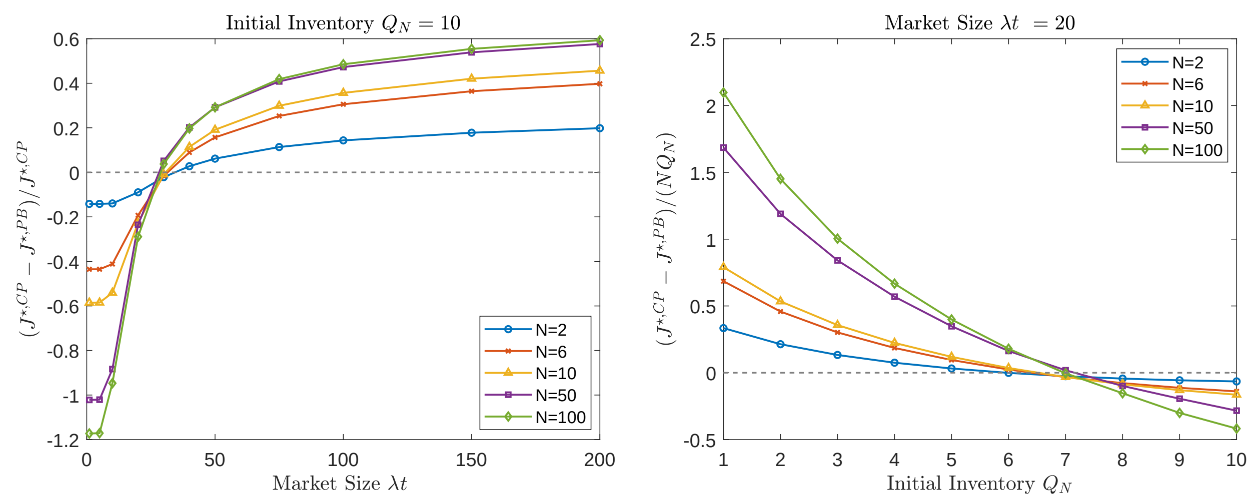


Figure 12 **Inventory-to-market-size ratio boundary under which dynamic component pricing policy starts dominating dynamic pure bundle pricing (Frechet domain of attraction)**

8.3. Exponential Distribution: Gumble Domain of Attraction

Our final set of numerics examines valuations that follow a standard exponential distribution belonging to the Gumbel attraction domain. This domain more broadly encompasses distributions with bounded support, light tails, and moderately heavy tails. Readers can refer to Abdallah and Reed (2025b) for more detailed information about this domain. Although our theoretical findings were established for the Weibull and Frechet domains of attraction, we numerically demonstrate that the insights regarding the diminishing value of bundles in the large market regime continue to hold for valuations that belong to the Gumbel domain of attraction. Similar to Section 8.2, we compare optimal dynamic component pricing (CP) effectiveness against optimal dynamic pure bundling (PB) since the purchase probabilities for mixed bundling policy do not admit a closed-form expression. We note that the optimal dynamic component pricing problem admits a closed-form solution (Gallego and Van Ryzin 1994), while the pure bundling problem does not. Therefore, we numerically solve the HJB equations to obtain the optimal value function for the optimal dynamic pure bundling policy.

We present two plots in Figure 13 that illustrate the differences in the relative value function between optimal dynamic component pricing (CP) and pure bundle pricing (PB) for different market sizes and initial inventory levels. We consider equal initial inventories for all item types for both plots, i.e., $Q_1 = ... = Q_N$. The findings from both plots align with theoretical and numerical insights regarding the Weibull and Frechet domains of attraction, revealing how the advantage of the component pricing strategy over pure bundling changes as market size increases or inventory becomes scarce.



(a) Relative expected revenue difference (b) Expected revenue difference per unit sold

Figure 13 Relative expected revenue from the optimal dynamic component pricing compared to the optimal dynamic pure bundle pricing for valuations that follow a standard exponential distribution (Gumbel domain of attraction)

Notably, the patterns shown in Figure 13 are strikingly similar to those in Figure 12 concerning the Pareto distribution. With high IMR, whether due to large inventory or small market size, pure bundling significantly outperforms component pricing, and this advantage grows as N increases. Conversely, component pricing dominates for lower IMR, characterized by either small inventory or large market size, especially as N becomes larger.

It appears that the switching point when component pricing begins to dominate pure bundling is more robust across different numbers of items than in the case of Pareto distribution. This is highlighted in Figure 14, where we numerically investigate the switching point when optimal dynamic component pricing dominates that of pure bundling. We observe that the inventory-to-market size boundary appears robust regardless of the number of item types. The transition to component pricing seems to occur when the IMR is around $30-36\%$. However, in contrast to Figure 12, a larger number of item types N leads to a transition at a slightly higher IMR boundary.

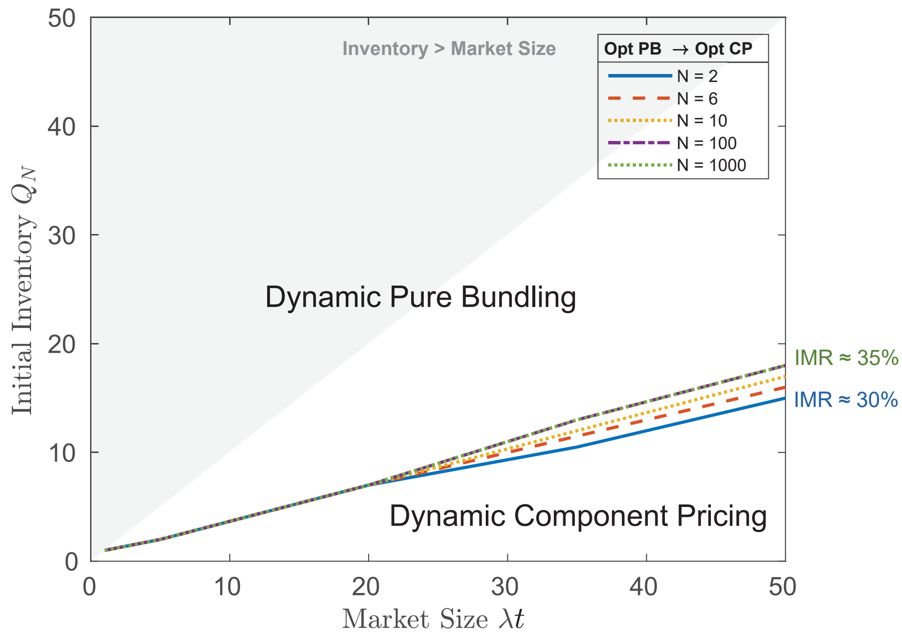


Figure 14 **Inventory-to-market-size ratio boundary under which dynamic component pricing policy starts dominating dynamic pure bundle pricing (Gumbel domain of attraction)**

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9. Conclusion

This paper examines the revenue implications of bundling and component pricing in inventory-constrained settings, focusing on how the inventory-to-market-size ratio (IMR) influences the optimal pricing strategy. Our results reveal that bundling, while traditionally viewed as a revenue-enhancing strategy, is, in fact, revenue-distorting in the large market regime. Specifically, we show that when inventory is sufficiently scarce relative to market size, dynamic component pricing outperforms commonly used bundling strategies such as bundle size pricing and pure bundling. In fact, we show that the optimal dynamic mixed bundling strategy mimics a simple dynamic component pricing policy in the large market regime. We also identify numerically the switching points that depend on IMR where firms should transition from bundling to component pricing as market conditions change. This insight challenges classical bundling theory and highlights the importance of considering inventory constraints when deploying bundling strategies.

While this study provides a structured framework for understanding bundling in inventory-constrained environments, there are several avenues for future research. First, our analysis assumes a monopolist setting; extending the model to competitive markets could provide deeper insights into how firms strategically adopt bundling when facing rivals. In particular, understanding how price competition interacts with inventory limitations could refine our conclusions on bundling's effectiveness. Second, our analysis is based on known demand distributions; future research could investigate learning-based pricing models that adjust prices in real time based on the observed data. Exploring how such methods interact with bundling decisions under uncertainty—where firms learn about demand distributions dynamically—could enhance the practical applicability of our framework. Finally, another future research direction is investigating how scaling different operational considerations, such as different scaling of inventory levels, market size, and number of items, impacts the optimal dynamic bundling strategy. This is important for firms whose market conditions are not properly approximated by the large market regime.

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Appendix.

In this Appendix, we present the proofs for the results stated in the main body of the paper. The appendix is organized as follows: In Section A, we present the proofs related to the properties of the multi-dimensional system of equations $\tilde{v}(q)$. In Section B and C, we establish the results for distribution that belong to the Weibull and Frechet domains of attractions, respectively.

A. Proof of Section 5: Properties of $\tilde{v}(q)$

In this section, we prove the properties of $\tilde{v}_q(\kappa)$, the solution to the multi-dimensional system of equations (17).

Proof of Proposition 2. The proof of properties (i) and (ii) is similar to the properties for v_{q_n} for $q_n \geq 0$ in the single item setting as established by Abdallah and Reed (2025a) and are omitted. We next show properties (iii)-(vi).

(iii) Let $\kappa > 0$ and recall that by definition $\tilde{v}_0(\kappa) = 0$. The statement is true for $\|q\|_1$ =0. Suppose it is true for all $\|q\|_1 \leq l$ for some $l \geq 0$ and fix q such that $\|q\|_1 = l$ and $q + e_n$ for some $n = 1, \ldots, N$, i.e, $\|q + e_n\|_1 = l + 1$. We consider two cases depending on q_n.

Case I ($q_n = 0$): the sytem of equations (17) can be written

$$\tilde{v}_{q+e_n}^{\kappa-1} = \frac{|\mathcal{N}(q+e_n)|^{-1/\kappa}}{\kappa} \left(|\mathcal{N}(q+e_n)|\tilde{v}_{q+e_n}^{\kappa} - \tilde{v}_q^{\kappa} - \sum_{m \in \mathcal{N}(q)} \tilde{v}_{q+e_n-e_m}^{\kappa} \right) \tag{A1}$$

Consider the function

$$g(x) = x^{k-1} - \frac{|\mathcal{N}(q+e_n)|^{-1/\kappa}}{\kappa} \left(|\mathcal{N}(q+e_n)|x^{\kappa} - \tilde{v}_q^{\kappa} - \sum_{m \in \mathcal{N}(q)} \tilde{v}_{q+e_n-e_m}^{\kappa} \right),$$

where $x \in [x_o, \infty)$ and $x_o = \sum_{m \in \mathcal{N}(q+e_n)} \tilde{v}_{q+e_n-e_m}^{\kappa} / |\mathcal{N}(q+e_n)|$. We have from Property (i) that g has a unique root at $x = \tilde{v}_{q+e_n}$. Moreover, it is straightforward to verify that $g(x_o) > 0$ and $g(x) \to -\infty$ as $x \to \infty$. It follows from the continuity of g that for $x \geq \tilde{v}_{q+e_n}$, then $g(x) < 0$.

Now suppose for a contradiction that $\tilde{v}_q \geq \tilde{v}_{q+e_n}$. This implies that $g(\tilde{v}_q) \leq 0$ and hence

$$\tilde{v}_q^{\kappa-1} \leq \frac{|\mathcal{N}(q+e_n)|^{-1/\kappa}}{\kappa} \left(|\mathcal{N}(q)|\tilde{v}_q^{\kappa} - \sum_{m \in \mathcal{N}(q)} \tilde{v}_{q+e_n-e_m}^{\kappa} \right). \tag{A2}$$

However, by the induction hypothesis we have $\sum_{m \in \mathcal{N}(q)} \tilde{v}_{q+e_n-e_m}^{\kappa} > \sum_{m \in \mathcal{N}(q)} \tilde{v}_{q-e_m}^{\kappa}$, hence

$$\tilde{v}_q^{\kappa-1} < \frac{|\mathcal{N}(q+e_n)|^{-1/\kappa}}{\kappa} \left(|\mathcal{N}(q)|\tilde{v}_q^{\kappa} - \sum_{m \in N(q)} \tilde{v}_{q-e_m}^{\kappa} \right) = \left(\frac{|\mathcal{N}(q+e_n)|}{|\mathcal{N}(q)|} \right)^{-1/\kappa} \tilde{v}_q^{\kappa-1}, \tag{A3}$$

which is a contradiction.

Case II ($q_n \geq 1$): we have $\mathcal{N}(q) = \mathcal{N}(q+e_n)$. Now notice that from (17), then $\tilde{v}_q$ and $\tilde{v}_{q+e_n}$ satisfy

$$\sum_{m \in \mathcal{N}(q)} \tilde{v}_{q-e_m}^{\kappa} = \tilde{v}_q^{\kappa} \left(1 - \frac{\kappa}{\mathcal{N}(q)^{(\kappa-1)/\kappa}} \tilde{v}_q^{-1} \right) \text{ for } q \in \mathbb{N}^N, \tag{A4}$$

$$\sum_{m \in \mathcal{N}(q)} \tilde{v}_{q+e_n-e_m}^{\kappa} = \tilde{v}_{q+e_n}^{\kappa} \left(1 - \frac{\kappa}{\mathcal{N}(q)^{(\kappa-1)/\kappa}} \tilde{v}_{q+e_n}^{-1} \right) \text{ for } q \in \mathbb{N}^N. \tag{A5}$$

Since, by the induction hypothesis $\sum_{m\in\mathcal{N}(q)} \tilde{v}^{\kappa}_{q+e_n-e_m} > \sum_{m\in\mathcal{N}(q)} \tilde{v}^{\kappa}_{q-e_m}$, it follows that $\tilde{v}_{q+e_n} > \tilde{v}_q$.

(iv) We again prove the statement by induction. Fix $N>1$, the statement is true when $\|q\|_1 = 0$. Now fix some $l \geq 1$ and suppose that $\tilde{v}_q$ is permutation invariant for all q such that $\|q\|_1 \leq l-1$. Next, consider q such that $\|q\|_1 = l$ and denote one of its permutations by $\sigma(q)$. Noting that $|\mathcal{N}(q)| = |\mathcal{N}(\sigma(q))|$, it follows from (17) that

$$\tilde{v}^{\kappa-1}_q = \frac{|\mathcal{N}(q)|^{(\kappa-1)/\kappa}}{\kappa}\left(\tilde{v}^{\kappa}_q - \frac{\sum_{n\in\mathcal{N}(q)} \tilde{v}^{\kappa}_{q-e_n}}{|\mathcal{N}(q)|}\right), \tag{A6}$$

$$\tilde{v}^{\kappa-1}_{\sigma(q)} = \frac{|\mathcal{N}(\sigma(q))|^{(\kappa-1)/\kappa}}{\kappa}\left(\tilde{v}^{\kappa}_{\sigma(q)} - \frac{\sum_{n\in\mathcal{N}(q)} \tilde{v}^{\kappa}_{\sigma(q)-e_n}}{|\mathcal{N}(q)|}\right). \tag{A7}$$

By the induction hypothesis, we have $\sum_{n\in\mathcal{N}(q)} \tilde{v}^{\kappa}_{\sigma(q)-e_n} = \sum_{n\in\mathcal{N}(q)} \tilde{v}^{\kappa}_{q-e_n}$. Therefore by the uniqueness of the solution to the system of equations (17), we have $\tilde{v}_q = \tilde{v}_{\sigma(q)}$.

(v) For $k=1$, we have $v_{q_n} = q_n$ for $n = 1,\ldots,N$ and it is straightforward to verify that $\tilde{v}(q) = \sum_{n=1}^N q_n$ satisfies (17).

Now let $k>1$ and fix $N \geq 1$. The statement is true for $\|q\|_1 = 0$. Now fix some $l \geq 2$ and suppose the statement is true for all $\|q\|_1 \leq l-1$. Consider q such that $\|q\|_1 = l$ and suppose for a contradiction that $\tilde{v}^{\kappa}_q < \sum_{n=1}^N v^{\kappa}_{q_n} = \sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n}$, then similar to the proof of property (iii) and by the uniqueness of the solution of (17), we have

$$\left(\frac{\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n}}{|\mathcal{N}(q)|}\right)^{(\kappa-1)/\kappa} < \frac{1}{\kappa}\left(\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n} - \frac{\sum_{n\in N(q)} \tilde{v}^{\kappa}_{q-e_n}}{|\mathcal{N}(q)|}\right). \tag{A8}$$

Note that $0 < (\kappa-1)/\kappa < 1$ and hense applying Jensen's inequality, we obtain

$$\left(\frac{\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n}}{|\mathcal{N}(q)|}\right)^{(\kappa-1)/\kappa} \geq \frac{1}{|\mathcal{N}(q)|}\sum_{n\in\mathcal{N}(q)} v^{\kappa-1}_{q_n}, \tag{A9}$$

Turning to the right-hand side of (A8), note that by the induction hypothesis, for $n\in\mathcal{N}(q)$, we have

$$\tilde{v}^{\kappa}_{q-e_n} \geq \sum_{k\in\mathcal{N}(q)} v^{\kappa}_{q_k} - \left(v^{\kappa}_{q_n} - v^{\kappa}_{q_n-1}\right).$$

which implies

$$\frac{1}{\kappa}\left(\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n} - \frac{\sum_{n\in N(q)} \tilde{v}^{\kappa}_{q-e_n}}{|\mathcal{N}(q)|}\right) \leq \frac{1}{\kappa}\left(\frac{\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n} - v^{\kappa}_{q_n-1}}{|\mathcal{N}(q)|}\right) = \frac{1}{|\mathcal{N}(q)|}\sum_{n\in\mathcal{N}(q)} v^{\kappa-1}_{q_n} \tag{A10}$$

and hence is a contradiction.

The case for $\kappa<1$ follows analogously where the inequalities are flipped, and instead of the concavity property, we have convexity.

(vi) The if part is straightforward, so we show the only if part. The statement is true for $|\mathcal{N}(q)| = 1$. First consider $k>1$ and Fix $q\in\mathbb{N}^N$ such that $|\mathcal{N}(q)| \geq 2$ and suppose $\tilde{v}^{\kappa}_q = \sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n}$. It follows from (17), that

$$\left(\frac{\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n}}{|\mathcal{N}(q)|}\right)^{(\kappa-1)/\kappa} = \frac{1}{\kappa}\left(\sum_{n\in\mathcal{N}(q)} v^{\kappa}_{q_n} - \frac{\sum_{n\in N(q)} \tilde{v}^{\kappa}_{q-e_n}}{|\mathcal{N}(q)|}\right) \leq \frac{1}{\mathcal{N}(q)}\sum_{n\in\mathcal{N}(q)} v^{\kappa-1}_{q_n} \tag{A11}$$

where the inequality is due to property (v) and (A10). However, similar to the proof of property (v), by Jensen's inequality for strictly concave functions, we obtain

$$\left(\frac{\sum_{n\in\mathcal{N}(q)} v_{q_n}^{\kappa}}{|\mathcal{N}(q)|}\right)^{(\kappa-1)/\kappa} \geq \frac{1}{\mathcal{N}(q)} \sum_{n\in\mathcal{N}(q)} v_{q_n}^{\kappa-1} \tag{A12}$$

with equality if and only if $v_{q_n} = v_{q_m}$ for all $n, m \in \mathcal{N}(q)$. Therefore, $\tilde{v}_q^{\kappa} = \sum_{n\in\mathcal{N}(q)} v_{q_n}^{\kappa}$ implies $v_{q_n} = v_{q_m}$ for all $n, m \in \mathcal{N}(q)$. But since v is strictly increasing in q, then $q_n = q_m$ for all $n, m \in \mathcal{N}(q)$.

Now pick any item $m \in \mathcal{N}(q)$, we have $\tilde{v}_q^{\kappa} = |\mathcal{N}(q)| v_{q_m}^{\kappa}$. In this case, (17) yields

$$v_{q_m}^{\kappa-1} = \frac{1}{\kappa}\left(|\mathcal{N}(q)| v_{q_m}^{\kappa} - \frac{\sum_{n\in N(q)} \tilde{v}_{q-e_n}^{\kappa}}{|\mathcal{N}(q)|}\right) \tag{A13}$$

However, by the permutation invariance property (iv), we have $\sum_{n\in N(q)} \tilde{v}_{q-e_n}^{\kappa} = |\mathcal{N}(q)| \cdot \tilde{v}_{q-e_m}$, and therefore

$$v_{q_m}^{\kappa-1} = \frac{1}{\kappa}\left(|\mathcal{N}(q)| v_{q_m}^{\kappa} - \tilde{v}_{q-e_m}^{\kappa}\right) \tag{A14}$$

which implies that $\tilde{v}_{q-e_m}^{\kappa} = |\mathcal{N}(q)| v_{q_m}^{\kappa} - (v_{q_m}^{\kappa} - v_{q_m-1}^{\kappa}) = \left(\sum_{n\neq m\in N(q)} v_n^{\kappa}\right) + v_{q_m-1}^{\kappa}$. Applying Jensen's inequality again implies that $q_n = q_k$ for all $n, k \in \mathcal{N}(q - e_m)$, which is impossible unless $m \notin \mathcal{N}(q - e_m)$ or in other words $q_m = 1$. Therefore, $q_n = 1$ for all $n \in \mathcal{N}(q)$.

The case $k < 1$ follows analogously and is skipped. □

Before starting the proofs of the Weibull and Frechet domain of attractions, we first establish Lemma A1, which allows us to limit our attention to optimal bundle prices that are greater than or equal to the marginal value of the components included in the bundle.

LEMMA A1. *For each $t \geq 0$ and $q \in \mathbb{N}_+^N$, there exists an optimal pricing policy $p^\star \in \mathcal{U}$ such that $p_b^\star(q,t) \geq \Delta J^\star(q,b,t)$ for each $b \in \{1, 2, ..., B_N\}$.*

Proof of Lemma A1. Let $p^\star$ be an optimal policy and fix $t \geq 0, q \in \mathbb{N}_+^N$. Now suppose that there exists some set $\mathcal{B} \subset \{1, 2, ..., B\}$ such that $p_b^\star(q,t) < \Delta J(q,b,t)$ for $b \in \mathcal{B}$, and $p_b^\star(q,t) \geq \Delta J(q,b,t)$ for $b \in \mathcal{B}^C$. Now consider an alternative policy $p \in \mathcal{U}$ such that $p_b(q,t) = \Delta J(q,b,t)$ for $b \in \mathcal{B}$ and $p_b(q,t) = p_b^\star(q,t)$ for $b \in \mathcal{B}^C$. It follows from the HJB equation (7) that it is sufficient to show that

$$\sum_{b=1}^{B} \lambda_b(q,p)\left(p_b(q,t) - \Delta J^\star(q,b,t)\right) \geq \sum_{b=1}^{B} \lambda_b(q,p^\star)\left(p_b^\star(q,t) - \Delta J(q,b,t)\right). \tag{A15}$$

Since $p_b(q,t) = p_b^\star(q,t)$ for $b \in \mathcal{B}^C$ and $\lambda_b(q,p)(p_b(q,t) - \Delta J(q,b,t)) = 0$ for $b \in \mathcal{B}$, then in order to show (A15) it suffices to prove that

$$\sum_{b\in\mathcal{B}^C} (\lambda_b(q,p) - \lambda_b(q,p^\star))(p_b^\star(q,t) - \Delta J(q,b,t)) \geq \sum_{b\in\mathcal{B}} \lambda_b(q,p^\star)(p_b^\star(q,t) - \Delta J(q,b,t)).$$

However, since $p_b^\star(q,t) < \Delta J(q,b,t)$ for $b \in \mathcal{B}$, it next suffices to prove that

$$\sum_{b\in\mathcal{B}^C} (\lambda_b(q,p) - \lambda_b(q,p^\star))(p_b^\star(q,t) - \Delta J(q,b,t)) \geq 0.$$

Moreover, since $p_b(q,t) > p_b^\star(q,t)$ for $b \in \mathcal{B}$ and $p_b(q,t) = p_b^\star(q,t)$ for $b \in \mathcal{B}^C$, it is straightforward to show that $\lambda_b(q,p) \geq \lambda_b(q,p^\star)$ for $b \in \mathcal{B}^C$. □

As a consequence of Lemma A1 and throughout the proofs, we can limit ourselves to admissible pricing policies $p \in \mathcal{U}$ such that for each $t \geq 0$ and $q \in \mathbb{N}_+^N$,

$$p_b(q,t) \geq \Delta J^\star(q,b,t). \tag{A16}$$

B. Proofs of Section 6: Weibull Domain of Attraction

In this section, we present and prove an equivalent version of Theorem 3 in addition to Corollary 1.

We start by introducing some new notation. Note that since F is in the Weibull domain of attraction, then it has a bounded support with $x_U < \infty$. Now for each $q \in \mathbb{N}^N$ and $t \geq 0$ set

$$\hat{J}^\star(q,t) = \left(\sum_{n=1}^{N} q_n\right) x_U - J^\star(q,t) \text{ and } \hat{p}_b^\star(q,t) \ = \ |\psi_b| x_U - p_b^\star(q,t). \tag{B1}$$

where $\psi_b \subseteq \{1,2,..,N\}$ the set of items contained in bundle b. Furthermore, denote by $\bar{\psi} = \{|\psi_b|, b = 1,\ldots,B\} \subseteq \{1,\ldots,N\}^B$ the vector corresponding to the number of items in each bundle.

Now for $b \in \mathcal{B}(q)$, let $\Delta\hat{J}^\star(q,b,t) = \hat{J}^\star(q,t) - \hat{J}^\star(q - H_b, t) = |\psi_b| x_U - \Delta J^\star(q,b,t)$, where $\hat{J}^\star(0,\cdot) = 0$. The HJB equations (7)-(9) may be written as

$$-\frac{\partial \hat{J}^\star(q,t)}{\partial t} = \sup_{\hat{p} \in \hat{\mathcal{P}}(q)} \left\{ \sum_{b=1}^{B} \lambda_b(q, \bar{\psi} x_U - \hat{p})(\Delta\hat{J}^\star(q,b,t) - \hat{p}_b) \right\}, \quad \forall t \geq 0,\ q \in \mathbb{N}^N : |\mathcal{N}(q)| \geq 1, \tag{B2}$$

$$\hat{J}^\star(q,0) = \left(\sum_{n=1}^{N} q_n\right) x_U, \quad q \in \mathbb{N}^N, \tag{B3}$$

$$\hat{J}^\star(0,t) = 0, \quad t \geq 0. \tag{B4}$$

Next note that from Lemma 1 that $1 - F^{*,|\psi_b|}(|\psi_b| x_U - 1/x) = x^{-|\psi_b|\alpha} L_b(x)$ and the extreme value theory centering and norming functions are given by $b^{*,|\psi_b|}(t) = |\psi_b| x_U$ and $a^{*,|\psi_b|}(t) = \left(|\psi_b| x_U - \overset{-1}{F^{*,\psi_b}}(1 - 1/t)\right) = t^{-1/(|\psi_b|\alpha)} L_{b1}(t)$. Therefore, by standard properties of slowly varying functions and for any b such that $|\psi_b| \geq 2$, we have

$$\frac{a(t)}{a^{*,|\psi_b|}(t)} \to 0, \quad \text{as } t \to \infty. \tag{B5}$$

Now, we can establish some important properties regarding the optimal policies in the Weibull domain of attraction. Denote by $\mathcal{B}(q)$ the set of bundles that are not stocked out given $q \in \mathbb{N}^N$, we have the following lemma.

Lemma B1. *Given any $q \in \mathbb{N}^N$, the following properties hold:*

1. *For every $b \in \mathcal{B}(q)$, we have*

$$\limsup_{\lambda t > 0} \frac{\hat{p}_b^\star(q,t)}{a(\lambda t)} \leq \limsup_{\lambda t > 0} \frac{\Delta\hat{J}^\star(q,b,t)}{a(\lambda t)} < \infty. \tag{B6}$$

2. *For every $b \in \mathcal{B}(q)$ such that $|\psi_b| \geq 2$,*

$$\limsup_{\lambda t > 0} \frac{\hat{p}_b^\star(q,t)}{a^{*,|\psi_b|}(\lambda t)} = \limsup_{t > 0} \frac{\Delta\hat{J}^\star(q,b,t)}{a^{*,|\psi_b|}(\lambda t)} = 0. \tag{B7}$$

3. *Letting β_n denote the singleton bundle containing only item n, then for every $n = 1,\ldots,N$ such that $q_n \geq 1$, we have*

$$\liminf_{\lambda t > 0} \frac{\Delta\hat{J}^\star(q,\beta_n,t)}{a(\lambda t)} > 0. \tag{B8}$$

Proof of Lemma B1. First, notice that (B6) together with (B5) imply (B7), so we establish (B6). From Lemma A1 and for every $b \in \mathcal{B}(q)$, we have

$$\frac{\hat{p}_b^\star(q,t)}{a(\lambda t)} \leq \frac{\Delta \hat{J}^\star(q,b,t)}{a(\lambda t)} \leq \frac{\hat{J}^\star(q,t)}{a(\lambda t)}. \tag{B9}$$

However, since component pricing (additive bundle pricing) is a feasible policy, it follows from Abdallah and Reed (2025a) that $\limsup_{\lambda t > 0} \hat{J}^\star(q,t)/a(\lambda t) < \infty$, which establishes (B6).

To establish (B8), denote by $Z_{n,M(\lambda t)} = \max\{X_{m,n} : m = 1,\ldots,M(\lambda t)\}$ the largest valuation of item n out of the $M(\lambda t)$ random customer arrivals. We have that $\Delta J^\star(q,\beta_n,t) \leq E[Z_{n,M(\lambda t)}]$ and therefore

$$\frac{\Delta \hat{J}(q,\beta_n,t)}{a(\lambda t)} \geq \frac{x_U - E[Z_{n,M(\lambda t)}]}{a(\lambda t)} \geq P\left(\frac{x_U - Z_{n,M(\lambda t)}}{a(\lambda t)} > 1\right) \tag{B10}$$

where the second inequality is due to Markov's inequality. However, by the Fisher-Tippet theorem for random indexed extremes (see, for example, Theorem 4.3.2 in (Embrechts et al. 2013)), we have that

$$P\left(\frac{x_U - Z_{n,M(\lambda t)}}{a(\lambda t)} > 1\right) \to 1/e > 0, \quad \text{as } t \to 0,$$

which implies (B8). $\square$

Now, letting $\grave{(\cdot)} := \hat{(\cdot)}/a(\lambda t)$, we are now ready to state and prove an equivalent statement to Theorem 3.

Theorem B1. *If F_X is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$,*

$$\lim_{\lambda t \to \infty} \grave{J}^\star(q,t) = \left(\sum_{n=1}^N w_{q_n}^{(\alpha+1)/\alpha}(\alpha)\right), \tag{B11}$$

and, for each bundle $b \in \{1,2,\ldots,B\}$, an optimal bundle pricing policy satisfies

$$\lim_{\lambda t \to \infty} \grave{p}_b^\star(q,t) = \sum_{n \in \psi_b} w_{q_n}^{1/\alpha}(\alpha). \tag{B12}$$

and,

$$\lim_{\lambda t \to \infty} \lambda t \left(1 - F^{*,|\psi_b|}\left(|\psi_b| x_U - a(\lambda t)\grave{p}_b^\star(q,t)\right)\right) = \begin{cases} w_{q_n}(\alpha) & \text{if } \psi_b = \{n\} \\ 0 & \text{if } |\psi_b| \geq 2, \end{cases} \tag{B13}$$

where $w_{q_n}(\alpha) = v_{q_n}((\alpha+1)/\alpha)$.

Proof of Theorem B1. Before proceeding with the proof we establish the equivalence of (B11) and (22). Noting that $a(\lambda t/x) = x_U - F^{-1}(1 - x/(\lambda t))$ and letting $\zeta(q_n) = (w_{q_n}(\alpha)/q_n)^\alpha w_{q_n}(\alpha)$, then (22) can be equivalently written as

$$J^\star(q,t) = \sum_{n=1}^N q_n \left(X_u - a(\lambda t/\zeta(q_n))\right) + o(a(\lambda t))$$

However, in the Weibull domain of attraction and by Corollary 1.2.10 in (Haan and Ferreira 2006), we have $a(\lambda t/\zeta(q_n)) = a(\lambda t)\zeta(q_n)^{1/\alpha} + o(a(\lambda t))$. Therefore, (22) is now equivalent to

$$\begin{aligned}\sum_{n=1}^N q_n x_U - J^\star(q,t) &= \sum_{n=1}^N q_n \zeta(q_n)^{1/\alpha} a(\lambda t) + o(a(\lambda t)) \\ &= \sum_{n=1}^N w_{q_n}^{(\alpha+1)/\alpha} a(\lambda t) + o(a(\lambda t))\end{aligned}$$

which is also equivalent to (B11). The rest of the equivalency follows analogously.

We now proceed with the proof. Note that it is sufficient to establish the result for $\lambda = 1$, so fix $q \in \mathbb{N}^N$ and assume without loss of generality that $q_n > 0$ for every $n = 1, \ldots, N$; otherwise, those items can be ignored. It follows from the equivalent HJB equations (B2)- (B4) that for a sufficiently large $t > 0$, $\grave{p}^\star(q,t)$ attains the supremum of

$$\begin{aligned}\grave{\Pi}^\star(q,N,t) &= \sup_{\grave{p}\in[0,x_U]^B} \left\{ \sum_{b=1}^{B} \lambda_b \left(q, \bar{\psi} x_U - a(t)\grave{p}\right) \left[\Delta \grave{J}^\star(q,b,t) - \grave{p}_b\right] \right\} \\ &= \sum_{b=1}^{B} \lambda_b \left(q, \bar{\psi} x_U - a(t)\grave{p}^\star(q,t)\right) \left[\Delta \grave{J}^\star(q,b,t) - \grave{p}^\star_b(q,t)\right]. \end{aligned} \tag{B14}$$

First, we establish the following convergence result

$$\lim_{t\to\infty} t \left(\grave{\Pi}^\star(q,N,t) - \sum_{n=1}^{N} (1 - F(x_U - a(t)\grave{p}^\star_{\beta_n}(q,t))) \left(\Delta \grave{J}^\star(q,\beta_n,t) - \grave{p}^\star_{\beta_n}(q,t)\right) \right) = 0. \tag{B15}$$

which, by the principle of optimality, implies

$$\lim_{t\to\infty} t \left(\grave{\Pi}^\star(q,N,t) - \sum_{n=1}^{N} \sup_{\grave{\rho}_n\in[0,x_U]} (1 - F(x_U - a(t)\grave{\rho}_n)) \left(\Delta \grave{J}^\star(q,\beta_n,t) - \grave{\rho}_n\right) \right) = 0. \tag{B16}$$

To this end, note that for $b = 1, \ldots, B$, we have that

$$\begin{aligned} t \cdot \lambda_b \left(q, \bar{\psi} x_U - a(t)\grave{p}^\star(q,t)\right) &\le t \left(1 - F^{*,\psi_b}(|\psi_b| x_U - a(t)\grave{p}^\star_b(q,t))\right) \\ &= t \left(1 - F^{*,\psi_b}\left(|\psi_b| x_U - a^{*,|\psi_b|}(t) \frac{a(t)}{a^{*,|\psi_b|}(t)} \grave{p}^\star_b(q,t)\right)\right) \end{aligned} \tag{B17}$$

Moreover, since by Lemma 1, F^{*,ψ_b} is in the Weibull domain of attraction, then by Proposition 1, then for $c > 0$, we obtain

$$t\left(1 - F^{*,\psi_b}(|\psi_b| x_U - a^{*,|\psi_b|}(t) \cdot c\right) \to c^\alpha. \tag{B18}$$

However, by Lemma B1 and for any bundle b such that $|\psi_b| \ge 2$, we have $\left(a(t)/a^{*,|\psi_b|}(t)\right) \grave{p}^\star_b(q,t) \to 0$ as $t \to \infty$. Therefore, for any b such that $|\psi_b| \ge 2$ we have

$$t \cdot \lambda_b \left(q, |\psi_b| x_U - a(t)\grave{p}^\star_b(q,t)\right) \to 0 \quad \text{as } t \to \infty, \tag{B19}$$

which establishes (B15) and therefore (B16). Also note that this establishes (B13) for $|\psi_b| \ge 2$.

Next, similar to the single-item case (Abdallah and Reed 2025a), it follows from the FOC of the optimization problem in (B16) and invoking the von-mises condition (19), that

$$\frac{\grave{\rho}^\star_n(q,t)}{\Delta \grave{J}^\star(q,\beta_n,t)} \to \frac{\alpha}{\alpha+1} \quad \text{as } t \to \infty. \tag{B20}$$

Moreover, since from Lemma B1 we have $\limsup_{t>0} \Delta \grave{J}^\star(q,\beta_n,t) < \infty$, then after some algebra and similar to the single-item case, we obtain

$$\lim_{t\to\infty} \left(t(1 - F(x_U - a(t)\grave{\rho}^\star(q,t))) \left(\Delta \grave{J}^\star(q,\beta_n,t) - \grave{\rho}^\star_n(q,t)\right) - \frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left(\Delta \grave{J}^\star(q,\beta_n,t)\right)^{\alpha+1} \right) = 0, \tag{B21}$$

which together with (B16) yields

$$\lim_{t\to\infty}\left(t\grave{\Pi}^\star(q,N,t)-\sum_{n=1}^{N}\frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\Delta\grave{J}^\star(q,\beta_n,t)\right)^{\alpha+1}\right)=0. \tag{B22}$$

On the other hand, note that differentiating $\grave{J}^\star(q,t)$ directly with respect to t, we obtain

$$\begin{aligned}\frac{\partial\grave{J}^\star(q,t)}{\partial t}&=\frac{1}{a(t)}\frac{\partial\hat{J}^\star(q,t)}{\partial t}-\frac{a'(t)}{a(t)}\grave{J}^\star(q,t)\\&=\frac{1}{t}\left(-t\grave{\Pi}^\star(q,N,t)-\frac{ta'(t)}{a(t)}\grave{J}^\star(q,t)\right).\end{aligned} \tag{B23}$$

We are now ready to show by induction that $\grave{J}^\star(q,t)\to\sum_{n=1}^{N}w_{q_n}^{(\alpha+1)/\alpha}(\alpha)$ as $t\to\infty$. The statement is true for $\|q\|_1=0$, now fix $l\geq 1$ and suppose the satement is true for all $\|q\|_1<=l-1$ and consider q such that $\|q\|_1=l$.

Note that after some algebra, we can rewrite (B23) as

$$\frac{\partial\grave{J}^\star(q,t)}{\partial t}=\frac{1}{\alpha t}\left(\grave{\epsilon}(t)+g\left(\grave{J}^\star(q,t)\right)\right). \tag{B24}$$

where

$$\begin{aligned}\grave{\epsilon}(t)=&\left(\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\sum_{n=1}^{N}\left(\grave{J}^\star(q,t)-\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)+\left(w_{q_n}^{(\alpha+1)/\alpha}(\alpha)-w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right)\right)^{1+\alpha}-\alpha t\grave{\Pi}^\star(q,N,t)\right)\\&+\grave{J}^\star(q,t)\left(-\alpha\frac{ta'(t)}{a(t)}-1\right)\end{aligned} \tag{B25}$$

$$g(x)=x-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\sum_{n=1}^{N}\left(x-\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)+\left(w_{q_n}^{(\alpha+1)/\alpha}-w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right)\right)^{1+\alpha}. \tag{B26}$$

Regarding $\grave{\epsilon}(t)$, note that by the inverse value theorem and von-Mises condition (19)

$$-t\frac{\partial a(t)/\partial t}{a(t)}\to\frac{1}{\alpha}\quad\text{as } t\to\infty.$$

One the other hand, by the induction hypothesis, then for $n=1,\dots,N$, we have

$$\grave{J}(q-e_n)\to\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)-\left(w_{q_n}^{(\alpha+1)/\alpha}(\alpha)-w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right)\quad\text{as } t\to\infty,$$

which together with (B22), yields

$$\grave{\epsilon}(t)\to 0\quad\text{as } t\to\infty.$$

Turning our attention to $g\left(\grave{J}^\star(q,t)\right)$, note that since component pricing (additive bundle pricing) is a feasible policy, it follows from Abdallah and Reed (2025a) that $\limsup_{t>0}\grave{J}^\star(q,t)\leq\sum_{n=1}^{N}w_{q_n}^{(\alpha+1)/\alpha}(\alpha)$. Moreover, by the induction hypothesis and (B8), for a sufficiently large t, we have $\grave{J}^\star(q,t)>\gamma(q,t)$ where

$$\gamma(q,t)=\max_{n=1,\dots,N}\left\{\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)-\left(w_{q_n}^{(\alpha+1)/\alpha}(\alpha)-w_{q_n-1}^{(\alpha-1)/\alpha}(\alpha)\right)\right\}\leq\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)$$

.

However, note that for $x\in[\gamma(q,t),\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)]$ and for any $n=1,\dots,N$, we have

$$w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\leq x-\sum_{k=1}^{N}w_{q_k}^{(\alpha+1)/\alpha}(\alpha)+w_{q_n}^{(\alpha+1)/\alpha}(\alpha)\leq w_{q_n}^{(\alpha+1)/\alpha}(\alpha).$$

It now follows from the properties of the recursion defining w_{q_n} and the uniqueness of the solution of v_{q_n} (see (Abdallah and Reed 2025a)) that

$$\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(x-\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha)+\left(w_{q_n}^{(\alpha+1)/\alpha}(\alpha)-w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right)\right)^{1+\alpha}$$
$$\leq x-\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha)+w_{q_n}^{(\alpha+1)/\alpha}(\alpha),$$

and summing over n, we obtain

$$\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\sum_{n=1}^{N}\left(x-\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha)+\left(w_{q_n}^{(\alpha+1)/\alpha}(\alpha)-w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right)\right)^{1+\alpha}$$
$$\leq x+(N-1)\left(x-\sum_{k=1}^{N} w_{q_n}^{(\alpha+1)/\alpha}(\alpha)\right),$$

and therefore for $x\in(\gamma(q,t),\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha))$, we have $g(x)>0$. Similarly, we can show that for $x\in\left(\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha),\infty\right)$, we have $g(x)<0$. Moreover, it can be verify that for $x=\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha)$ we have $g(x)=0$ and hence $x=\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha)$ is a unique root of $g(x)$ on $(\gamma(q,t),\infty)$. Therefore, since $\grave{\epsilon}(t)\to 0$ as $t\to\infty$, it can be shown using the comparison theorem (Arnold 1992) that $\grave{J}^\star(q,t)\to\sum_{k=1}^{N} w_{q_k}^{(\alpha+1)/\alpha}(\alpha)$ as $t\to\infty$.

To establish (B12), note that from (B20) together with (B11), we have that as $t\to\infty$

$$\grave{p}^\star_{\beta_n}(q,t)\to\frac{\alpha}{\alpha+1}\left(w_{q_n}^{(\alpha+1)/\alpha}(\alpha)-w_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right)=w_{q_n}^{1/\alpha}(\alpha) \tag{B27}$$

where the last equality is due to (15) and the fact that $w_{q_n}(\alpha)=v_{q_n}((\alpha+1)/\alpha)$.

Finally, (B13) for $|\psi_b|=1$, follows from (B12) together with Propoistion 1 for the Weibull domain of attraction. □

Proof of Corollary 3. From Theorem B1 and its analogous version for the pure bundling policy, we obtain

$$J^\star(q,t)-J^{\star,\mathsf{PB}}(q,t) = \left(w_{q_N}(N\alpha)^{(N\alpha+1)/N\alpha}\right)a^{*,N}(\lambda t)-\left(Nw_{q_N}(\alpha)^{(\alpha+1)/\alpha}\right)a(\lambda t)$$
$$+\ o(a^{*,N}(\lambda t))+o(a(\lambda t)).$$

and the remaining results follow properties of slowly varying functions, which for $N\geq 2$

$$\frac{a(\lambda t)}{a^{*,N}(\lambda t)}\to 0 \quad \text{as } \lambda t\to\infty.$$

□

B.1. Proofs for Bundle Size Pricing: Weibull Domain of Attraction

We first state a similar result to Lemma A1 whose proof is similar and so is skipped.

Lemma B2. *For each $t\geq 0$ and $q\in\mathbb{N}^N$ such that $|\mathcal{N}(q)|\geq 1$, there exists a $\rho^\star(q,t)$ such that $\rho^\star_k(q,t)\geq\sum_{b\in\mathcal{B}_{|\mathcal{N}(q)|}}\Delta J^{\star,\mathsf{BSP}}(q,b,t)/|\mathcal{B}_k(q)|$ for each $k\leq|\mathcal{N}(q)|$.*

Now for each $q \in \mathbb{N}^N$ and $t \geq 0$ set

$$\hat{J}^{\star,\mathsf{BSP}}(q,t) \; = \; \left(\sum_{n=1}^{N} q_n\right) x_U - J^{\star,\mathsf{BSP}}(q,t) \text{ and } \hat{\rho}^{\star}_k(q,t) \; = \; kx_U - \rho^{\star}_k(q,t). \tag{B28}$$

Also set $\Delta\hat{J}^{\star,\mathsf{BSP}}(q,b,t) = \hat{J}^{\star,\mathsf{BSP}}(q,t) - \hat{J}^{\star,\mathsf{BSP}}(q-H_b,t) = kx_U - \Delta J^{\star,\mathsf{BSP}}(q,b,t)$, where $\hat{J}^{\star}(0,\cdot) = 0$.

The HJB equations (36)-(38) may be written as

$$-\frac{\partial \hat{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} \; = \; \sup_{\hat{\rho}\in[0,x_U]^N} \left\{ \sum_{k=1}^{|\mathcal{N}(q)|} \lambda_k(q, kx_U - \hat{\rho}) \left(\frac{1}{|\mathcal{B}_k(q)|} \sum_{b\in\mathcal{B}_k(q)} \Delta\hat{J}^{\star,\mathsf{BSP}}(q,b,t) - \hat{\rho}_k \right) \right\}, \; \forall t \geq 0, \; |\mathcal{N}(q)| \geq 1, \tag{B29}$$

$$\hat{J}^{\star,\mathsf{BSP}}(q,0) \; = \; \left(\sum_{n=1}^{N} q_n\right) x_U, \quad q \in \mathbb{N}^N, \tag{B30}$$

$$\hat{J}^{\star,\mathsf{BSP}}(0,t) \; = \; 0, \quad t \geq 0. \tag{B31}$$

Next, we state a Lemma that is similar to Lemma B1, and so its proof is skipped.

Lemma B3. *For any $q \in \mathbb{N}^N$ we have*

1. *for $|\mathcal{N}(q)| \geq 1$, then for every $k = 1, \ldots, |\mathcal{N}(q)|$, we have*

$$\limsup_{t>0} \frac{\hat{\rho}^{\star}_k(q,t)}{a(\lambda t)} \leq \limsup_{t>0} \frac{1}{|\mathcal{B}_k(q)|} \frac{\sum_{b\in\mathcal{B}(q)} \Delta\hat{J}^{\star}(q,b,t)}{a(\lambda t)} < \infty. \tag{B32}$$

2. *for $|\mathcal{N}(q)| \geq 2$, then for every $k = 2, \ldots, |\mathcal{N}(q)|$,*

$$\limsup_{t>0} \frac{\hat{\rho}^{\star}_k(q,t)}{a^{*,|\psi_b|}(\lambda t)} = \limsup_{t>0} \frac{1}{|\mathcal{B}_k(q)|} \frac{\sum_{b\in\mathcal{B}(q)} \Delta\hat{J}^{\star,\mathsf{BSP}}(q,b,t)}{a^{*,|\psi_b|}(\lambda t)} = 0. \tag{B33}$$

3. *Letting β_n denote singleton bundle containing item n, then for $|\mathcal{N}(q)| \geq 1$, we have*

$$\liminf_{t>0} \frac{1}{|\mathcal{B}_1(q)|} \frac{\sum_{n:\beta_n\in\mathcal{B}_1(q)} \Delta\hat{J}^{\star}(q,\beta_n,t)}{a(\lambda t)} > 0. \tag{B34}$$

We now state and prove an equivalent statement to Theorem 4.

Theorem B2. *If F is in the Weibull domain of attraction with index $\alpha > 0$ and satisfies the von-Mises condition (19), then for each $q \in \mathbb{N}^N$ with $\mathcal{N}(q) \geq 1$ the optimal BSP value function is given by*

$$\lim_{\lambda t\to\infty} \hat{J}^{\star,\mathsf{BSP}}(q,t) = \tilde{w}_q^{(\alpha+1)/\alpha}(\alpha). \tag{B35}$$

Moreover, the only relevant price is the price of the bundles of size 1 where,

$$\lim_{\lambda t\to\infty} \hat{\rho}^{\star}_1(q,t) = \frac{1}{N^{1/(\alpha+1)}} \tilde{w}_q^{1/\alpha}(\alpha) \tag{B36}$$

and the purchasing probabilities for bundle sizes $k = 1, \ldots, N$ are given by

$$\lim_{\lambda t\to\infty} t\lambda_k(q,\rho^{\star}) = \begin{cases} \frac{\tilde{w}_q(\alpha)}{N^{\alpha/(\alpha+1)}} & \text{for } k = 1, \\ 0 & \text{for } k = 2, \ldots, N, \end{cases} \tag{B37}$$

where $\tilde{w}_q(\alpha) = \tilde{v}_q((\alpha+1)/\alpha)$.

Proof of Theorem B2. It is sufficient to establish the result for $\lambda = 1$. Now fix $q \in \mathbb{N}^N$ and assume without loss of generality that $q_n > 0$ for every $n = 1, \dots, N$; otherwise, those items can be ignored. It follows from the (modified) HJB equation (B29) that for a sufficiently large $t > 0$, $\grave{\rho}^\star(q,t)$ attains the supremum of

$$\grave{\Pi}^{\star,\mathsf{BSP}}(q,N,t) = \sup_{\grave{\rho}\in[0,\infty)^N} \left\{ \sum_{k=1}^K \lambda_k(q, kx_U - \grave{\rho}) \left(\frac{1}{|\mathcal{B}_k(q)|} \sum_{b\in\mathcal{B}_k(q)} \Delta \grave{J}^{\star,\mathsf{BSP}}(q,b,t) - \grave{\rho}_k \right) \right\}$$
$$= \sum_{k=1}^K \lambda_k(q, kx_U - \grave{\rho}^\star(q,t)) \left(\frac{1}{|\mathcal{B}_k(q)|} \sum_{b\in\mathcal{B}_k(q)} \Delta \grave{J}^{\star,\mathsf{BSP}}(q,b,t) - \grave{\rho}^\star_k(q,t) \right). \quad \text{(B38)}$$

Now similar to the proof of Theorem B1, we can establish that for $k \geq 2$

$$t\lambda_k\left(q, kx_U - \grave{\rho}^\star(q,t)\right) \to 0 \quad \text{as } t \to \infty,$$

which implies (B37) for $k \geq 2$. It now follows that for $k = 1$,

$$t\left(\lambda_1(q, kx_U - \grave{\rho}^\star(q,t)) - P\left(\max\{X_1,\dots,X_N\} > x_u - a(t)\grave{\rho}^\star_1(q,t)\right)\right) \to 0 \quad \text{as } t \to \infty,$$

On the other hand, given the i.i.d. assumption, then

$$P\left(\max\{X_1,\dots,X_N\} > x_U - a(t)\grave{\rho}^\star_1(q,t)\right) = \left(1 - F\left(x_U - a(t)\grave{\rho}^\star_1(q,t)\right)\right) \sum_{n=0}^{N-1} \left(F\left(x_U - a(t)\grave{\rho}^\star_1(q,t)\right)\right)^n.$$

and since $a(t)\grave{\rho}^\star_1(q,t) \to 0$ as $t \to \infty$, we have

$$\frac{P\left(\max\{X_1,\dots,X_N\} > x_U - a(t)\grave{\rho}^\star_1(q,t)\right)}{N\left(1 - F(x_U - a(t)\grave{\rho}^\star_1(q,t)\right)} \to 1 \quad \text{as } t \to \infty,$$

Therefore, together with (B38) and Lemma B3 property 1), we obtain

$$\lim_{t\to\infty} t \left(\grave{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \sup_{\grave{\rho}_1\geq 0} \left\{ \left(1 - F(x_U - a(t)\grave{\rho}_1)\right) \left(\sum_{n=1}^N \Delta \grave{J}^{\star,\mathsf{BSP}}(q,\beta_n,t) - N\grave{\rho}_1 \right) \right\} \right) = 0. \quad \text{(B39)}$$

Next note that for a sufficiently large t, we have that the supremum inside the lefthand side of (B39) is attained by $\grave{\rho}^\star_1(q,t)$ that satisfies the FOC. After some algebra and invoking the von Mises condition (19), it follows that

$$\frac{N\grave{\rho}^\star_1(q,t)}{\sum_{n=1}^N \Delta \grave{J}^{\star,\mathsf{BSP}}(q,\beta_n,t)} \to \frac{\alpha}{\alpha+1} \quad \text{as } t \to \infty. \quad \text{(B40)}$$

However, since by Lemma B3, $\limsup_{t>0} \sum_{n=1}^N \Delta \grave{J}^{\star,\mathsf{BSP}} < \infty$, then after some algebra and similar to (B21), we obtain

$$\lim_{t\to\infty} \left(t\grave{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \frac{N}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left(\frac{\sum_{n=1}^N \Delta \grave{J}^{\star,\mathsf{BSP}}(q,\beta_n,t)}{N} \right)^{\alpha+1} \right) = 0. \quad \text{(B41)}$$

.

Now note that, differentiating $\grave{J}^{\star,\mathsf{BSP}}(q,t)$ directly with respect to t yields

$$\frac{\partial \grave{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} = \frac{1}{a(t)} \frac{\partial \hat{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} - \frac{a'(t)}{a(t)} \grave{J}^{\star,\mathsf{BSP}}(q,t)$$
$$= \frac{1}{t}\left(-t\grave{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \frac{ta'(t)}{a(t)} \grave{J}^{\star,\mathsf{BSP}(q,t)}(q,t) \right). \quad \text{(B42)}$$

We are now ready to show by induction that $\grave{J}^\star(q,t) \to \tilde{w}_q^{(\alpha+1)/\alpha}$ as $t\to\infty$. The statement is true for $||q||_1 = 0$. Fix $l \geq 2$ and suppose the statement is true for all $||q||_1 \leq l-1$ and consider an arbitrary $q \in \mathbb{N}^N$ such that $||q||_1 = l$.

Observe that after some algebra, we can rewrite (B42) as

$$\frac{\partial \grave{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} = \frac{1}{\alpha t}\left(\grave{\epsilon}(t) + g\left(\grave{J}^{\star,\mathsf{BSP}}(q,t)\right)\right). \tag{B43}$$

where

$$\begin{aligned}\grave{\epsilon}(t) &= \left(N\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\grave{J}^{\star,\mathsf{BSP}}(q,t) - \frac{\sum_{n=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}}{N}\right)^{1+\alpha} - \alpha t \grave{\Pi}^{\star,\mathsf{BSP}}(q,N,t)\right)\\ &+ \grave{J}^{\star,\mathsf{BSP}}(q,t)\left(-\alpha\frac{ta'(t)}{a(t)} - 1\right) \qquad\qquad (B44)\\ g(x) &= x - N\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(x - \frac{\sum_{n=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}}{N}\right)^{1+\alpha}. \qquad\qquad (B45)\end{aligned}$$

Regarding $\grave{\epsilon}(t)$, note that by the inverse value theorem and the von-Mises condition (19)

$$-t\frac{a'(t)}{a(t)} \to \frac{1}{\alpha} \quad \text{as } t\to\infty, \tag{B46}$$

Furthermore, by the induction hypothesis and (B41), have that for $n = 1,\dots,N$, then

$$\grave{J}^{\star,\mathsf{BSP}}(q-e_n,t) \to \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha} \quad \text{as } t\to\infty. \tag{B47}$$

It now follows from (B41), (B46) and (B47) that

$$\grave{\epsilon}(t) \to 0 \quad \text{as } t\to\infty.$$

Turning our attention to $g\left(\grave{J}^{\star,\mathsf{BSP}}(q,t)\right)$, note that by the induction hypothesis and (B34). for a sufficiently large t, we have $\grave{J}^\star(q,t) > \max_n\{\tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}\}$ and hence

$$\grave{J}^\star(q,t) > \frac{\sum_{n=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}}{N}.$$

On the other hand, from the properties of the recursion defining $\tilde{w}_q$, it follows that for $x \in [\sum_{k=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}/N, \infty)$, $g(x)$ has a unique root given by $x = \tilde{w}_q^{(\alpha+1)/\alpha}$. Moreover, by the uniqueness of the root, it is straightforward to verify that $g(x) > 0$ for $x \in [\sum_{k=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}/N, \tilde{w}_q^{(\alpha+1)/\alpha})$ and $g(x) < 0$ for $x \in (\tilde{w}_q^{(\alpha+1)/\alpha}, \infty)$. Therefore, since $\grave{\epsilon}(t) \to 0$ as $t\to\infty$, then again using the comparison theorem (Arnold 1992) it can be shown that $\grave{J}^{\star,\mathsf{BSP}}(q,t) \to \tilde{w}_q^{(\alpha+1)/\alpha}$ as $t\to\infty$.

To establish (B36), note that from (B40) together with (B35), we have that as $t\to\infty$

$$\grave{\rho}_1^\star(q,t) \to \frac{\alpha}{\alpha+1}\left(\tilde{w}_q^{(\alpha+1)/\alpha}(\alpha) - \frac{\sum_{n=1}^N w_{q-e_n}^{(\alpha+1)/\alpha}(\alpha)}{N}\right) = \frac{1}{N^{1/(\alpha+1)}} w_{q_n}^{1/\alpha}(\alpha) \tag{B48}$$

where the last equality is due to (17) and the fact that $\tilde{w}_{q_n}(\alpha) = \tilde{v}_q((\alpha+1)/\alpha)$.

Finally, (B37) for $k = 1$, follows from (B36) together with (1) for the Weibull domain of attraction. □

Proof of Proposition 3. It suffices to establish the case for $\lambda = 1$. Assume that F satisfies the von Mises condition (19) then F is absolutely continuous on (x_0, x_U). This then implies that $1 - F(F^{-1}(1-r)) = r$ for any $r < 1 - F(x_0) = r_0$. Now, let $\rho \in \mathcal{U}^{\mathsf{BSP}}$ where $\rho_1(q,t) = F^{-1}(1 - 1/(N^{\alpha/(\alpha+1)})\left(\tilde{w}_q(\alpha)/t\right)$ and $\rho_k(q,t) = kx_U$ for $k \geq 1$. Without loss of generality, we assume $q \in \mathbb{N}_+^N$ with $q_N \geq 1$, i.e., $|\mathcal{N}(q)| = N$ since otherwise the stocked-out products can be ignored. It then follows by standard theory that for each $q \in \mathbb{N}_+^N$ and $t > \tilde{w}_q(\alpha)/(r_0 N^{\alpha/(\alpha+1)})$, we have

$$-\frac{1}{a(t)}\frac{\partial \hat{J}_\rho^{\mathsf{BSP}}(q,t)}{\partial t} = P\left(\max\{X_1,\ldots,X_N\} > x_U - a(t)\grave{\rho}_1(q,t)\right)\left(\frac{\sum_{n=1}^N \Delta \grave{J}_\rho^{\mathsf{BSP}}(q,\beta_n,t)}{N} - \grave{\rho}_1(q,t)\right).$$

Denote by $\grave{\Pi}^{\mathsf{BSP}}(q,N,t) = -1/a(t)\left(\partial \hat{J}_\rho^{\mathsf{BSP}}(q,t)/\partial t\right)$, then similar to (B39) and after some algebra, we have

$$\lim_{t\to\infty} t\left(\grave{\Pi}^{\mathsf{BSP}}(q,N,t) - \left(1 - F(x_U - a(t)\grave{\rho}_1(q,t))\right)\left(\sum_{n=1}^N \Delta \grave{J}_\rho^{\mathsf{BSP}}(q,\beta_n,t) - N\grave{\rho}_1(q,t)\right)\right) = 0, \tag{B49}$$

which similar to (B41) yields

$$\lim_{t\to\infty}\left(t\grave{\Pi}^{\mathsf{BSP}}(q,N,t) - N\grave{\rho}_1(q,t)^{\alpha+1}\left(\frac{\sum_{n=1}^N \Delta \grave{J}_\rho^{\mathsf{BSP}}(q,\beta_n,t)}{N\grave{\rho}_1(q,t)} - 1\right)\right) = 0 \tag{B50}$$

However, since $\grave{\rho}_1(q,t) = \grave{\rho}_1^\star(q,t) - o(a(t))$ with $a(t) \to 0$ as $t \to \infty$, it follows that

$$\lim_{t\to\infty}\grave{\rho}_1(q,t) = \lim_{t\to\infty}\grave{\rho}_1^\star(q,t) = \frac{\tilde{w}_q^{1/\alpha}}{N^{1/(\alpha+1)}},$$

which, after some algebra, implies

$$\lim_{t\to\infty}\left(t\grave{\Pi}^{\mathsf{BSP}}(q,N,t) - \tilde{w}_q^{\alpha+1/\alpha}\left(N^{1/(\alpha+1)}\left(\grave{J}_\rho^{\mathsf{BSP}}(q,t) - \sum_{n=1}^N \grave{J}_\rho^{\mathsf{BSP}}(q-e_n,t)/N\right)\frac{1}{\tilde{w}_q^{1/\alpha}} - 1\right)\right) = 0 \tag{B51}$$

We are now ready to show by induction that $\grave{J}_\rho^{\mathsf{BSP}}(q,t) \to \tilde{w}_q^{(\alpha+1)/\alpha}$ as $t \to \infty$. The statement is true for $||q||_1 = 0$, now suppose it is true for all $||q||_1 \leq l-1$ for some $l \geq 2$ and consider an arbitrary $q \in \mathbb{N}^N$ such that $||q||_1 = l$.

Note that similar to (B42), we have

$$\frac{\partial \grave{J}_\rho^{\mathsf{BSP}}(q,t)}{\partial t} = \frac{1}{\alpha t}\left(-\alpha t\grave{\Pi}^{\mathsf{BSP}}(q,N,t) - \alpha\frac{ta'(t)}{a(t)}\grave{J}_\rho^{\mathsf{BSP}(q,t)}(q,t)\right), \tag{B52}$$

which can be written as

$$\frac{\partial \grave{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} = \frac{1}{\alpha t}\left(\grave{\epsilon}(t) + g\left(\grave{J}_\rho^{\mathsf{BSP}}(q,t)\right)\right). \tag{B53}$$

where

$$\begin{aligned}\grave{\epsilon}(t) &= \alpha\tilde{w}_q^{\alpha+1/\alpha}\left(N^{1/(\alpha+1)}\left(\grave{J}_\rho^{\mathsf{BSP}}(q,t) - \sum_{n=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}/N\right)\frac{1}{\tilde{w}_q^{1/\alpha}} - 1\right) - \alpha t\grave{\Pi}^{\star,\mathsf{BSP}}(q,N,t) \\ &\quad + \grave{J}_\rho^{\mathsf{BSP}}(q,t)\left(-\alpha\frac{ta'(t)}{a(t)} - 1\right) \end{aligned} \tag{B54}$$

$$g(x) = x - \tilde{w}_q^{\alpha+1/\alpha}\left(N^{1/(\alpha+1)}\left(x - \sum_{n=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}/N\right)\frac{1}{\tilde{w}_q^{1/\alpha}} - 1\right). \tag{B55}$$

Note that by (B46), (B51), and the induction hypothesis we have $\grave{\epsilon}(t) \to 0$ as $t \to \infty$. As for $g(x)$, and from the properties of $\tilde{w}_q(\alpha)$, it is straightforward to verify that $g(x)$ has a unique root $x = \tilde{w}_q^{(\alpha+1)/\alpha}$, where $g(\tilde{w}_q^{(\alpha+1)/\alpha}) = 0$. Meanwhile, for $x > \tilde{w}_q^{(\alpha+1)/\alpha}$, then $g(x) < 0$ and for $x \in [\sum_{n=1}^N \tilde{w}_{q-e_n}^{(\alpha+1)/\alpha}/N, \tilde{w}_q^{(\alpha+1)/\alpha})$, we have $g(x) > 0$. Therefore, again by the comparison theorem we have that $\grave{J}_\rho^{\mathsf{BSP}}(q,t) \to \tilde{w}_q^{(\alpha+1)/\alpha}$ as $t \to \infty$. □

C. Proofs of Section 7: Frechet Domain of Attraction

Let $\beta_n \in \{1,2,\ldots,B\}$ denote the singleton bundle that only contains item n, that is $\psi_{\beta_n} = \{n\}$. Given $q \in \mathbb{N}^N$, denote by $\mathcal{B}(q) = \{b \in \mathcal{B} : q_n \geq 1 \text{ for all } n \in \psi_b\}$ be the set of bundles that are not out of stock and let $\bar{(\cdot)} := (\cdot)/a(\lambda t)$. We establish the following intermediate Lemma and then state and prove an equivalent statement to Theorem 5.

Lemma C1. *Given an optimal policy $p^\star \in \mathcal{U}$, then for every $q \in \mathbb{N}^N$ and bundle $b \in \mathcal{B}(q)$,*

$$\limsup_{\lambda t > 0} \bar{p}^\star_b(q,t) < \infty \tag{C1}$$

Proof of Lemma C1. It is sufficient to establish the case for $\lambda = 1$. Let $p^\star \in \mathcal{U}$ be an optimal policy and fix $q \in \mathbb{N}^N$ such that $|\mathcal{N}(q)| \geq 1$. Suppose for a contradiction that there exists a bundle $b \in \mathcal{B}(q)$

$$\limsup_{t>0} \bar{p}^\star_b(q,t) = \infty,$$

and denote by $\sigma_n(q) \subseteq \mathcal{B}(q)$ the set of bundles that contain an item $n \in \mathcal{N}(q)$. Then, by the subadditivity and monotonicity of the pricing policies over bundles, there exists at least one item n such that for every $b \in \sigma_n(q)$

$$\limsup_{t>0} \bar{p}^\star_b(q,t) = \infty. \tag{C2}$$

Fix this item n and let $\mathcal{B}^\star(q) = \{b \in \mathcal{B}(q) : \limsup_{t>0} \bar{p}^\star_b(q,t) < \infty\}$. Note that $\mathcal{B}^\star(q) \cap \sigma_n(q) = \phi$.

We now show that (C2) yields

$$\begin{aligned} &\liminf_{t>0} \sum_{b=1}^{B} t\lambda_b(q, a(t)\bar{p}^\star(q,t)) \left(\bar{p}^\star_b(q,t) - \Delta \bar{J}^\star(q,b,t)\right) \\ &= \liminf_{t>0} \sum_{b \in \mathcal{B}^\star(q)} t\lambda_b(q, a(t)\bar{p}^\star(q,t)) \left(\bar{p}^\star_b(q,t) - \Delta \bar{J}^\star(q,b,t)\right). \end{aligned} \tag{C3}$$

To see this, note that

$$\begin{aligned} \sum_{b \in \mathcal{B}(q)\setminus\mathcal{B}^\star(q)} t\lambda_b(q, a(t)\bar{p}^\star_b(q,t)) \left(\bar{p}^\star_b(q,t) - \Delta \bar{J}^\star(q,b,t)\right) &\leq \sum_{b \in \mathcal{B}(q)\setminus\mathcal{B}^\star(q)} t\left(1 - F^{*,|\psi_b|}(a(t)\bar{p}^\star_b(q,t))\right) \bar{p}^\star_b(q,t) \\ &\leq \sum_{b \in \mathcal{B}(q)\setminus\mathcal{B}^\star(q)} Nt\left(1 - F\left(a(t)\bar{p}^\star_b(q,t)/N\right)\right) \bar{p}^\star_b(q,t). \end{aligned} \tag{C4}$$

However, from Proposition 1 for the Frechet domain of attraction then for any $0 < c < \infty$, we have

$$t\left(1 - F(a(t)c)\right)c = c^{(1-\alpha)} \quad \text{as } t \to \infty, \tag{C5}$$

and since $\alpha > 1$ then for each $b \in \mathcal{B}(q)\setminus\mathcal{B}^\star(q)$

$$\liminf_{t>0} t\left(1 - F(a(t)\bar{p}^\star(q,t)/N)\right) \bar{p}^\star_b(q,t) = 0, \tag{C6}$$

which implies (C3).

Next, consider, without loss of generality, an alternative pricing vector $\breve{p}$ such that $\breve{p}_b(q,t)=\bar{p}^\star_b(q,t)$ for $b\in\mathcal{B}^\star(q)$ and $\breve{p}_b(q,t)=\infty$ otherwise. We have that

$$\begin{aligned}&\liminf_{t>0}\sum_{b=1}^{B} t\lambda_b(a(t)\breve{p}(q,t))\left(\breve{p}_b(q,t)-\Delta\bar{J}^\star(q,b,t)\right)\\ &=\liminf_{t>0}\sum_{b\in\mathcal{B}^\star(q)} t\lambda_b(a(t)\bar{p}^\star(q,t))\left(\bar{p}^\star_b(q,t)-\Delta\bar{J}^\star(q,b,t)\right)\end{aligned}\tag{C7}$$

Denote by $\sigma^\star_n(q)=\{b\in\sigma_n(q):\text{there exits } b'\in\mathcal{B}^\star(q)\cup\{0\}\text{ such that }\psi_b\backslash\{n\}=\psi_{b'}\}$ where 0 is the empty bundle that represents a no-purchase. In words, each bundle in $\sigma^\star_n(q)$ has an adjacent bundle that is either empty or in $\mathcal{B}^\star(q)$ that consists of the same items except for item n. Also denote by $\zeta(b)\in\mathcal{B}^\star(q)\cup\{0\}$ this adjacent bundle where $\psi_{\zeta(b)}=\psi_b\backslash\{n\}$.

Now consider the another alternative pricing vector p such that

$$\bar{p}_b(q,t)\;=\begin{cases}2\bar{J}^\star(q,t)+\breve{p}_{\zeta(b)}(q,t) & \text{if } b\in\sigma^\star_n(q)\\ \breve{p}_b(q,t) \ \ \text{if } b\in\mathcal{B}\backslash\sigma^\star_n(q)\end{cases}\tag{C8}$$

where $p_0(q,t)=0$.

It is straightforward to verify that p is consistent with the subadditivity and monotonicity rules and, hence, admissible.

Note that for $b\in\mathcal{B}(q)\backslash\left(\mathcal{B}^\star(q)\cup\sigma^\star_n(q)\right)$, we have $\bar{p}(q,t)=\breve{p}(q,t)=\infty$ and hence the pruchasing probabilities are given by $\mathbb{E}[y^\star_b(q,t,a(t)\bar{p})]=\mathbb{E}[y^\star_b(q,t,a(t)\breve{p})]=0$. Therefore, it is sufficient to compare the purchasing probabilities for $b\in\sigma^\star_n(q)$ and $b\in\mathcal{B}^\star(q)$.

Case I $-$ $E=\{X_n<2J^\star(q,t)\}$: Consider the event E, then for any $b\in\sigma^\star_n(q)$ we have

$$\sum_{k\in\psi_b}X_k-2J^\star(q,t)-a(t)\breve{p}_{\zeta(b)}(q,t)<\sum_{k\in\psi_{\zeta(b)}}X_k-a(t)\breve{p}_{\zeta(b)}(q,t).$$

Therefore $y^\star_b(q,X,a(t)\bar{p})=y^\star_b(q,X,a(t)\breve{p})=0$, for every $b\in\sigma^\star_n(q)$. Similarly, for every $b\in\mathcal{B}^\star(q)$, $y^\star_b(q,X,a(t)\bar{p})=y^\star_b(q,X,a(t)\breve{p})$ given E.

Noting that $\lambda_b(q,a(t)\bar{p}(q,t))=\lambda\mathbb{E}\left[y^\star_b(q,X,a(t)\bar{p}(q,t))\right]$, then by conditioning on E, we obtain

$$\begin{aligned}\sum_{b\in\mathcal{B}}t\lambda_b(q,a(t)\bar{p}(q,t)|E)\left(\bar{p}_b(q,t)-\Delta\bar{J}^\star(q,b,t)\right)&=\sum_{b\in\mathcal{B}^\star(q)}t\lambda_b(q,a(t)\breve{p}(q,t)|E)\left(\breve{p}_b(q,t)-\Delta\bar{J}^\star(q,b,t)\right)\\ &=\sum_{b\in\mathcal{B}^\star(q)}t\lambda_b(q,a(t)\breve{p}(q,t))\left(\breve{p}_b(q,t)-\Delta\bar{J}^\star(q,b,t)\right),\end{aligned}\tag{C9}$$

where the last inequality holds since the purchasing decisions under $\breve{p}$ is independent of X_n. In particular, since by construction $\breve{p}_b(q,t)=\infty$ for all $b\in\sigma_n(q)$.

Case II–$E^c=\{X_n\geq 2J^\star(q,t)\}$: Similar to the arguments in Case I, when $\{X_n\geq 2J^\star(q,t)\}$, we have $y^\star_b(q,X,a(t)\bar{p})=0$ given E^c for every $b\in\mathcal{B}^\star$. However, $y^\star_b(q,X,a(t)\bar{p})=y^\star_{\zeta(b)}(q,X,a(t)\breve{p})$ almost surely for every $b\in\sigma^\star_n(q)$ since adding item n increases the utility. Note that when if $\{X_n=2J^\star(q,t)\}$, the customer is indifferent between $b\in\sigma_n(q)^\star$ and $\zeta(b)$. However, by the tie-breaking rule, the customer prefers the larger bundle and hence prefers $b\in\sigma^\star_n(q)$.

Next by conditioning on E^c and recalling that that the purchasing decisions under $\breve{p}\in\mathcal{P}$ are independent of X_n, we obtain

$$\begin{aligned}
&\sum_{b=1}^{B} t\lambda_b(q, a(t)\bar{p}(q,t)|E^c) \left(\bar{p}_b(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \\
&= \sum_{b\in\sigma_n^\star(q)} t\lambda_{\zeta(b)}(q, a(t)\breve{p}(q,t)) \left(2\bar{J}(q,t) + \breve{p}_{\zeta(b)}(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \\
&= \sum_{b\in\mathcal{B}^\star(q)\cup\{0\}} t\lambda_b(q, a(t)\breve{p}(q,t)) \left(2\bar{J}(q,t) + \breve{p}_b(q,t) - \Delta\bar{J}^\star(q,\zeta^{-1}(b),t)\right) \qquad \text{(C10)}
\end{aligned}$$

where $\zeta^{-1}(b) \in \sigma_n^\star(q)$ is such that $\psi_{\zeta^{-1}(b)} = \psi_b \cup \{n\}$ and the last inequality is due to the fact that $\bar{J}(q,t) \geq \Delta\bar{J}(q,b,t)$ for all $b \in \mathcal{B}$, $q \in \mathbb{N}^+$ and $t > 0$.

Putting (C9) and (C10) together, we have

$$\begin{aligned}
&\sum_{b=1}^{B} t\lambda_b(q, a(t)\bar{p}(q,t)) \left(\bar{p}_b(q,t) - \Delta\bar{J}^\star(q,b,t)\right) - \sum_{b=1}^{B} t\lambda_b(a(t)\breve{p}(q,t)) \left(\breve{p}_b(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \\
&= (1 - F(2J^\star(q,t))) \left(\sum_{b\in\mathcal{B}(q)^\star\cup\{0\}} t\lambda_b(a(t)\breve{p}(q,t)) \left(2\bar{J}^\star(q,t) + \Delta\bar{J}^\star(q,b,t) - \Delta\bar{J}^\star(q,\zeta^{-1}(b),t)\right) \right) \\
&> (1 - F(2J^\star(q,t))) \left(t\bar{J}^\star(q,t)\right) \qquad \text{(C11)}
\end{aligned}$$

where the last inequality is due to the fact that $\sum_{b\in\mathcal{B}(q)^\star\cup\{0\}} \lambda_b(a(t)\breve{p}(q,t)) = 1$ and $\bar{J}^\star(q,t) \geq \Delta\bar{J}^\star(q,b,t)$ for all $b \in \mathcal{B}$.

Next note that

$$\liminf_{t>0} \bar{J}^\star(q,t) \;=\; \kappa(q) \;>\; 0 \quad \text{and} \quad \limsup_{t>0} \bar{J}^\star(q,t) \;=\; \eta(q) < \infty,$$

where the inequalities follow from Abdallah and Reed (2025a). Hence, since $1/t = 1 - F(a(t))$ and for each $c > 0$, $(1 - F(cx))/(1 - F(x)) \to c^{-\alpha}$ as $x \to \infty$ uniformly for all c in a compact interval, it follows that

$$\begin{aligned}
\liminf_{t>0} (1 - F(2J^\star(q,t))) \left(t\bar{J}^\star(q,t)\right) &\geq \liminf_{t>0} \bar{J}^\star(q,t) \liminf_{t>0} t\left(1 - F(2a(t)\bar{J}^\star(q,t))\right) \\
&= \gamma(q) \;>\; 0, \qquad \text{(C12)}
\end{aligned}$$

where $\gamma(q) = \kappa(q)(2\eta(q))^{-\alpha}$.

Putting (C7), (C11), and (C12) together, yields

$$\liminf_{t>0} \sum_{b=1}^{B} t\lambda_b(a(t)\bar{p}(q,t)) \left(\bar{p}_b(q,t) - \Delta\bar{J}^\star(q,b,t)\right) > \liminf \sum_{b=1}^{B} t\lambda_b(a(t)\bar{p}^\star(q,t)) \left(\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right)$$

which is a contradiction to the optimality of $p^\star(q,t)$ in the HJB equations. □

We now prove the following equivalent theorem to Theorem 5.

THEOREM C1. *If F_X is in the Frechet domain of attraction with index $\alpha > 1$ and satisfies the von Mises condition (45), then for each $q \in \mathbb{N}^N$,*

$$\lim_{\lambda t\to\infty} J^\star(q,t)/a(\lambda t) = \left(\sum_{n=1}^{N} \phi_{q_n}^{(\alpha-1)/\alpha} \right), \qquad \text{(C13)}$$

and, for each bundle $b \in \{1,2,...,B\}$, an optimal bundle pricing policy satisfies

$$\lim_{\lambda t\to\infty} p_b^\star(q,t)/a(\lambda t) = \sum_{n\in\psi_b} \phi_{q_n}^{-1/\alpha} \tag{C14}$$

and,

$$\lim_{\lambda t\to\infty} \lambda t \left(1 - F^{*,|\psi_b|}\left(a(\lambda t)\bar{p}_b^\star(q,t)\right)\right) = \begin{cases} \phi_{q_n}(\alpha) & \text{if } \psi_b = \{n\} \\ 0 & \text{if } |\psi_b| \geq 2, \end{cases} \tag{C15}$$

where $\phi_{q_n} = v_{q_n}((\alpha-1)/\alpha)$.

Proof of Theorem C1. It is sufficient to establish the result for $\lambda = 1$. We proceed by induction. The statement is true for $||q||_1 = 0$; now, suppose it is true for all $||q||_1 \leq l-1$ and consider $q \in \mathbb{N}^N$ such that $||q||_1 = l$. Assume without loss of generality that $q_n \geq 1$ for each $n = 1,\dots,N$, since otherwise, we can ignore such items as they cannot be sold. Note that according to the HJB optimality conditions and the von Mises condition (45), for any $t>0$, $\bar{p}^\star(q,t)$ attains the supremum of

$$\begin{aligned} \bar{\Pi}^\star(q,N,t) &= \sup_{\bar{p}\in\bar{\mathcal{P}}(q)} \left\{ \sum_{b=1}^{B} \lambda_b(a(t)\cdot\bar{p}) \left[\bar{p} - \Delta\bar{J}^\star(q,b,t)\right] \right\} \\ &= \sum_{b=1}^{B} \lambda_b\left(a(t)\bar{p}^\star(q,t)\right) \left[\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right] \end{aligned} \tag{C16}$$

where $\bar{(\cdot)} := (\cdot)/a(\lambda t)$.

First note that since additive bundle pricing (component pricing) is a feasible policy, it follows from Abdallah and Reed (2025a) we have $\liminf_{t>0} \bar{J}^\star(q,t) \geq \sum_{n=1}^{N} \phi_{q_n}^{(\alpha-1)/\alpha}$. Therefore, it follows by the induction hypothesis that for every $b = 1,\dots,B$, we have $\liminf_{t>0} \Delta\bar{J}^\star(q,b,t) > 0$. Moreover, by Lemmas A1 and C1, we obtain that

$$0 < \liminf_{t>0} \Delta\bar{J}^\star(q,b,t) \leq \limsup_{t>0} \Delta\bar{J}^\star(q,b,t) < \infty \tag{C17}$$

and

$$0 < \liminf_{t>0} \bar{p}_b^\star(q,t) \leq \limsup_{t>0} \bar{p}_b^\star(q,t) < \infty \tag{C18}$$

We start by establishing the following asymptotic upper-bound

$$\limsup_{t>0}\ t\cdot\left(\bar{\Pi}^\star(q,N,t) - \sum_{n=1}^{N}(1 - F(a(t)\bar{p}_{\beta_n}^\star(q,t)))\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}^\star(q,\beta_n,t)\right)\right) \leq 0, \tag{C19}$$

where $\beta_n \in \{1,...,B\}$ is the singleton bundle that contains only item n.

Now for each $b = 1,2,...,B$, and $p \in \mathbb{R}$, let

$$1\left\{\sum_{n\in\psi_b} X_n > p\right\} = \sum_{n\in\psi_b} 1\{X_n > p\} + \varepsilon(b,X,p). \tag{C20}$$

Taking expectations, we have

$$E[\varepsilon(b,X,p)] = (1 - F^{*,|\psi_b|}(p)) - |\psi_b|(1 - F(p)). \tag{C21}$$

where $F^{*,|\psi_b|}$ is the convolution distribution of bundle b's valuation, i,e, $\sum_{n\in\psi_b} X_n$. Next, note that for $p \in \mathbb{R}_+^B$ and $b = 1,2,...,B$,

$$y_b^\star(q,X,p) = 1\left\{\sum_{n\in\psi_b} X_n > p_b\right\} y_b^\star(q,X,p),$$

and so we may write

$$y_b^\star(q,X,p) = \left(\sum_{n\in\psi_b} 1\{X_n > p_b\}\right) y_b^\star(q,X,p) + \varepsilon(b,X,p_b) y_b^\star(q,X,p).$$

This then implies that

$$\lambda_b(q,p) = \sum_{n\in\psi_b} E\left[1\{X_n > p_b\} y_b^\star(q,X,p)\right] + E[\varepsilon(b,X,p_b) y_b^\star(q,X,p)].$$

Substituting the previous expression into $\bar{\Pi}^\star(q,N,t)$ and interchanging the order of summation, we obtain after some algebra that

$$\begin{aligned} t\cdot\bar{\Pi}^\star(q,N,t) &= \sum_{n=1}^{N}\left(\sum_{b\in\sigma_n} tE\left[1\{X_n > p_b^\star(q,t)\} y_b^\star(q,X,p^\star)\right]\left(\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right)\right) \\ &+ \sum_{b=1}^{B} tE[\varepsilon_b(q,X,p^\star) y_b^\star(q,X,p^\star)]\left(\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right), \end{aligned} \tag{C22}$$

where $\sigma_n \subseteq \{1,2,...,B\}$ denotes the set of bundles that contain item n.

Regarding the second term on the right-hand side of (C22), note that

$$tE[\varepsilon_b(q,X,p^\star) y_b^\star(q,X,p^\star)] \le t(1 - F(a(t)\bar{p}_b^\star(q,t))\left(\frac{1 - F^{*,|\psi_b|}\left(a(t)\bar{p}_b^\star(q,t)\right)}{1 - F\left(a(t)\bar{p}_b^\star(q,t)\right)} - |\psi_b|\right)$$

However, since by (C18) we have $a(t)\bar{p}_b^\star(q,t) \to \infty$ as $t \to \infty$, it then follows by a standard result by Feller (1991) regarding convolutions of regularly varying functions, that

$$\frac{1 - F^{*,|\psi_b|}\left(a(t)\bar{p}_b^\star(q,t)\right)}{1 - F\left(a(t)\bar{p}_b^\star(q,t)\right)} \to |\psi_b| \quad \text{as } t \to \infty. \tag{C23}$$

and by Proposition 1 for the Frechet domain of attraction together with (C18), we have

$$\limsup_{t>0} t\left(1 - F\left(a(t)\bar{p}_b^\star(q,t)\right)\right) < \infty,$$

and therefore

$$\sum_{b=1}^{B} tE[\varepsilon_b(q,X,p^\star) y_b^\star(q,X,p^\star)]\left(\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \to 0 \quad \text{as } t \to \infty. \tag{C24}$$

Now setting

$$g_n(q,t,p^\star) = \sum_{b\in\sigma_n(q)} E\left[1\{X_n > p_b^\star(q,t)\} y_b^\star(X,p^\star)\right]\left(\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right), \tag{C25}$$

for $n = 1,\dots,N$, it follows from (C22) and (C24) that

$$t\left(\bar{\Pi}^\star(q,N,t) - \sum_{n=1}^{N} g_n(q,t,p^\star)\right) \to 0 \text{ as } \; t \to \infty. \tag{C26}$$

Hence, to establish (C19) it is sufficient to establish following asymptotic upper bound on g_n,

$$\limsup_{t>0}\; t\cdot\left(g_n(q,t,p^\star) - (1 - F(a(t)\bar{p}_{\beta_n}^\star(q,t)))\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}^\star(q,\beta_n,t)\right)\right) \le 0. \tag{C27}$$

Fix $n \in \{1,2,...,N\}$ and note that if $q_n = 0$ the statement is trivially true. Assume $q_n > 0$, then by the monotonicity property of the bundle pricing policies, we have that for each $b \in \sigma_n(q)$ we may write $\bar{p}_b^\star(q,t) = \bar{p}_{\beta_n}^\star(q,t) + \bar{\epsilon}_b^\star(q,t)$ with $\bar{\epsilon}_b^\star(q,t) \geq 0$, and

$$\begin{aligned} & tE\left[1\{\bar{X}_n > \bar{p}_b^\star(q,t)\}y_b^\star(\bar{X},\bar{p}^\star)\right]\left(\bar{p}_b^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \\ = & \ tE\left[1\{\bar{X}_n > \bar{p}_b^\star(q,t)\}y_b^\star(\bar{X},\bar{p})\right]\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \\ & + tE\left[1\{\bar{X}_n > \bar{p}_b^\star(q,t)\}y_b^\star(\bar{X},\bar{p})\right]\bar{\epsilon}_b^\star(q,t). \end{aligned} \tag{C28}$$

Regarding the second term on the right-hand side, note that since $\bar{p}_b^\star(q,t) = \bar{p}_{\beta_n}^\star(q,t) + \bar{\epsilon}_b^\star(q,t)$ with $\bar{\epsilon}_b^\star(q,t) \geq 0$ it follows that

$$1\{\bar{X}_n > \bar{p}_b^\star(q,t)\}y_b^\star(\bar{X},\bar{p}) \leq 1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}1\left\{\sum_{\substack{k\in\psi_b \\ k\neq n}} \bar{X}_k \geq \bar{\epsilon}_b^\star(q,t)\right\}. \tag{C29}$$

By the independence of the components of $\bar{X}$, we have that

$$\begin{aligned} & tE\left[1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}1\left\{\sum_{k\in\psi_b,k\neq n} \bar{X}_k \geq \bar{\epsilon}_b^\star(q,t)\right\}\right]\bar{\epsilon}_b^\star(q,t) \\ = & \ t(1 - F(a(\lambda)\bar{p}_{\beta_n}^\star(q,t)))P\left(\sum_{k\in\psi_b,k\neq n} X_k \geq a(t)\bar{\epsilon}_b^\star(q,t)\right)\bar{\epsilon}_b^\star(q,t). \end{aligned} \tag{C30}$$

However, since $0 < \liminf_{t>0} \bar{p}_{\beta_n}^\star(q,t) < \limsup_{t>0} \bar{p}_{\beta_n}^\star(q,t) < \infty$, we have that $\limsup_{t>0} t(1 - F(a(t)\bar{p}_{\beta_n}^\star(q,t))) < \infty$. Moreover, since by Lemma C1 we have $\limsup_{t>0} \bar{\epsilon}_b^\star(q,t) < \infty$, it follows from (C30) that

$$tE\left[1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}1\left\{\sum_{k\in\psi_b,k\neq n} \bar{X}_k > \bar{\epsilon}_b^\star(q,t)\right\}\right]\bar{\epsilon}_b^\star(q,t) \to 0 \quad \text{as} \quad t \to \infty. \tag{C31}$$

From (C25) through (C31) it now follows that

$$\lim_{t\to\infty} t \cdot \left(g_n(q,t,p^\star) - \sum_{b\in\sigma_n(q)} E\left[1\{\bar{X}_n > \bar{p}_b^\star(q,t)\}y_b^\star(\bar{X},\bar{p})\right]\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right)\right) = 0. \tag{C32}$$

Now note that since $\bar{p}_b^\star(q,t) = \bar{p}_{\beta_n}^\star(q,t) + \bar{\epsilon}_b^\star(q,t)$ with $\bar{\epsilon}_b^\star(q,t) \geq 0$ for each $b \in \sigma_n(q)$ it follows that

$$1\{\bar{X}_n > \bar{p}_b^\star(q,t)\}y_b^\star(\bar{X},\bar{p}) \leq 1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}y_b^\star(\bar{X},\bar{p}), \tag{C33}$$

and so (C32) implies

$$\limsup_{t>0} \left(tg_n(q,t,p^\star) - \sum_{b\in\sigma_n(q)} tE\left[1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}y_b^\star(\bar{X},\bar{p})\right]\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right)\right) \leq 0. \tag{C34}$$

Interchanging summation and expectation we now have that

$$\begin{aligned} & \sum_{b\in\sigma_n(q)} tE\left[1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}y_b^\star(\bar{X},\bar{p})\right]\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}^\star(q,b,t)\right) \\ = & \ tE\left[1\{\bar{X}_n > \bar{p}_{\beta_n}^\star(q,t)\}\sum_{b\in\sigma_n(q)} y_b^\star(\bar{X},\bar{p})\left(\bar{p}_{\beta_n}^\star(q,t) - \Delta\bar{J}(q,b,t)\right)\right], \end{aligned} \tag{C35}$$

and so since

$$\sum_{b\in\sigma_n(q)} y_b^\star(\bar{X},\bar{p}) \leq 1,$$

with $y_b^\star(\bar{X},\bar{p})\geq 0$ for $b\in\sigma_n(q)$ and $\Delta\bar{J}^\star(q,\beta_n,t)\geq\Delta\bar{J}^\star(q,b,t)$ for $b\in\sigma_n$, it follows that

$$\begin{aligned} &tE\left[1\{\bar{X}_n>\bar{p}^\star_{\beta_n}(q,t)\}\sum_{b\in\sigma_n(q)} y_b^\star(\bar{X},\bar{p})\left(\bar{p}^\star_{\beta_n}(q,t)-\Delta\bar{J}^\star(q,b,t)\right)\right] \\ &\leq t(1-F(a(t)\bar{p}^\star_{\beta_n}(q,t)))\left(\bar{p}^\star_{\beta_n}(q,t)-\Delta\bar{J}^\star(q,\beta_n,t)\right). \end{aligned} \tag{C36}$$

Putting (C36), (C35), and (C34) together yields the asymptotic upper bound in (C27). Therefore, we now have that

$$\limsup_{t>0}\ t\cdot\left(\bar{\Pi}^\star(q,N,t)-\sum_{n=1}^{N}(1-F(a(t)\bar{p}^\star_{\beta_n}(q,t)))\left(\bar{p}^\star_{\beta_n}(q,t)-\Delta\bar{J}^\star(q,\beta_n,t)\right)\right)\leq 0. \tag{C37}$$

Next, we show that component pricing (additive bundle pricing) asymptotically achieves this upper bound. To this end, consider an additive bundle pricing policy $\rho(q,t)\in\mathbb{R}_+^B$ such that

$$\rho_b(q,t) = a(t)\sum_{n\in\psi_b}\bar{p}^\star_{\beta_n}(q,t)\quad\text{for } b=1,\ldots,B. \tag{C38}$$

By the optimality principle, we have

$$\bar{\Pi}^\star(q,N,t)\geq\sum_{b=1}^{b}\lambda_b(q,a(t)\bar{\rho}(q,t)\left[\bar{\rho}_b(q,t)-\Delta\bar{J}^\star(q,b,t),\right]. \tag{C39}$$

where $\lambda_b(q,a(t)\bar{\rho}(q,t))=E[y_b^\star(\bar{X},\bar{\rho})]$.

However, it follows from the additive pricing assumption that

$$1\{y_b^\star(\bar{X},\bar{\rho})=1\}\leq\Pi_{n\in\psi_b}1\{X_n\geq a(t)\bar{p}_{\beta_n}\}, \tag{C40}$$

since $1\{X_n<a(t)\bar{p}_{\beta_n}\}$ implies the customer strictly prefers the smaller bundle without item n.

Next, recall by the von-Mises condition that for a sufficiently large t (and hence large $a(t)\bar{p}_{\beta_n}$), the distribution F_X admits a density. Therefore, by taking expectations on both sides and considering the independence of the components of X, it follows that for a sufficiently large t

$$E[y_b^\star(\bar{X},\bar{p})]\leq\Pi_{n\in\psi_b}\left(1-F(a(t)\bar{p}_{\beta_n}(q,t))\right)\}. \tag{C41}$$

It now follows from Proposition 1 that for any bundle b that includes more than one item, $|\psi_b|\geq 2$, we have

$$t\cdot E[y_b^\star(\bar{X},\bar{p})]\to 0\quad\text{as } t\to\infty. \tag{C42}$$

Therefore, the following convergence result holds

$$\lim_{t\to\infty} t\left(\sum_{b=1}^{B}\lambda_b(a(t)\bar{p}(q,t)\left[\bar{p}_b(q,t)-\Delta\bar{J}^\star(q,b,t)\right]-\sum_{n=1}^{N}(1-F(a(t)\bar{p}^\star_{\beta_n}(q,t)))\left(\bar{p}^\star_{\beta_n}(q,t)-\Delta\bar{J}^\star(q,\beta_n,t)\right)\right)=0,$$

which together with (C37) and (C39) implies

$$\lim_{t\to\infty} t\left(\bar{\Pi}^\star(q,N,t)-\sum_{n=1}^{N}(1-F(a(t)\bar{p}^\star_{\beta_n}(q,t)))\left(\bar{p}^\star_{\beta_n}(q,t)-\Delta\bar{J}^\star(q,\beta_n,t)\right)\right)=0.$$

and in particular, by the optimality of $\bar{\Pi}^{\star}(q,N,t)$ we have

$$\lim_{t\to\infty} t\left(\bar{\Pi}^{\star}(q,N,t)-\sum_{n=1}^{N}\sup_{\bar{\rho}_n\geq 0}\left(1-F(a(t)\bar{\rho}_n)\right)\left(\bar{\rho}_n-\Delta\bar{J}^{\star}(q,\beta_n,t)\right)\right)=0. \tag{C43}$$

Note that while the pricing policies converge to an additive bundle pricing policy (i.e., component pricing), this may not guarantee the convergence of the value function. So, we proceed to show that the optimal value function indeed converges to optimal dynamic component pricing.

We next show that

$$\lim_{t\to\infty}\left(t\sup_{\bar{\rho}_n\geq 0}\left(1-F(a(t)\bar{\rho}_n)\right)\left(\bar{\rho}_n-\Delta\bar{J}^{\star}(q,\beta_n,t)\right)-\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\left(\Delta\bar{J}^{\star}(q,\beta_n,t)\right)^{1-\alpha}\right)=0. \tag{C44}$$

For t sufficiently large, the supremum is attained, and it is straightforward to show that the maximizer $\bar{\rho}_n^{\star}(q,t)$ satisfies the FOC. Invoking the von Mises condition (45) to the FOC and after some algebra, we obtain

$$\frac{\bar{\rho}_n^{\star}(q,t)}{\Delta\bar{J}^{\star}(q,\beta_n,t)}=C_0(q,n,t)=\left(1-\frac{(1-F(a(t)\bar{\rho}_n^{\star}(q,t)))}{a(t)\bar{\rho}_n^{\star}(q,t)f(a(t)\bar{\rho}_n^{\star}(q,t))}\right)^{-1}\to\frac{\alpha}{\alpha-1}\quad\text{as } t\to\infty, \tag{C45}$$

where the convergence holds by the von-Mises condition.

Noting that $1/t=1-F(a(t))=a(t)^{1/\alpha}L(a(t)$, then after some algebra, we have

$$\begin{aligned}&t\sup_{\bar{\rho}_n\geq 0}\left(1-F(a(t)\bar{\rho}_n)\right)\left(\bar{\rho}_n-\Delta\bar{J}^{\star}(q,\beta_n,t)\right)\\&=C_0(q,n,t)^{-\alpha}\left(C_0(q,n,t)-1\right)\frac{L(a(t)\bar{\rho}_n^{\star}(q,t))}{L(a(t)}\left(\Delta\bar{J}^{\star}(q,\beta_n,t)\right)^{1-\alpha}.\end{aligned} \tag{C46}$$

Since by the definition of slowly varying functions, we have $L(a(t)\bar{\rho}_n^{\star}(q,t))/L(a(t)\to 1$ as $t\to\infty$, then (C45) and (C46) imply (C44) which together with (C43) yields

$$\lim_{t\cdot\to\infty}\left(t\bar{\Pi}^{\star}(q,N,t)-\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\sum_{n=1}^{N}\left(\Delta\bar{J}^{\star}(q,\beta_n,t)\right)^{1-\alpha}\right)=0. \tag{C47}$$

On the other hand, differentiating $\bar{J}^{\star}(q,t)$ directly with respect to t, yields

$$\frac{\partial\bar{J}^{\star}(q,t)}{\partial t}=\frac{1}{a(t)}\frac{\partial J^{\star}(q,t)}{\partial t}-\frac{a'(t)}{a(t)}\bar{J}^{\star}(q,t) \tag{C48}$$

$$=\frac{1}{t}\left(t\bar{\Pi}^{\star}(q,t)-\frac{ta'(t)}{a(t)}\bar{J}^{\star}(q,t)\right), \tag{C49}$$

which, after some algebra, we may write

$$\frac{\partial\bar{J}^{\star}(q,t)}{\partial t}=\frac{1}{t}\left(\bar{\epsilon}(t)-g\left(\bar{J}^{\star}(q,t)\right)\right), \tag{C50}$$

where

$$\begin{aligned}\bar{\epsilon}(t)=&\left(\alpha t\bar{\Pi}^{\star}(q,N,t)-\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\sum_{n=1}^{N}\left(\bar{J}^{\star}(q,t)-\sum_{k=1}^{N}\phi_{q_k}^{(\alpha-1)/\alpha}+\left(\phi_{q_n}^{(\alpha-1)/\alpha}-\phi_{q_n-1}^{(\alpha-1)/\alpha}\right)^{1-\alpha}\right)\right)\\&+\bar{J}^{\star}(q,t)\left(1-\alpha\frac{ta'(t)}{a(t)}\right)\end{aligned} \tag{C51}$$

$$g(x)=\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\sum_{n=1}^{N}\left(x-\sum_{k=1}^{N}\phi_{q_k}^{(\alpha-1)/\alpha}+\left(\phi_{q_n}^{(\alpha-1)/\alpha}-\phi_{q_n-1}^{(\alpha-1)/\alpha}\right)\right)^{1-\alpha}-x. \tag{C52}$$

for $x \in (\gamma(q,t),\infty)$ where $\gamma(q,t) = \max\{\sum_{k=1}^{N} \phi_{q_k}^{(\alpha-1)/\alpha} - \left(\phi_{q_n}^{(\alpha-1)/\alpha} - \phi_{q_n-1}^{(\alpha-1)/\alpha}\right)\} \le \sum_{k=1}^{N} \phi_{q_k}^{(\alpha-1)/\alpha}$.

However, for $x \in (\gamma(q,t),\infty)$, $g(x)$ is strictly decreasing in x, and by definition of the recursion for ϕ_q, then $g(x)$ has a unique root at $x = \sum_{k=1}^{N} \phi_{q_k}^{(\alpha-1)/\alpha}$. Moreover, Noting that $ta'(t)/a(t) \to 1/\alpha$ and $\limsup_{t>0} \bar{J}^\star(q,t) < \infty$, it follows from (C47) and the induction hypothesis that $\bar{\epsilon}(t) \to 0$ as $t \to \infty$ and therefore it can be shown using the comparison theorem Arnold (1992) that $\bar{J}^\star(q,t) \to \sum_{k=1}^{N} \phi_{q_k}^{(\alpha-1)/\alpha}$ as $t \to \infty$.

To establish (C14), note that from (C45) together with (C13), we have that as $t \to \infty$

$$\bar{p}^\star_{\beta_n}(q,t) \to \frac{\alpha}{\alpha-1}\left(\phi_{q_n}^{(\alpha-1)/\alpha}(\alpha) - \phi_{q_n-1}^{(\alpha+1)/\alpha}(\alpha)\right) = \phi_{q_n}^{-1/\alpha}(\alpha) \tag{C53}$$

where the last equality is due to (15) and the fact that $\phi_{q_n}(\alpha) = v_{q_n}((\alpha-1)/\alpha)$. Finally, (C15) for $\psi_b = \{n\}$ follows from (C14) together with Proposition 1. □

C.1. Proofs for Pure Bundling: Frechet Domain of Attraction

First, we show that if F satisfies von Mises condition (45), then so does $F^{*,N}$.

LEMMA C2. *If F satisfies von Mises condition for the Frechet domain of attraction (45), then so does $F^{*,N}$. In particular,*

$$\lim_{x\to\infty} \frac{x f^{*,N}(x)}{1 - F^{*,N}(x)} = \alpha.$$

where $f^{*,N}$ is the density function of $F^{*,N}$.

Proof of Lemma C2. Since F_X satisfies the von-Mises condition, it follows from Proposition A3.8(b) in Embrechts et al. (2013) that $f_X \in RV_{-1-\alpha}$. Moreover, by the convolution closure of regularly varying densities by Bingham et al. (2006). we get that also $f_X^{N\star} \in RV_{-1-\alpha}$. Finally, from Proposition A3.8(c) in Embrechts et al. (2013), we get that $F_X^{N\star}$ also satisfies von Mises condition (45). □

Proof of Proposition 4. It suffices to prove the result for $\lambda = 1$. First, from Lemma C2, $F^{*,N}$ belongs to the Frechet domain of attraction with the same index α of F. It now follows from Proposition 1 that for $x > 0$

$$t(1 - F^{*,N}(a^{*,N}(t)x) \to x^{-\alpha} \quad \text{as } t \to \infty, \tag{C54}$$

where $a^{*,N}(t) = \overset{-1}{F}_{*,N}(1-1/t)$. Moreover, it now follows from Abdallah and Reed (2025a) that

$$J^{\star,PB}(q,t) = q_N a^{*,N}\left(t\phi_{q_N}^{\alpha-1}/q_N^{\alpha}\right) + o(a^{*,N}(t)), \text{and} \tag{C55}$$

$$p^\star(q,t) = a^{*,N}(t/\phi_{q_N}) \tag{C56}$$

However, from a standard result for convolutions of functions with regularly varying tails (Feller 1991) that

$$\frac{1 - F^{*,N}(x)}{N(1 - F(x))} \to 1 \quad \text{as } x \to \infty$$

which together with (C54) yields

$$\frac{1}{t\,(1 - F(a^{*,N}(t)x))} \to N x^{\alpha} \quad \text{as } t \to \infty.$$

Inverting both sides and applying the convergence of inverses of monotone functions, we obtain,

$$\frac{F^{-1}(1 - \frac{1}{tx})}{a^{*,N}(t)} \to \left(\frac{x}{N}\right)^{1/\alpha} \quad \text{as } t \to \infty.$$

Noting that $a(t) = F^{-1}(1 - 1/t)$, we obtain that

$$\frac{a(t)}{a^{*,N}(t)} \to N^{-1/\alpha} \quad \text{as } t \to \infty,$$

and together with (C55) and (C56) establishes the desired result. □

C.2. Proofs for Bundle Size Pricing: Frechet Domain of Attraction

First, recall that the HJB equations for the optimal dynamic BSP policy are given by

$$\frac{\partial J^{\star,\mathsf{BSP}}(q,t)}{\partial t} = \sup_{\rho\in[0,\infty)^N}\left\{\sum_{\substack{k=1\\|\mathcal{B}_k(q)|>1}}^{K}\lambda_k(q,p)\left(\rho_k(q,t)-\frac{1}{|\mathcal{B}_k(q)|}\sum_{b\in\mathcal{B}_k(q)}\Delta J^{\star,\mathsf{BSP}}(q,b,t)\right)\right\},\ \forall t\geq 0,\ q\in\mathbb{N}^N, \tag{C57}$$

$$J^{\star,\mathsf{BSP}}(q,0) = 0,\quad q\in\mathbb{N}^N, \tag{C58}$$

$$J^{\star,\mathsf{BSP}}(0,t) = 0,\quad t\geq 0, \tag{C59}$$

We first state a similar Lemma to Lemma C1 whose proof is similar and hence is skipped.

LEMMA C3. *Given an policy $\rho^\star\in\mathcal{P}^{\mathsf{BSP}}$, then every $q\in\mathbb{N}_+^N$ and bundle $k\leq|\mathcal{N}(q)|$,*

$$\limsup_{t>0}\bar{\rho}_k^\star(q,t) < \infty \tag{C60}$$

We now state and prove an equivalent version of Theorem 6.

THEOREM C2. *If F is in the Frechet domain of attraction with index $\alpha>1$ and satisfies the von-Mises condition (45), then under the optimal dynamic bundle size pricing policy for each $q\in\mathbb{N}^N$ with $\mathcal{N}(q)\geq 1$ and, we have*

$$\lim_{\lambda t\to\infty}\bar{J}^{\star,\mathsf{BSP}}(q,t) = \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha) \tag{C61}$$

Moreover, the only relevant price is the price of the bundles of size 1 where,

$$\lim_{\lambda t\to\infty}\bar{\rho}_1^\star(q,t) = N^{1/(\alpha-1)}\tilde{\phi}_q^{-1/\alpha}(\alpha) \tag{C62}$$

and the purchasing probabilities for bundle sizes $k=1,\ldots,N$ are given by

$$\lim_{\lambda t\to\infty}t\lambda_k(q,\rho^\star) = \begin{cases}\frac{1}{N^{\alpha/(\alpha-1)}}\tilde{\phi}_q(\alpha) & \text{for } k=1,\\ 0 & \text{for } k=2,\ldots,N,\end{cases} \tag{C63}$$

where $\tilde{\phi}_q(\alpha)=\tilde{v}_q((\alpha-1)/\alpha)$.

Proof of Theorem 6. it is sufficient to establish the result for $\lambda=1$. Fix $q\in\mathbb{N}^N$ and assume without loss of generality that $q_n>0$ for every $n=1,\ldots,N$; otherwise, those items can be ignored. It follows from the (normalized) HJB equation (C57) that for a sufficiently large $t>0$, $\bar{\rho}^\star(q,t)$ attains the supremum of

$$\begin{aligned}\bar{\Pi}^{\star,\mathsf{BSP}}(q,N,t) &= \sup_{\bar{\rho}\in[0,\infty)^N}\left\{\sum_{k=1}^{K}\hat{\lambda}_k(q,a(t)\bar{\rho})\left(\bar{\rho}_k-\frac{1}{|\mathcal{B}_k(q)|}\sum_{b\in\mathcal{B}_k(q)}\Delta\bar{J}^{\star,\mathsf{BSP}}(q,b,t)\right)\right\}\\ &= \sum_{k=1}^{K}\hat{\lambda}_k(q,a(t)\bar{\rho}^\star(q,t))\left(\bar{\rho}_k^\star(q,t)-\frac{1}{|\mathcal{B}_k(q)|}\sum_{b\in\mathcal{B}_k(q)}\Delta\bar{J}^{\star,\mathsf{BSP}}(q,b,t)-\right).\end{aligned} \tag{C64}$$

Similar to the proof of Theorem C1, we can show that for $k\geq 2$

$$t\hat{\lambda}_k\left(q,a(t)\bar{\rho}^\star(q,t)\right)\to 0\ \text{ as } t\to 0$$

and for $k=1$,

$$t\left(\hat{\lambda}_1(q,a(t)\bar{\rho}^\star(q,t)) - P\left(\max\{X_1,\ldots,X_N\} > a(t)\bar{\rho}_1^\star(q,t)\right)\right) \to 1 \quad \text{as } t\to 0$$

whereby the i.i.d. assumption implies

$$P\left(\max\{X_1,\ldots,X_N\} > a(t)\bar{\rho}_1^\star(q,t)\right) = \left(1-F\left(a(t)\bar{\rho}_1^\star(q,t)\right)\right)\sum_{n=0}^{N-1}\left(F\left(a(t)\bar{\rho}_1^\star(q,t)\right)\right)^n$$

and since $a(t)\bar{\rho}_1^\star(q,t)\to\infty$ as $t\to\infty$, we have

$$\frac{P\left(\max\{X_1,\ldots,X_N\} > a(t)\bar{\rho}_1^\star(q,t)\right)}{N\left(1-F(a(t)\bar{\rho}_1^\star(q,t)\right)} \to 1 \quad \text{as } t\to\infty$$

which together with (C64) yields the following convergence result

$$\lim_{t\to\infty} t\left(\bar{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \sup_{\bar{\rho}_1\geq 0}\left\{(1-F(a(t)\bar{\rho}_1))\left(\sum_{n=1}^N \Delta\bar{J}^{\star,\mathsf{BSP}}(q,\beta_n,t) - N\bar{\rho}_1\right)\right\}\right) = 0. \tag{C65}$$

Now note that for a sufficiently large t, we have that the supremum is attained by $\bar{\rho}_1^\star(q,t)$ that satisfies the FOC. Invoking the von Mises condition (45) to the FOC, then after some algebra we obtain

$$\frac{N\bar{\rho}_1^\star(q,t)}{\sum_{n=1}^N \Delta\bar{J}^{\star,\mathsf{BSP}}(q,\beta_n,t)} \to \frac{\alpha}{\alpha-1} \quad \text{as } t\to\infty, \tag{C66}$$

and since $\limsup_{t>0}\sum_{n=1}^N \Delta\bar{J}^{\star,\mathsf{BSP}} < \infty$, then after some algebra, we obtain

$$\lim_{t\to\infty}\left(t\bar{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \frac{N}{\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{1-\alpha}\left(\frac{\sum_{n=1}^N \Delta\bar{J}^{\star,\mathsf{BSP}}(q,\beta_n,t)}{N}\right)^{1-\alpha}\right) = 0. \tag{C67}$$

.

On the other hand, differentiating $\bar{J}^{\star,\mathsf{BSP}}(q,t)$ directly with respect to t yields

$$\begin{aligned}\frac{\partial\bar{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} &= \frac{1}{a(t)}\frac{\partial J^{\star,\mathsf{BSP}}(q,t)}{\partial t} - \frac{a'(t)}{a(t)}\bar{J}^{\star,\mathsf{BSP}}(q,t)\\ &= \frac{1}{t}\left(t\bar{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \frac{ta'(t)}{a(t)}\bar{J}^{\star,\mathsf{BSP}(q,t)}(q,t)\right).\end{aligned} \tag{C68}$$

We are now ready to show by induction that $\bar{J}^\star(q,t)\to\tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)$ as $t\to\infty$. The statement is true for $||q||_1=0$, now suppose it is true for all $||q||_1 <= l-1$ and consider $||q||_1 = l$.

Note that we can rewrite (C68) as

$$\frac{\partial\bar{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} = \frac{1}{\alpha t}\left(\bar{\epsilon}(t) + g\left(\bar{J}^{\star,\mathsf{BSP}}(q,t)\right)\right). \tag{C69}$$

where

$$\begin{aligned}\bar{\epsilon}(t) &= \left(\alpha t\bar{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - N\left(\frac{\alpha}{\alpha-1}\right)^{1-\alpha}\left(\bar{J}^{\star,\mathsf{BSP}}(q,t) - \frac{\sum_{n=1}^N\tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)}{N}\right)^{1-\alpha}\right)\\ &\quad + \bar{J}^{\star,\mathsf{BSP}}(q,t)\left(1-\alpha\frac{ta'(t)}{a(t)}\right) \qquad \text{(C70)}\\ g(x) &= N\left(\frac{\alpha}{\alpha-1}\right)^{1-\alpha}\left(x - \frac{\sum_{n=1}^N\tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)}{N}\right)^{1-\alpha} - x. \qquad \text{(C71)}\end{aligned}$$

Regarding $\bar{\epsilon}(t)$, note that by the inverse value theorem and von Mises condition (45)

$$t\frac{a'(t)}{a(t)} \to \frac{1}{\alpha} \quad \text{as } t\to\infty, \tag{C72}$$

Furthermore, by the induction hypothesis and (C67), then after some algebra, we obtain

$$\bar{\epsilon}(t) \to 0 \quad \text{as } t\to\infty.$$

Moreover, by the induction hypothesis and uniqueness of the system of equations defining $\tilde{v}_q$, it follows that $g(x)$ has a unique root $x=\tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)$ on the interval $\left[\sum_{k=1}^N \tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)/N,\infty\right)$. Furthermore, $g(x)<0$ for $x\in\left[\sum_{k=1}^N \tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)/N, \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)\right)$ and $g(x)>0$ for $x\in(\tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha),\infty)$. Therefore, since $\bar{\epsilon}(t)\to 0$ as $t\to\infty$, and again using the comparison theorem (Arnold 1992) it can be shown that $\bar{J}^{\star,\mathsf{BSP}}(q,t)\to\tilde{\phi}_q^{(\alpha-1)/\alpha}$ as $t\to\infty$.

Regarding the convergence of $\bar{\rho}_1^{\star}(q,t)$, note that given the convergence of $\bar{J}^{\star,\mathsf{BSP}}(q,t)$, then (C66) now yields

$$\begin{aligned}\lim_{\lambda t\to\infty}\bar{\rho}_1^{\star}(q,t) &= \frac{\alpha}{\alpha-1}\left(\tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)-\frac{\sum_{n=1}^N \tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)}{N}\right)\\ &= N^{1/(\alpha-1)}\tilde{\phi}_q^{-1/\alpha}(\alpha)\end{aligned} \tag{C73}$$

where the last equality follows from the fact that $\phi_q(\alpha)=\tilde{v}((\alpha-1)/\alpha)$. Finally, (C63) for $k=1$ follows from (C62) together with Proposition 1. $\square$

Proof of Proposition 5. It suffices to establish the case for $\lambda=1$. Since F satisfies the von Mises condition (45), then F is absolutely continuous on (x_0,∞). Let $\rho\in\mathcal{P}^{\mathsf{BSP}}$ where $\bar{\rho}_1(q,t)=F^{-1}\left(1-N^{\alpha/(1-\alpha)}\tilde{\phi}_q(\alpha)/t\right)$. Without loss of generality we assume $q\in\mathbb{N}_+^N$ with $|\mathcal{N}(q)|=N$ since otherwise the stocked out items can be ignored. It then follows by standard theory that for each $q\in\mathbb{N}_+^N$ and $t>\tilde{\phi}_q(\alpha)/N^{\alpha/(\alpha+1)}$,

$$\frac{1}{a(t)}\frac{\partial J_\rho^{\mathsf{BSP}}(q,t)}{\partial t} = P\left(\max\{X_1,\dots,X_N\}>a(t)\bar{\rho}_1(q,t)\right)\left(\bar{\rho}_1(q,t)-\frac{\sum_{n=1}^N \Delta\bar{J}_\rho^{\mathsf{BSP}}(q,\beta_n,t)}{N}\right).$$

Denote by $\bar{\Pi}^{\mathsf{BSP}}(q,N,t)=1/a(t)\left(\partial J_\rho^{\mathsf{BSP}}(q,t)/\partial t\right)$, then similar to (C65) and after some algebra, we have

$$\lim_{t\to\infty} t\left(\bar{\Pi}^{\mathsf{BSP}}(q,N,t)-\left\{(1-F(a(t)\bar{\rho}_1(q,t)))\left(N\grave{\rho}_1(q,t)-\sum_{n=1}^N \Delta\bar{J}_\rho^{\mathsf{BSP}}(q,\beta_n,t)\right)\right\}\right)=0 \tag{C74}$$

which yields

$$\lim_{t\to\infty}\left(t\bar{\Pi}^{\mathsf{BSP}}(q,N,t)-N\bar{\rho}_1(q,t)^{1-\alpha}\left(1-\frac{\sum_{n=1}^N \Delta\bar{J}_\rho^{\mathsf{BSP}}(q,\beta_n,t)}{N\bar{\rho}_1(q,t)}\right)\right)=0 \tag{C75}$$

However, since $\rho_1(q,t)=\rho_1^{\star}(q,t)-o(a(\lambda t))$, it follows that

$$\lim_{t\to\infty}\bar{\rho}_1(q,t)=\lim_{t\to\infty}\bar{\rho}_1^{\star}(q,t)=N^{1/(\alpha-1)}\tilde{\phi}_q^{-1/\alpha}(\alpha).$$

which, after some algebra, implies

$$\lim_{t\to\infty}\left(t\bar{\Pi}^{\mathsf{BSP}}(q,N,t)-\tilde{\phi}_q^{\alpha-1/\alpha}(\alpha)\left(1-N^{1/(1-\alpha)}\tilde{\phi}_q^{1/\alpha}(\alpha)\left(\bar{J}_\rho^{\mathsf{BSP}}(q,t)-\sum_{n=1}^N \bar{J}_\rho^{\mathsf{BSP}}(q-e_n,t)/N\right)\right)\right)=0 \tag{C76}$$

We are now ready to show by induction that $\bar{J}^{\mathsf{BSP}}_{\rho}(q,t) \to \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)$ as $t \to \infty$. The statement is true for $||q|| = 0$, now suppose it is true for all $||q|| <= l-1$.

Note that similar to (C68), we have

$$\frac{\partial \bar{J}^{\mathsf{BSP}}_{\rho}(q,t)}{\partial t} = \frac{1}{\alpha t}\left(\alpha t \bar{\Pi}^{\mathsf{BSP}}(q,N,t) - \alpha \frac{ta'(t)}{a(t)} \bar{J}^{\mathsf{BSP}(q,t)}_{\rho}(q,t)\right), \tag{C77}$$

which can be written as

$$\frac{\partial \bar{J}^{\star,\mathsf{BSP}}(q,t)}{\partial t} = \frac{1}{\alpha t}\left(\bar{\epsilon}(t) + g\left(\bar{J}^{\mathsf{BSP}}_{\rho}(q,t)\right)\right). \tag{C78}$$

where

$$\bar{\epsilon}(t) = \alpha t \bar{\Pi}^{\star,\mathsf{BSP}}(q,N,t) - \alpha \tilde{\phi}_q^{\alpha-1/\alpha}(\alpha)\left(1 - N^{1/(1-\alpha)}\tilde{\phi}_q^{1/\alpha}(\alpha)\left(\bar{J}^{\mathsf{BSP}}_{\rho}(q,t) - \sum_{n=1}^{N}\tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)/N\right)\right)$$
$$+ \grave{J}^{\mathsf{BSP}}_{\rho}(q,t)\left(1 - \alpha\frac{ta'(t)}{a(t)}\right) \tag{C79}$$

$$g(x) = \alpha \tilde{\phi}_q^{\alpha-1/\alpha}(\alpha)\left(1 - N^{1/(1-\alpha)}\tilde{\phi}_q^{1/\alpha}(\alpha)\left(x - \sum_{n=1}^{N}\tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)/N\right)\right) - x. \tag{C80}$$

Note that by (C76), (C72), and the induction hypothesis we have $\bar{\epsilon}(t) \to 0$ as $t \to \infty$. As for $g(x)$, and the from the properties of $\tilde{w}_q(\alpha)$, it is straightforward to verify that $g(x)$ has a unique root $x = \tilde{\phi}_q^{(\alpha-1)/\alpha}$. Meanwhile, for $x > \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)$, then $g(x) > 0$ and for $x \in \left[\sum_{n=1}^{N}\tilde{\phi}_{q-e_n}^{(\alpha-1)/\alpha}(\alpha)/N, \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)\right)$, we have $g(x) < 0$. Therefore, again by the comparison Theorem we have that $\bar{J}^{\mathsf{BSP}}_{\rho}(q,t) \to \tilde{\phi}_q^{(\alpha-1)/\alpha}(\alpha)$ as $t \to \infty$. □