CHAPTER EIGHT
DIFFERENTIAL CALCULUS

8.1 FUNCTIONS AND LIMITS
(Background reading: sections 2.2 and 3.5)

Most of this book is concerned with the relationships among mathematical variables and numbers. The natures of these relationships are defined by *functions*. A function is a rule that assigns to each number in a set a unique second number. Functions are generally represented by equations, graphs, and tables. The following example is a "generic" functional relationship in equation form: $y = f(x)$, which reads "y is a function of x." For each value of x, the function assigns a unique value for y. If y increases as x increases, we say that y is a direct, or increasing, function of x. The following are examples where y is an increasing function of x:

(a) $y = 10x$;
(b) $y = 2x + 1$;
(c) $y = \frac{1}{2}x$;
(d) $y = 3e^x$;
(e) $y = 5x^2 + 3x + 1$ (when $x > -0.3$);
(f) $y = 9x^3 + 3x^2 + 2x + 1$.

Functions (a), (b), and (c) are linear; graphs depicting the relationships between x and y would be represented by lines. Equation (d) represents an exponential function. Equation (e) is a quadratic function (it is a polynomial of order 2) and equation (f) is a cubic function (it is a polynomial of order 3). If y decreases as x increases, we say that y is a decreasing, or inverse, function of x. The following are examples where y is a decreasing function of x:

$$y = \frac{1}{x} \quad \text{(where } x > 0\text{)}, \qquad y = -x + 5,$$

$$y = -2x^2 - 4x \quad \text{(where } x > -1\text{)}, \qquad y = 2e^{-x},$$

$$y = \frac{2}{5x} \quad \text{(where } x > 0\text{)}, \qquad y = -5(x^2).$$

Now consider a function $y = f(x)$. As x approaches (gets closer to) some value a (without actually equaling a), causing y to approach L, we say that the *limit* of $f(x)$ as x approaches a equals L. The limit is expressed as follows:

$$\lim_{x\to a} f(x) = L. \tag{8.1}$$

The limit of $f(x)$ as x approaches a is L. Consider the following examples of limits:

(a) $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0;$ (b) $\lim_{n\to\infty}\sum_{t=1}^{n}\left(\frac{1}{(1+k)^t}\right) = \frac{1}{k};$

(c) $\lim_{x\to\infty}\left(\frac{3x}{5x^2}\right) = 0;$ (d) $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = \mathrm{e} \approx 2.71828;$

(e) $\lim_{m\to\infty}\left(1+\frac{i}{m}\right)^{m\cdot n} = \mathrm{e}^{i\cdot n};$ (f) $\lim_{h\to\infty}\frac{2(x+h)-2x}{h} = 2.$

Thus, the limits of functions (a) and (c) are 0; the limit of function (b) is $1/k$, the limit of function (d) is e; the limit of function (e) is e^{in}; and the limit of function (f) is 2.

APPLICATION 8.1: THE NATURAL LOG (Background reading: sections 2.5, 2.11, 4.4, 4.5, and 8.1 and application 2.7)

The number e is most useful for growth, time value, and probability-based models in finance. This number e is defined as a limit as follows:

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = \mathrm{e} \approx 2.71828.$$

Thus, as m approaches infinity, the value of function $(1 + 1/m)^m$ approaches number e, which is approximated at 2.71828. Notice the likeness of the above limit to a standard compounded interest formula from section 4.4. The number e can also be derived as follows:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{i!} = \left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\right) = \mathrm{e}.$$

As n approaches infinity, the value of this function approaches the number e.

8.2 Slopes, Derivatives, Maxima, and Minima
(Background reading: section 8.1)

Among the most useful financial applications of differential calculus are finding rates of change and growth and determining maximum and minimum values for functions. Let y be a function of x; that is, $y = f(x)$. A change in x may affect a change in y. For example, if $y = 3x$, a change in x by one will result in a change in y by 3. Therefore, the slope of this function is 3. Because the slope of this function does not change as x changes, this function is said to be linear. Thus, y is a linear function of x. The slope m of any line is defined as follows:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}, \tag{8.2}$$

where $x_1 - x_0 = \Delta x$ does not equal zero. In our example where $y = 3x$, if $x_0 = 2$ and $x_1 = 4$, then $y_0 = 6$ and $y_1 = 12$. We find that $\Delta x = 2$ and $\Delta y = 6$. Clearly, $\Delta y \div \Delta x = 3$, the slope of the function. This slope or rate of change in y is constant with respect to x. Thus, this function can be represented by a line.

Consider a second function: $y = 3x^2$, which is represented by figure 8.1. Clearly, the slope of the function y in figure 8.1 changes as x changes. Because the slope of function y is not constant, equation (8.2) cannot be used to determine its slope over a finite range, except where the change in x approaches zero. Now, we will define h as Δx as Δx approaches zero. Thus, h is defined as the limit of Δx as Δx approaches zero.

We can use the calculus concept of a derivative to measure rates of change in functions or slopes in graphs. When functions have slopes that are continuously changing, the derivative is used to find an instantaneous rate of change. That is, the derivative provides the change in y induced by an infinitesimal change in x. Let y be given as a function of x. If x were to increase by a small (infinitesimal – that is,

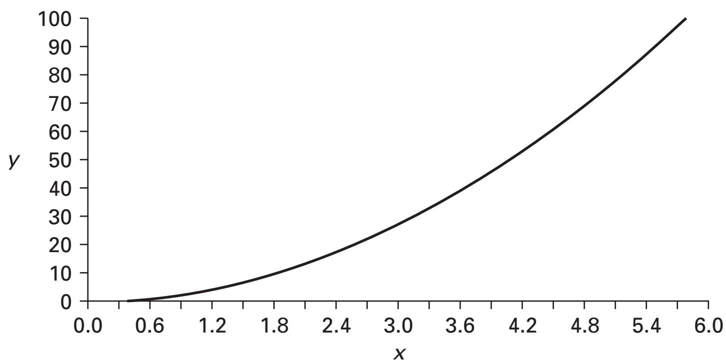

Figure 8.1 Changing slope: $y = 3x^2$.

approaching though never equaling zero) amount h, how much would we expect y to change? This rate of change is given by the derivative of y with respect to x, which is defined using the limit function as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{8.3}$$

Thus, if an infinitesimal value h (that is, a value approaching zero) were to be added to x, y would change by the derivative of y with respect to x multiplied by the amount of change in x:

$$\mathrm{d}y = \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] \cdot \mathrm{d}x.$$

Now, consider a second function $y = f(x)$. The derivative of this function with respect to x, $\mathrm{d}y/\mathrm{d}x = f'(x)$, is itself a function of x. The slope of the curve representing the function is positive whenever the derivative is positive. Whenever this derivative is positive, an infinitesimal increase in x will lead to an increase in y by $f'(x) \cdot \Delta x$. The slope of the function represented by figure 8.1 is positive throughout (since $x > 0$ at all points in the figure). In figure 8.2, the slope is positive to the right of the minimum point and negative to the left. In figure 8.3, the slope is positive to the left of the maximum point and negative to the right. Whenever the derivative is negative, an infinitesimal increase in x will lead to a decrease in y. A zero derivative implies that an infinitesimal change in x will lead to no change in y. A zero derivative *may* imply a minimum or maximum value for y, as is the case in figures 8.2 and 8.3. One may frequently find the minimum or maximum points in a function by determining when its derivative is equal to zero.

The function $f'(x)$ representing the derivative (or first derivative) indicates the slope of the original function $f(x)$. The function representing the derivative of the derivative $f''(x)$ (the second derivative) indicates the slope of the first derivative function and the

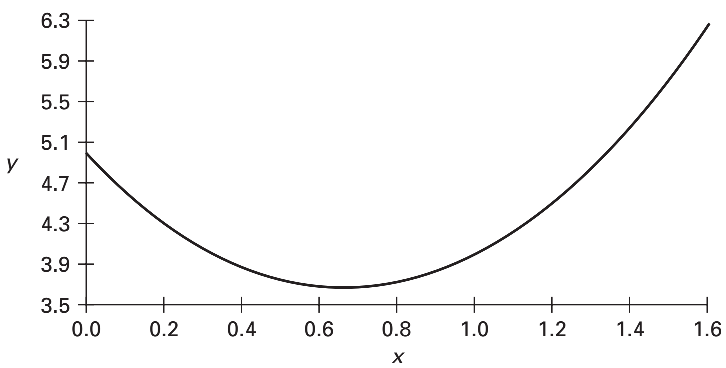

Figure 8.2 Concave up function: $y = 3x^2 - 4x + 5$.

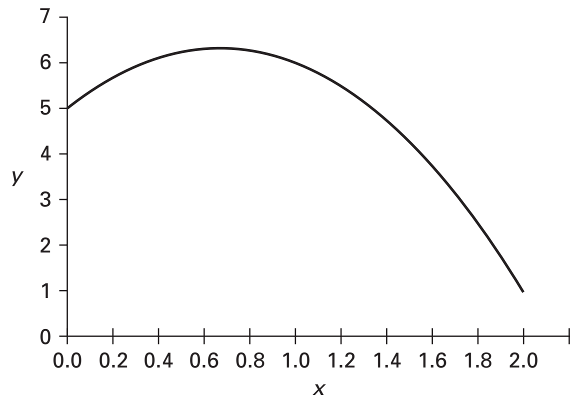

Figure 8.3 Concave down function: $y = -3x^2 + 4x + 5$.

concavity (change in the slope) of the original function $f(x)$. Notice that the slopes in figures 8.1 through 8.3 change as x changes. The rate of change in a slope is determined by the second derivative:

$$\frac{d^2y}{dx^2} = f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}. \tag{8.4}$$

The second derivative $f''(x)$ of a function is simply the derivative of the first derivative. The function represented in figure 8.1 has a positive second derivative when $x > 0$, as suggested by the fact that it appears concave up, indicating that its slope increases as x increases. Note that the first derivative of this function is always positive. The function represented in figure 8.2 has a positive second derivative, as indicated by its upwards concavity. Thus, the slope of this function is increasing as x increases. The slope of this function is negative when x is small, is zero when the function is minimized, and becomes positive when x rises above that level which minimizes y. The function represented in figure 8.3 has a negative second derivative, consistent with the downward concavity in figure 8.3. Thus, the slope of this function decreases as x increases. Its slope is positive when x is small, is zero when the function is maximized, and becomes negative when x rises above that level which maximizes y.

8.3 Derivatives of Polynomials
(Background reading: section 8.2)

The polynomial function is among the most commonly used in financial modeling. The polynomial function specifies variable y in terms of a coefficient c (or series of coefficients c_j), variable x (or series of variables x_j), and an exponent n (or series of exponents n_j). While all the exponents in a polynomial equation will be nonnegative integers, the rules that we discuss here will still apply when the exponents assume negative or noninteger values. Where there exists a single coefficient, variable, and exponent, the polynomial function is represented as follows:

$$y = c \cdot x^n. \tag{8.5}$$

For example, let $c = 3$ and $n = 2$. Our polynomial function is written as follows: $y = 3x^2$. The derivative of y with respect to x in equation (8.5) is the following function:[1]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = c \cdot n \cdot x^{n-1}. \tag{8.6}$$

The derivative of y with respect to x is for this polynomial is found by multiplying the original coefficient c by the original exponent n and subtracting 1 from the original exponent. Taking the derivative of y with respect to x in our example, we obtain $\mathrm{d}y/\mathrm{d}x = 3 \cdot 2 \cdot x^{2-1} = 6x$. Note that this particular derivative is always positive when $x > 0$, implying that the slope of this curve is always positive when $x > 0$. Consider a second polynomial with more than one term (m terms in total). In this second case, there will be one variable x, m coefficients (c_j), and m exponents (n_j):

$$y = \sum_{j=1}^{m} c_j \cdot x^{n_j}. \tag{8.7}$$

The derivative of such a function y with respect to x is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{j=1}^{m} c_j \cdot n_j \cdot x^{n_j-1}. \tag{8.8}$$

That is, the derivative of equation (8.7) is found by simply taking the derivative of each term in y with respect to x and then summing these derivatives. Consider a second example, a second order (the largest exponent is 2) polynomial function given by $y = 3x^2 - 4x + 5$. The derivative of this function with respect to x is $\mathrm{d}y/\mathrm{d}x = 6x - 4$. This function is plotted in figure 8.2. Note that the function is maximized when $x = \frac{2}{3}$. Also note that when $x = \frac{2}{3}$, the slope of the curve equals zero. We can demonstrate that the derivative equals zero as follows:

$$\begin{aligned} 6x - 4 &= 0, \\ 6x &= 4, \\ x &= \tfrac{2}{3}. \end{aligned}$$

[1] This rule is derived in appendix 8.A.

This derivative is positive when $x > \frac{2}{3}$, zero when $x = \frac{2}{3}$, and negative when $x < \frac{2}{3}$. Thus, when $dy/dx > 0$, y increases as x increases; when $dy/dx < 0$, y decreases as x increases; and when $dy/dx = 0$, y may be either minimized or maximized. The slopes in figure 8.2 are consistent with these derivatives. When $x = \frac{2}{3}$, $dy/dx = 0$ and the value of $y = f(x)$ is minimized at $3\frac{2}{3}$.

In many instances, derivatives can be used to find minimum and maximum values of functions. To ensure that we have found a minimum (rather than a maximum), we determine the second derivative. The second derivative of a function indicates its concavity. A positive second derivative indicates upward concavity, which indicates that the function $f(x)$ either increases in x at an increasing rate or decreases in x at a decreasing rate. The function depicted in figure 8.2 exhibits positive concavity. A negative second derivative indicates downward concavity, which indicates that the function $f(x)$ either increases in x at a decreasing rate or increases in x at a decreasing rate.

The second derivative is found by taking the derivative of the first derivative. If the first derivative equals zero and the second derivative is greater than zero, we have a minimum value for y (the function is concave up). If the first derivative is zero and the second derivative is less than zero, we have a maximum (the function is concave down). If the second derivative is zero, we have neither a minimum nor a maximum. The second derivative in the above example is given by $d^2y/dx^2 = 6$, also written $f''(x) = 6$. Since the second derivative 10 is greater than zero, the function $f(x)$ is concave up and we have found a minimum value for y. In many cases, more than one "local" minimum or maximum value will exist.

Consider a third example where $y = -3x^2 + 4x + 5$. The first derivative is $dy/dx = -6x + 4$. Setting the first derivative equal to zero, we find our maximum as follows:

$$-6x + 4 = 0,$$
$$-6x = -4,$$
$$x = \tfrac{2}{3}.$$

We check second order conditions (the second derivative) to ensure that this is a maximum. The second derivative is $d^2y/dx^2 = -6$. Since -6 is less than zero, the function $f(x)$ is concave up and we have a maximum at $\frac{2}{3}$.

APPLICATION 8.2: MARGINAL UTILITY
(Background reading: section 8.3 and application 3.9)

In Application 3.9, we defined a utility of wealth function for a particular individual as follows:

$$U = f(W).$$

Now, let us consider the following more specific utility function:

$$U = 1{,}000W - 0.002W^2 \qquad \text{for } 0 \le W < 250{,}000.$$

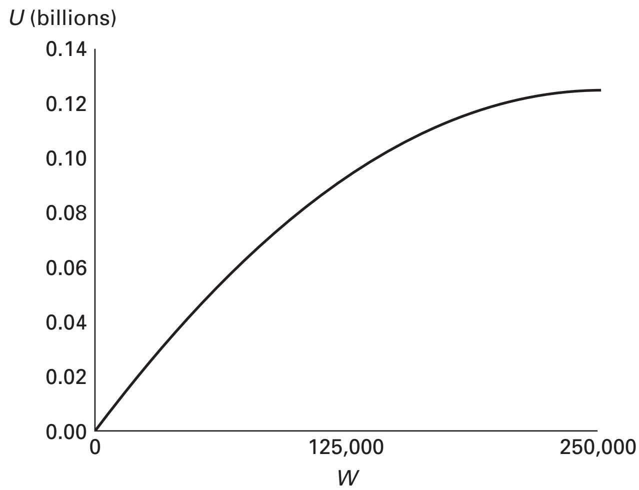

Figure 8.4 Utility of wealth and risk aversion. Utility-of-wealth function for risk-averse individual: $f'(W) > 0$; $f'(W) < 0$. For example, $U_w = 1{,}000W - 0.002W^2$ for $0 < W < 250{,}000$.

The curve for this function is plotted in figure 8.4. In the range from $W = 0$ to $W = 250{,}000$, utility increases as wealth increases. For example, when wealth equals 40,000, utility equals 36,800,000; when wealth equals 50,000, utility equals 45,000,000. Using the polynomial rule, we find the first derivative with respect to W of the utility function as follows:

$$f'(W) = 1{,}000 - 0.004W > 0 \qquad \text{for } 0 \le W < 250{,}000.$$

This derivative is positive, indicating that utility increases as wealth increases – as long as wealth remains within the specified range. When wealth equals 40,000, the derivative of utility with respect to wealth equals 840. This derivative is positive as long as wealth is between 0 and 250,000.

The second derivative of the utility function is found:

$$f''(W) = -0.004 < 0.$$

Thus, this utility function is concave down, indicating diminishing marginal utility with respect to wealth. This implies that rates of increase in utility resulting from increases in wealth become smaller as investors grow wealthier. That is, an individual derives smaller increases in utility when wealthier than when with less wealth. Each additional unit of wealth decreases the level of utility increase resulting from additional wealth by 0.004.

Downward concavity of a utility of wealth function indicates risk aversion. This is because potential wealth increases associated with actuarially fair gambles result in smaller utility changes than potential wealth decreases. This means that a gamble that results in zero change in expected wealth reduces expected utility, because potential lost utility from losing a given amount of wealth exceeds potential the utility gain from an identical increase in wealth. Investors are risk averse because the "pain" derived from a possible wealth loss exceeds the satisfaction derived from an equal possible wealth gain. Consider a gamble in which the investor assumes a risk where his \$40,000 wealth level may either increase to \$50,000 or decrease to \$30,000. Recall that the investor's utility level before the gamble is 36,800,000. Potential utility levels associated with the gamble are obtained as follows:

$$U = 1{,}000 \cdot 50{,}000 - 0.002 \cdot (50{,}000)^2 = 45{,}000{,}000,$$

$$U = 1{,}000 \cdot 30{,}000 - 0.002 \cdot (30{,}000)^2 = 28{,}200{,}000.$$

The utility gain associated with the potential wealth gain of \$50,000 – \$40,000 = \$10,000 is 45,000,000 – 36,800,000 = 8,200,000. The utility loss associated with the potential wealth loss of \$40,000 – \$30,000 = \$10,000 is 36,800,000 – 28,200,000 = 8,400,000. Note that the potential utility loss exceeds the potential utility gain.

APPLICATION 8.3: DURATION AND IMMUNIZATION
(Background reading: sections 4.8, 4.9, 5.4, and 8.3 and application 4.4)

United States Treasury bonds, notes, and bills are generally regarded to be free of default risk and to have very low liquidity risk. However, these bonds, particularly those with longer terms to maturity, are still subject to price fluctuations when they are traded in secondary markets. The primary sources of these fluctuations are changes in interest rates offered on new issues. Such interest rate changes frequently have some element of unpredictability. One should expect that as interest rates on newly issued bonds increase, values of existing bonds will decrease. Similarly, interest rate decreases affecting new bond issues will increase the values of bonds which are already outstanding. The duration model is intends to measure the proportional change in the value of an existing bond induced by a change in interest rates or yields of new issues.

The simple present-value model is most useful for the valuation of bonds and other fixed-income instruments. Yields to maturity of priced issues are frequently used as discount rates to value new issues and to value other issues with comparable terms. It is important for analysts to know how changes in new-issue interest rates or yields will affect values of other fixed-income instruments. Bond *duration* measures the proportional price sensitivity of a bond to changes in the market rate of interest (or yields at which comparable bonds are selling). Consider a two-year 8% coupon Treasury issue that is currently selling for \$982.41. The yield to maturity y of this bond is 9%. Since default risk and liquidity risk are presumed to be insignificant, interest rate risk is of primary concern. Assume that this bond's yield or discount rate is consistent with market yields of comparable Treasury issues. Further assume that bonds of all terms to maturity have the same yield and that the market prices all bonds at their present values. Thus, discount rates k for all bonds equal their yields to maturity y and these yields are invariant with respect to bond terms to maturity. If market interest rates and yields were to rise for new Treasury issues, then the yield of this bond would rise accordingly. However, since the contractual terms of this bond will not change, the bond's market price must drop to accommodate a yield consistent with the market. More generally, the value of an n-year bond paying interest at a rate of c on face value F is determined by a present-value model with the yield y of comparable issues serving as the discount rate k:

$$PV = \sum_{t=1}^{n} \frac{cF}{(1+y)^t} + \frac{F}{(1+y)^n}. \tag{8.9}$$

Assume that the terms of the bond contract, n, F, and c, are constant. We want to measure the proportional change in the price of a bond induced by a proportional change in market interest rates (actually, a proportional change in $[1 + y]$). This proportional change or elasticity may be approximated by the bond's Macaulay Simple Duration Formula, as follows:

$$\frac{\Delta PV}{PV} \div \frac{\Delta(1+y)}{1+y} \approx Dur = \frac{\mathrm{d}PV}{PV} \div \frac{\mathrm{d}(1+y)}{1+y} = \frac{\mathrm{d}PV}{\mathrm{d}(1+y)} \cdot \frac{1+y}{PV}. \tag{8.10}$$

Equation (8.10) generally provides a reasonably good approximation of the proportional change in the value of a bond in a market meeting the assumptions described above, induced by an infinitesimal proportional change in $1 + y$. To compute the bond's sensitivity, we first rewrite equation (8.9) in polynomial form (to take derivatives later) and substitute y for k (since they are assumed to be equal):

$$PV = \sum_{t=1}^{n} \frac{cF_t}{(1+y)^t} + \frac{F}{(1+y)^n} = \sum_{t=1}^{n} cF(1+y)^{-t} + F(1+y)^{-n}. \tag{8.11}$$

We find the derivative of PV with respect to $(1 + y)$:

$$\frac{\mathrm{d}PV}{\mathrm{d}(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1}. \tag{8.12}$$

Equation (8.12) is rewritten

$$\frac{\mathrm{d}PV}{\mathrm{d}(1+y)} = \frac{\sum_{t=1}^{n} -tcF(1+y)^{-t} - nF(1+y)^{-n}}{1+y}. \tag{8.13}$$

Since the market rate of interest is assumed to equal the bond yield to maturity, the bond's price will equal its present value. We multiply both sides of equation (8.14) by $(1 + y) \div P_0$ to maintain consistency with equation (8.10), and obtain the duration formula as follows:

$$Dur = \frac{\mathrm{d}PV}{\mathrm{d}(1+y)} \cdot \frac{1+y}{P_0} = \frac{\sum_{t=1}^{n} -tcF(1+y)^{-t} - nF(1+y)^{-n}}{P_0}. \tag{8.14}$$

Thus, duration is defined as the proportional price change of a bond induced by an infinitesimal proportional change in $(1 + y)$ or 1 plus the market rate of interest:

$$Dur = \frac{\mathrm{d}PV}{\mathrm{d}(1+y)} \cdot \frac{1+y}{P_0} = \frac{\sum_{t=1}^{n} \frac{-tcF}{(1+y)^t} + \frac{-nF}{(1+y)^n}}{P_0}. \tag{8.15}$$

Since the market rate of interest will likely determine the yield to maturity of any bond, the duration of the bond described above is determined as follows from equation (8.15):

$$Dur = \frac{\frac{-1 \cdot 0.08 \cdot 1{,}000}{1+0.09} + \frac{-2 \cdot 0.08 \cdot 1{,}000}{(1+0.09)^2} + \frac{-2 \cdot 1{,}000}{(1+0.09)^2}}{982.41} = -1.925.$$

This duration level of -1.925 suggests that the proportional decrease in the value of this bond would equal 1.925 times the proportional increase in market interest rates. This duration level also implies that this bond has exactly the same interest rate sensitivity as a *pure discount bond* (a bond making no coupon payments, also known as a zero coupon bond) which matures in 1.925 years.

Application of the Simple Macaulay Duration model does require several important assumptions, some of which were described above. The accuracy of the model will be impaired by violations in these assumptions. First, the model assumes that yields are invariant with respect to maturities of bonds; that is, the yield curve is flat. Second, the model assumes that the investor's reinvestment rate for coupon payments will be identical to the bond's yield to maturity. Finally, any change in interest rates will be infinitesimal and will also be invariant with respect to time.

In chapter 7, we discussed bond portfolio dedication, which is concerned with exactly matching payouts required to satisfy liabilities with cash flows to be derived from bond portfolios. This process assumes that the portfolio is not adjusted by adding or selling bonds over time and that cash flows associated with liabilities will remain as originally anticipated. With shifting interest rates over time, these assumptions will not hold for many institutions. Alternatively, one may hedge fixed income portfolio risk by using *immunization* strategies, which are concerned with matching the present values of asset portfolios with the present values of cash flows associated with future liabilities. In particular, immunization strategies are primarily concerned with matching the duration of the asset side of the institution with the duration of the liability side of the portfolio. If institutional asset and liability durations are matched, it is expected that the net fund value (the fund's equity or surplus) will not be affected by a shift in interest rates. Thus, overall fund risk is minimized as fund asset and liability value changes offset each other. This simple immunization strategy requires the same assumptions as the Simple Duration Model:

1 Changes in $1 + y$ are infinitesimal.
2 The yield curve is flat (yields do not vary over terms to maturity).
3 All yields change by the same amount, regardless of term to maturity.
4 Only interest rate risk is significant.

APPLICATION 8.4: PORTFOLIO RISK AND DIVERSIFICATION (Background reading: sections 6.1, 6.4, 6.5 and 8.2)

Differential calculus can be used to demonstrate that the risk of a portfolio decreases as the number of securities in the portfolio increases. First, in section 6.1, we defined portfolio variance as follows:

$$\sigma_{\mathrm{p}}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}$$

$$= \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} w_i w_j \sigma_{i,j} + \sum_{i=1}^{n} w_i^2 \sigma_i^2. \tag{A}$$

We will argue here that, as n increases, σ_{p}^2 decreases. For the sake of simplicity, we will assume the following:

1 The portfolio is equally weighted in n securities; that is, $w_i = w_j = 1/n$ for each security.
2 All securities have the same variance, σ_i^2.
3 Each security has the same covariance $\sigma_{i,j}$ with every other security. This covariance will be equal to the average covariance between pairs of securities.
4 Returns on component securities are not perfectly correlated.

Now we rewrite equation (A), substituting $1/n$ for w_i and w_j:

$$\sigma_{\mathrm{p}}^2 = \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{1}{n}\frac{1}{n}\sigma_{i,j} + \sum_{i=1}^{n} \frac{1}{n^2}\sigma_i^2. \tag{B}$$

By definition, $\sum_{i=1}^{n}(1/n)\sigma_i^2$ is the mean of the security variances. Thus, the right-side term following the "+" is the mean security variance divided by n. There will be a total of $n(n-1)$ covariance terms. The average covariance between pairs of securities is written

$$\bar{\sigma}_{1,j} = \sum_{j=1}^{n} \frac{1}{n}\sigma_{i,j}. \tag{C}$$

Since the average covariance term will be added $n - 1$ times, we rewrite portfolio variance as follows:

$$\sigma_p^2 = \sum_{j=1}^{n-1} \frac{1}{n}\bar{\sigma}_{i,j} + \frac{1}{n}\bar{\sigma}_i^2 = \frac{(n-1)}{n}\bar{\sigma}_{i,j} + \frac{1}{n}\bar{\sigma}_i^2 = \bar{\sigma}_{i,j} - \frac{1}{n}\bar{\sigma}_{i,j} + \frac{1}{n}\bar{\sigma}_i^2$$

$$= \bar{\sigma}_{i,j} - n^{-1}\bar{\sigma}_{i,j} + n^{-1}\bar{\sigma}_i^2 = \bar{\sigma}_{i,j} + n^{-1}(\bar{\sigma}_i^2 - \bar{\sigma}_{i,j}). \tag{D}$$

To demonstrate that portfolio variance decreases as n (the number of securities) increases, we simply show that the derivative of σ_p^2 with respect to n is negative:

$$\frac{d\sigma_p^2}{dn} = -n^{-2}(\bar{\sigma}_i^2 - \bar{\sigma}_{i,j}) < 0, \tag{E}$$

which will be true whenever the average security variance exceeds the average covariance between different securities. This must hold whenever the correlation coefficient between security returns is less than one.

We can see from the final part of equation (D) that as the number of securities in the portfolio approaches infinity, the portfolio's risk approaches the average covariance between pairs of securities. Individual security variances are insignificant except to the extent that they affect covariances. Thus, only covariance is significant for large, well-diversified portfolios. If security returns are entirely independent ($\sigma_{ij} = 0$), portfolio risk approaches zero as the number of securities included in the portfolio approaches infinity.

8.4 Partial and Total Derivatives
(Background reading: section 8.3)

Thus far, we have focused on univariate functions, where a dependent variable y is expressed as a function of a single independent variable x. A multivariate function is expressed as a dependent variable y and a series of independent variables (e.g., $y = f[x_1, x_2]$). The partial derivative (e.g., $\partial y/\partial x_1$) of a function expresses the rate of change in the dependent variable induced by change in one of its independent variables, while holding other variables constant. The following represents a multivariate function and its relevant partial derivatives:

$$y = 10x_1^4 + 3x_2^3 + 2x_1^2x_2,$$

$$\frac{\partial y}{\partial x_1} = 40x_1^3 + 4x_1x_2,$$

$$\frac{\partial y}{\partial x_2} = 9x_2^2 + 2x_1^2.$$

Finance practitioners frequently deal with changes in the dependent variable induced by simultaneous changes in independent variables. One may use the total derivative dy to indicate changes in the dependent variable y induced by changes in one or more of n independent variables x_i:

$$dy = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} dx_i.$$

In the above example, the total derivative would be determined as follows:

$$\mathrm{d}y = \frac{\partial y}{\partial x_1}\mathrm{d}x_1 + \frac{\partial y}{\partial x_2}\mathrm{d}x_2 = (40x_1^3 + 4x_1x_2)\mathrm{d}x_1 + (9x_2^2 + 2x_1^2)\mathrm{d}x_2.$$

This total derivative dy is expressed as a function of derivatives of y with respect to each of the independent variables x_i and infinitesimal changes in each of these independent variables.

8.5 THE CHAIN RULE, PRODUCT RULE, AND QUOTIENT RULE
(Background reading: sections 8.3 and 8.4)

Each of the functions discussed in section 8.3 are written in polynomial form. Many functions are not or cannot be written in this manner. Other rules must be derived to find derivatives for these functions. The chain rule may be used to find derivatives for some of them. For example, consider the following function:

$$y = 7(5 + 3x)^2.$$

Although this function can be written in polynomial form ($y = 63x^2 + 210x + 175$; its derivative using the polynomial rule is $126x + 210$), we will apply the chain rule to find its derivative. The first step here in applying the chain rule is to define a function u such that $u = (5 + 3x)$. Now we write y as $y = 7u^2$. Also note that du/dx = 3 and dy/du = 14u. The chain rule is quite simple, although it does have useful and powerful implications:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{8.16}$$

That is, if y can be written as a function of function u, which itself is a function x, then the derivative of y with respect to x equals the derivative of y with respect to function u multiplied by the derivative of function u with respect to x. In our example, we find the derivative of y with respect to x as follows:

$$\frac{\partial y}{\partial x} = \underset{c}{7} \cdot \underset{n}{2} \cdot \underset{u}{(5 + 3x)}^{\underset{n-1}{1}} \cdot \underset{\frac{\partial u}{\partial x}.}{3} = 14(5 + 3x)\cdot 3 = 210 + 126x,$$

In the calculations above, $c \cdot n \cdot u^{n-1}$ = dy/du and 3 is du/dx.

Another highly useful tool from calculus is the product rule, which may be applied to a function such as $y = u \cdot v$. Consider the function $(3x + 5)(7x + 4)$, where function u is $3x + 5$ and function v is $7x + 4$. The product rule, defined as follows, may be applied to find the derivative of function y, where function y equals function u times function v ($y = u \cdot v$):

$$\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} v + \frac{\partial v}{\partial x} u. \tag{8.17}$$

In our example, the derivative of y with respect to x may be found as follows:

$$\frac{\partial y}{\partial x} = 3(7x + 4) + 7(3x + 5) = 42x + 47,$$

$$\frac{\partial u}{\partial x} \quad v \quad \frac{\partial v}{\partial x} \quad u.$$

Another highly useful tool from calculus is the quotient rule, which may be applied to a function such as $y = u \div v$. Consider the function $y = (3x + 5)/(7x + 4)$. Again, define a function u (the numerator) as $3x + 5$ and a function v (the denominator) as $7x + 4$. The quotient rule, defined as follows, may be applied to find the derivative of function y:

$$\frac{\partial y}{\partial x} = \left[\frac{\partial u}{\partial x} v - \frac{\partial v}{\partial x} u\right] \div v^2 \tag{8.18}$$

The quotient rule states that the derivative of y with respect to x equals the derivative of the numerator with respect to x times the denominator minus the derivative of the denominator with respect to x times the numerator all divided by the denominator squared. Thus, the derivative of y with respect to x is found as follows:

$$\frac{dy}{dx} = \left[\frac{3(7x + 4) - 7(3x + 5)}{(7x + 4)^2}\right] = \left[\frac{-1}{49x^2 + 56x + 16}\right].$$

APPLICATION 8.5: PLOTTING THE CAPITAL MARKET LINE (Background reading: sections 6.4, 7.4, and 8.5)

In section 6.4, we discussed the addition of securities with varying correlation coefficients to a portfolio and the impact of these additional securities on the risk and efficiency of that portfolio. More efficient combinations of securities can reduce portfolio risk without decreasing expected portfolio returns. As we add securities to a portfolio, the fact that they are not perfectly correlated with other securities in that portfolio will lead to reductions in portfolio risk and increases in portfolio efficiency.

Portfolio efficiency improves when portfolio risk is reduced without corresponding reductions in portfolio expected return. In this application, we are concerned with obtaining portfolio weights for the most efficient combinations of assets available in the economy. We will map out the most efficient combinations of securities at varying expected return levels. These most efficient portfolio combinations will be those which have the lowest risk of any potential portfolio with the same expected return. The set of portfolios which have the lowest levels of risk given their expected return levels have risk/expected returns lying on the Efficient Frontier. The Efficient Frontier represents the risk–return combinations (in standard deviation–expected-return space) of this set of most efficient portfolios available in the market. It contains the risk–return coordinates of each portfolio that minimizes risk given an expected portfolio return level. Similarly, the Efficient Frontier contains the risk–return coordinates of each portfolio that maximizes return given a risk level.

In any market with more than one security, there are infinitely many ways to combine those securities into portfolios. While investors cannot affect the characteristics (return, risk, and covariances) of the individual securities in which they invest, they are able to control portfolio weights. Investors should select securities and assign them weights in their personal portfolios such that the portfolios are as efficient as possible.

Each investor in the market intends to minimize risk by selecting each of the securities available in the market with the most appropriate portfolio weightings. Each security is considered for each investor's portfolio, with weights greater than zero (the security is purchased), equal to zero (the security is left out of the portfolio), or less than zero (the security is sold short).

Suppose that an investor is able to invest in some combination of risky assets (stocks) and a riskless asset (bond). Any portfolio consisting of a portfolio of risky assets and a risk-free asset will have an expected return and standard deviation combination lying on a line as in figure 8.5. This line represents the portfolio possibilities frontier

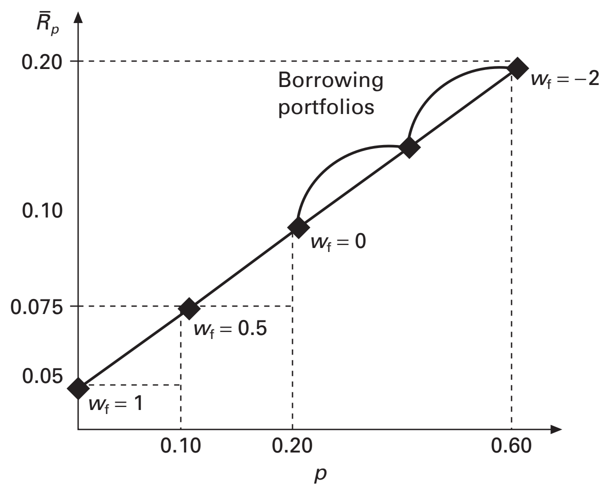

Figure 8.5 Combining a portfolio of risky assets with the riskless asset.

for the combination of risky assets and the riskless asset. Consider a portfolio of risky assets with an expected return equal to 0.10 and a standard deviation of returns equal to 0.20. Suppose that this portfolio of risky assets is to be combined with a riskless asset with a return of 5%. The expected return and standard deviation combination of this resultant portfolio will be determined by the portfolio weights that the investor associates with the portfolio of risky assets (w_R) and the riskless asset (w_f) :

$$\mathrm{E}[R_\mathrm{p}] = w_R \cdot 0.10 + w_f \cdot 0.05, \tag{A}$$

$$\sigma_\mathrm{p} = \sqrt{w_R^2 \cdot 0.20^2 + w_f^2 \cdot 0^2 + 2(w_R \cdot w_f \cdot 0.1 \cdot 0 \cdot 0)}, \tag{B}$$

$$\sigma_\mathrm{p} = \sqrt{w_R^2 \cdot \sigma_R^2 + 0 + 0}, \tag{C}$$

$$\sigma_\mathrm{p} = w_R \cdot \sigma_R. \tag{D}$$

By rewriting equation (D) as $w_R = \sigma_\mathrm{p} \div \sigma_R$, and substituting it into equation (A) for w_R, we find that the expected return of a portfolio combining risky assets and the riskless asset is a linear function of the return standard deviation of the portfolio of risky assets as follows:

$$\mathrm{E}[R_\mathrm{p}] = \frac{\sigma_\mathrm{p}}{\sigma_R} \cdot 0.10 + w_f \cdot 0.05, \tag{E}$$

$$\mathrm{E}[R_\mathrm{p}] = \frac{\sigma_\mathrm{p}}{\sigma_R} \cdot 0.10 + \left(1 - \frac{\sigma_\mathrm{p}}{\sigma_R}\right) \cdot 0.05, \tag{F}$$

$$\mathrm{E}[R_\mathrm{p}] = 0.05 + \frac{\sigma_\mathrm{p}}{\sigma_R} \cdot (0.10 - 0.05). \tag{G}$$

More generally, equation (G) can be written as follows:

$$\mathrm{E}[R_\mathrm{p}] = r_f + \frac{\sigma_\mathrm{p}}{\sigma_R} \cdot (\mathrm{E}[R_\mathrm{p}] - r_f). \tag{8.19}$$

Defining Θ_p as $\{\mathrm{E}[R_\mathrm{p}] - r_f\} \div \sigma_R$, we obtain the linear relationship between portfolio expected return and standard deviation as follows:

$$\mathrm{E}[R_\mathrm{p}] = r_f + \frac{\mathrm{E}[R_\mathrm{p}] - r_f}{\sigma_R} \cdot \sigma_\mathrm{p} = r_f + \Theta_\mathrm{p} \cdot \sigma_\mathrm{p}.$$

Theta (Θ_p) is the slope of the portfolio possibilities frontier depicted in figure 8.5. If there is a riskless asset in the economy, all of the most efficient portfolios will have a risk–return combination lying on a portfolio possibilities frontier known as the Capital

Market Line. The Capital Market Line is the portfolio possibilities frontier with the greatest slope. The investor's objective is to choose that portfolio of risky assets enabling him to maximize the slope of this line; that is, the investor should pick that portfolio with the largest possible Θ_p as defined by:

$$\Theta_p = \frac{E[R_p] - r_f}{\sigma_p}. \tag{8.20}$$

Consider an investor who has the opportunity to combine any one of an infinity of portfolios with the riskless asset. However, only one of these portfolios (if none are perfectly correlated) can be used to comprise the Capital Market Line. Hence, the investor's portfolio objective is to select portfolio weights w_i such that Θ_p is maximized:

$$\Theta_p = \frac{E[R_p] - r_f}{\sigma_p} = \frac{\sum_{i=1}^{n} w_i(E[R_i] - r_f)}{\sqrt{\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (w_i w_j \sigma_{i,j})}}.$$

This portfolio that maximizes Θ_p is known as the market portfolio of risky assets. Any combination of the market portfolio and the riskless asset will lie on the Capital Market Line and will dominate (be more efficient than) any other asset or portfolio with equal expected return. Thus, any investor should invest in some combination of this dominant portfolio of risky assets (stocks) and the riskless asset (bonds). The investor's first problem is to select the portfolio weights which comprise the market portfolio of stocks.

Suppose that the investor can invest in any combination of two risky securities 1 and 2. The expected returns, standard deviations, and covariance are given as follows:

$$E[R_1] = 0.11, \qquad \sigma_1 = 0.20,$$
$$E[R_2] = 0.15, \qquad \sigma_2 = 0.40, \qquad \sigma_{1,2} = -0.01.$$

The Treasury bill (riskless) rate in this economy is 0.06. We will now derive the set of equations that the investor needs to find his optimal portfolio of riskless assets.

Since the investor's objective is to select a portfolio of the two risky assets such that the slope of the Capital Market Line is maximized, we will select stock portfolio weights such that Θ_p is maximized. To accomplish this, we will find partial derivatives of Θ_p with respect to weights of each of each of the two stocks, set the partial derivatives equal to zero and solve for the weight values w_1 and w_2. First, we write Θ_p for the simple two-stock portfolio as follows:

$$\Theta_p = \frac{E[R_p] - r_f}{\sigma_p} = \frac{w_1(E[R_1] - r_f) + w_2(E[R_2] - r_f)}{(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12})^{1/2}}. \tag{A}$$

Next, we use the quotient rule to begin the process of finding the partial derivatives of Θ_p with respect to w_1 and w_2:

$$\frac{\partial \Theta_p}{\partial w_1} = \frac{\dfrac{\partial(\mathrm{E}[R_p] - r_f)}{\partial w_1}\sigma_p - \dfrac{\partial \sigma_p}{\partial w_1}(\mathrm{E}[R_p] - r_f)}{\sigma_p^2} = 0, \tag{B1}$$

$$\frac{\partial \Theta_p}{\partial w_2} = \frac{\dfrac{\partial(\mathrm{E}[R_p] - r_f)}{\partial w_2}\sigma_p - \dfrac{\partial \sigma_p}{\partial w_2}(\mathrm{E}[R_p] - r_f)}{\sigma_p^2} = 0. \tag{B2}$$

Before completing this process, we use the chain rule to find the partial derivatives $\partial\sigma_p/\partial w_1$ and $\partial\sigma_p/\partial w_1$ (from the denominator of equation (A)):

$$\frac{\partial \sigma_p}{\partial w_1} = \tfrac{1}{2}(\sigma_p^2)^{-1/2}(2w_1\sigma_1^2 + 2w_2\sigma_{12}) = \frac{w_1\sigma_1^2 + w_2\sigma_{12}}{\sigma_p}, \tag{C1}$$

$$\frac{\partial \sigma_p}{\partial w_2} = \tfrac{1}{2}(\sigma_p^2)^{-1/2}(2w_2\sigma_2^2 + 2w_1\sigma_{12}) = \frac{w_2\sigma_2^2 + w_1\sigma_{12}}{\sigma_p}. \tag{C2}$$

We can find from equation (A) that the derivative of $(\mathrm{E}[R_p] - r_f)$ with respect to w_i equals $(\mathrm{E}[R_i] - r_f)$. Next, we substitute our results of equations (C1) and (C2) into equations (B1) and (B2), to obtain:

$$\frac{\partial \Theta_p}{\partial w_1} = \frac{(\mathrm{E}[R_1] - r_f)\sigma_p - (\mathrm{E}[R_p] - r_f)(w_1\sigma_1^2 + w_2\sigma_{12})/\sigma_p}{\sigma_p^2} = 0, \tag{D1}$$

$$\frac{\partial \Theta_p}{\partial w_2} = \frac{(\mathrm{E}[R_2] - r_f)\sigma_p - (\mathrm{E}[R_p] - r_f)(w_2\sigma_2^2 + w_1\sigma_{12})/\sigma_p}{\sigma_p^2} = 0. \tag{D2}$$

Because the derivatives from equations (D1) and (D2) are both set equal to zero, we may multiply the numerator of each by σ_p and maintain the equalities. Next, we rewrite the equations as follows:

$$\begin{aligned} \mathrm{E}[R_1] - r_f &= \frac{(\mathrm{E}[R_p] - r_f)(w_1\sigma_1^2 + w_2\sigma_{12})}{\sigma_p^2}, \\ \mathrm{E}[R_2] - r_f &= \frac{(\mathrm{E}[R_p] - r_f)(w_1\sigma_{12} + w_2\sigma_2^2)}{\sigma_p^2}; \end{aligned} \tag{E1}$$

$$\begin{aligned} \mathrm{E}[R_1] - r_f &= \frac{(\mathrm{E}[R_p] - r_f)(w_1\sigma_1^2 + w_2\sigma_{12})}{\sigma_p^2}, \\ \mathrm{E}[R_2] - r_f &= \frac{(\mathrm{E}[R_p] - r_f)(w_1\sigma_{12} + w_2\sigma_2^2)}{\sigma_p^2}. \end{aligned} \tag{E2}$$

To continue the process of simplification, define the variable z_i to be $w_i(\mathrm{E}[R_\mathrm{p}] - r_f)/\sigma_\mathrm{p}^2$ and rewrite equations (E1) and (E2) as follows:

$$\mathrm{E}[R_1] - r_f = z_1\sigma_1^2 + z_2\sigma_{1,2},$$
$$\mathrm{E}[R_2] - r_f = z_1\sigma_{2,1} + z_2\sigma_2^2.$$

Substituting numerical values from our example, we have

$$0.11 - 0.06 = 0.2^2 z_1 - 0.01z_2,$$
$$0.15 - 0.06 = 0.01z_1 + 0.4^2 z_2,$$

or

$$0.05 = 0.04z_1 - 0.01z_2,$$
$$0.09 = -0.01z_1 + 0.16z_2.$$

Solving the above simultaneously yields $z_1 = 1.41$ and $z_2 = 0.65$. Since $\mathrm{E}[R_\mathrm{p}]$, r_f, and σ_p are the same for both z_1 and z_2, the ratio between w_1 and w_2 must be the same as the ratio between z_1 and z_2. Therefore, portfolio weights w_1 and w_2 are determined as follows:

$$w_1 = z_1 \div (z_1 + z_2) = 0.68,$$
$$w_2 = z_2 \div (z_1 + z_2) = 0.32.$$

Thus, 68% and 32% of this investor's wealth will be invested in securities 1 and 2. This represents the market portfolio (m) to the investor and is the most efficient combination of risky assets given the prevailing riskless rate of return at 6%. The return and risk levels of the portfolio (m) with two risky stocks are simply

$$\mathrm{E}[R_\mathrm{m}] = 0.68 \cdot 0.11 + 0.32 \cdot 0.15 = 0.12,$$
$$\sigma_\mathrm{m} = [0.68^2 \cdot 0.04 + 0.32^2 \cdot 0.16 + 2 \cdot 0.68 \cdot 0.32 \cdot (-0.01)]^{0.5} = 0.17.$$

The equation for the Capital Market Line is as follows:

$$\begin{aligned}\mathrm{E}[R_\mathrm{p}] &= r_f + \frac{\mathrm{E}[R_\mathrm{m}] - r_f}{\sigma_\mathrm{m}} \cdot \sigma_\mathrm{p} \\ &= 0.06 + \frac{0.12 - 0.06}{0.17} \cdot \sigma_\mathrm{p} = 0.06 + 0.35\sigma_\mathrm{p}.\end{aligned}$$

These results enable the investor to determine weightings for his optimal portfolio of risky assets (stocks). The investor allocates funds to this most efficient portfolio of risky assets and bonds based on his own attitudes toward return and risk. Finally, this derivation can easily be extended to include as many securities as exist in the market. Matrix mathematics such as the methodologies discussed in section 7.4 may simplify computations of larger systems.

8.6 Logarithmic and Exponential Functions

(Background reading: sections 2.5, 2.6, 8.1, and 8.2)

Logarithmic and exponential functions and derivatives of these functions are most useful in finance for modeling growth. Consider the following function:

$$f(x) = \mathrm{e}^{tx}.$$

The derivative of this exponential function is found as follows:

$$f'(x) = \lim_{h \to 0} \frac{\mathrm{e}^{t(x+h)} - \mathrm{e}^{tx}}{h} = \mathrm{e}^{tx} \lim_{h \to 0} \frac{\mathrm{e}^{th} - 1}{h}.$$

It is easy to demonstrate (or verify on a hand calculator) that as h approaches zero, $(\mathrm{e}^{th} - 1)/h$ will approach t. Thus, the derivative of the exponential function is found as follows:

$$\frac{\mathrm{d}\mathrm{e}^{tx}}{\mathrm{d}x} = t\mathrm{e}^{tx}. \tag{8.21}$$

Thus, the derivative of $\mathrm{e}^{0.05x}$ with respect to x is simply $0.05\mathrm{e}^{0.05x}$. If we accept the special case of equation (8.21) that $\mathrm{d}\mathrm{e}^x/\mathrm{d}x = \mathrm{e}^x$, the following more general expression for the derivative of an exponential function can be verified using the chain rule:

$$\frac{\mathrm{d}\mathrm{e}^{g(x)}}{\mathrm{d}x} = g'(x)\mathrm{e}^{g(x)}. \tag{8.22}$$

From equation (8.22), we can derive a function for the derivative of a logarithmic function $y = \ln(x)$. First, by definition, $\mathrm{e}^{\ln(x)} = x$, implying that $\mathrm{d}\mathrm{e}^{\ln(x)}/\mathrm{d}x = 1$. Now, consider the following special case of equation (8.22):

$$\frac{\mathrm{d}\mathrm{e}^{\ln(x)}}{\mathrm{d}x} = \frac{\mathrm{d}\ln(x)}{\mathrm{d}x}\mathrm{e}^{\ln(x)}.$$

This equation is written

$$1 = \frac{\mathrm{d}\ln(x)}{\mathrm{d}x} \cdot x,$$

which implies

$$\frac{\mathrm{d}\ln(x)}{\mathrm{d}x} = \frac{1}{x}. \tag{8.23}$$

8.7 Taylor Series Expansions

(Background reading: section 8.3)

The derivative was used in section 8.3 to determine the rate of change in $f(x)$ induced by a change in x. Unfortunately, when $f(x)$ is not linear in x, the estimates of change based on this derivative is normally accurate only for infinitesimal changes in x. The Taylor series approximation or expansion may be used for finite changes in x. Taylor series expansions are frequently used to evaluate a function $f(x_1)$ at a point x_1 that differs from a starting point x_0 at which $f(x_0)$ has already been evaluated. That is, the Taylor series may be used to approximate a rate of change in $f(x)$ induced by a change in x. An nth order Taylor series is defined as follows for a function $f(x)$ that is differentiable n times:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x + \frac{1}{2!} \cdot f''(x_0) \cdot (\Delta x)^2 + \frac{1}{3!} \cdot f'''(x_0) \cdot (\Delta x)^3 + \ldots + \frac{1}{n!} \cdot f^n(x_0) \cdot (\Delta x)^n. \tag{8.24}$$

For example, consider the function $y = 10x^3$. Let $x_0 = 2$, such that we have $f(x_0) = 80$, $f'(x_0) = 30x_0^2 = 120$, $f''(x_0) = 60x_0 = 120$, and $f'''(x_0) = 60$, and all higher order derivatives are equal to zero. Now, suppose that we wish to increase x by $\Delta x = 3$ to $x_1 = 5$. The Taylor series expansion may be used to evaluate x_1 as follows:

$$f(2 + 3) = f(2) + f'(2) \cdot 3 + \frac{1}{2!} \cdot f''(2) \cdot 3^2 + \frac{1}{3!} \cdot f'''(2) \cdot 3^3$$
$$= 80 + 120 \cdot 3 + \tfrac{1}{2} \cdot 120 \cdot 9 + \tfrac{1}{6} \cdot 60 \cdot 27 = 80 + 360 + 540 + 270 = 1{,}250.$$

This third order expansion provided an exact solution for $f(5) = 1{,}250$. This third order expansion provided an exact solution for $f(x)$ because $f(x)$ was differentiable only three times (fourth and higher derivatives equal zero) and our approximation used all three nonzero derivatives. In many cases, we will be able to obtain reasonable approximations, although not precise solutions, where not all nonzero derivatives are used. In this example, first order approximation results in $f(5) \approx 440$ and the second order approximation results in $f(5) \approx 980$:

$$f(2 + 3) \approx f(2) + f'(2) \cdot 3 = 80 + 120 \cdot 3 = 80 + 360 = 440,$$

$$f(2 + 3) \approx f(2) + f'(2) \cdot 3 + \frac{1}{2!} \cdot f''(2) \cdot 3^2$$
$$= 80 + 120 \cdot 3 + \tfrac{1}{2} \cdot 120 \cdot 9 = 80 + 360 + 540 = 980.$$

Note that the second order approximation based on first and second derivatives is superior to the first order approximation based only on the first derivative. Generally, Taylor series approximations will improve as order of the approximating equation increases (as n increases). If the equation is differentiable n or fewer times, the Taylor series approximation of the nth order will be precise.

APPLICATION 8.6: CONVEXITY AND IMMUNIZATION (Background reading: section 8.7 and application 8.4)

The duration model in application 8.4 was used to approximate the change in a bond's value resulting from a change in interest rates $1 + y$. However, the accuracy of the duration model is reduced when interest rates change by finite amounts. This is important because interest rate changes can be quite sudden and quite large. Duration may be regarded as yielding a first order approximation (it only uses the first derivative) of the change in the value of a bond resulting from a change in interest rates. As we saw above, a second order approximation will probably yield superior estimates to the first order approximation. *Convexity* is determined by the second derivative of the bond's value with respect to $1 + y$. Recall that the first derivative of the bond's price with respect to $1 + y$ is

$$\frac{\partial P_0}{\partial(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1}. \tag{8.25}$$

We find the second derivative by solving for the derivative of equation (8.25) as follows:

$$\begin{aligned}\frac{\partial^2 P_0}{\partial(1+y)^2} &= \left[\sum_{t=1}^{n} -t(-t-1)cF(1+y)^{-t-2}\right] - [n(-n-1)F(1+y)^{-n-2}]\\ &= \left[\sum_{t=1}^{n} \frac{(t^2+t)cF}{(1+y)^{t+2}}\right] + \left[\frac{(n^2+n)F}{(1+y)^{n+2}}\right].\end{aligned} \tag{8.26}$$

This second derivative divided by the bond's price as defined as the bond's convexity. Thus, convexity is the second derivative of P_0 with respect to $1 + y$ divided by P_0. The two derivatives given by equations (8.25) and (8.26) may be used in a Taylor series expansion to approximate new bond prices affected by changes in interest rates:

$$P_1 \approx P_0 + f(1+y_0) \cdot [\Delta(1+y)] + \frac{1}{2!} \cdot f''(1+y_0) \cdot [\Delta(1+y)]^2,$$

$$\begin{aligned}P_1 \approx P_0 &+ \left[\sum_{t=1}^{n} \frac{-tcF_t}{(1+y_0)^{t+1}} - \frac{nF}{(1+y_0)^{n+1}}\right] \cdot [\Delta y]\\ &+ \tfrac{1}{2}\left[\sum_{t=1}^{n} \frac{(t^2+t)\cdot cF_t}{(1+y_0)^{t+2}} + \frac{(n^2+n)\cdot F}{(1+y_0)^{n+2}}\right] \cdot [\Delta y]^2.\end{aligned} \tag{8.27}$$

Consider an example involving a three-year 6% $1,000-face-value coupon bond currently selling at par (face value). The yield to maturity of this bond is computed to be $y_0 = 0.06$. The first derivative of the bond's market value with respect to $1 + y$ at $y_0 = 0.06$ is found by equation (8.25) to be $-3{,}003.39$ (duration is $-3.00339 \cdot 1.06 \div 1{,}000 = -2.83339$); the second derivative is found from equation (8.26) to be 9,891.03 (convexity is $9{,}891.03 \div 1{,}000 = 9.89103$). If yields on comparable bonds were to increase from 0.06 to 0.09, the actual value of this bond would decrease to 924.06, as determined from a standard present-value model. If we were to use a first order series (the duration model) to estimate the new value of the bond, our approximation is 919.81. This approximation is not likely to be acceptable for most purposes. If we use the second order series (the convexity model), our approximation is 924.26:

$$924.26 = 1{,}000 - 2{,}83339 \cdot 0.03 \cdot \frac{1{,}000}{1+0.06} + \tfrac{1}{2} \cdot 1{,}000 \cdot 9.89103 \cdot 0.03^2. \quad (8.28)$$

While the estimate provided by this second order approximation is not precise, it does yield a new bond value that is closer to the bond's actual new value as determined by the present-value model. Therefore, the duration and immunization (application 8.4 above) applications can be substantially improved by using second order approximations of bond prices. Although duration and convexity provide only approximations of bond prices after interest rate shifts, their real value is in their application in bond risk management. The fixed income manager intending to hedge portfolio risk should match the convexities of fund assets and liabilities as well as the durations of fund assets and liabilities. This combination provides for a superior hedge.

8.8 The Method of Lagrange Multipliers
(Background reading: sections 7.4 and 8.4)

Differential calculus is particularly useful for determining minimums or maximums of functions of many types. Many optimization problems require constraints or limitations on functions to be minimized or maximized. For example, a portfolio manager may structure security weights in a portfolio so as to minimize portfolio risk. The most efficient portfolio minimizes risk at a given expected return level. The method of Lagrange multipliers enables function optimization subject to constraints. This method of Lagrange multipliers supplements the original function to be optimized by adding one term for each constraint to be considered. This extra term for each constraint is the product of a function of the constraint and a Lagrange multiplier λ. The method of Lagrange multipliers is most useful for nonlinear functions with more than one independent variable. For example, suppose that we wish to minimize the function $y = 5x^2 + 3z^2 + 6xz$ subject to the constraint that $3x + 2z = 50$. This problem may be written as follows:

$$OBJ\text{: } Min\ y = 5x^2 - 3z^2 + 6xz$$

$$s.t.\text{: } 3x + 2z = 50.$$

This problem statement reads. "Objective function: Minimize $y = 5x^2 - 12z^2 + 6xz$ subject to $3x + 2z = 50$." The Lagrange function combines the original function and the constraints as follows:

$$L = 5x^2 + 12z^2 + 6xz + \lambda(50 - 3x - 2z).$$

Notice that we have added a second function to the original function to be minimized. This second function is the product of a Lagrange multiplier λ and the constraint. This product will have a numerical value equal to zero. The Lagrange multiplier by itself may be interpreted as the change in y that will be induced by a change in the constraint numerical value (in this case, 50); that is, λ is a sensitivity variable. If the constraint is not binding, λ will equal zero. If the constraint is binding in this example, $3x + 2z$ will equal 50. Since either the contents within the parentheses or the Lagrange multiplier will equal zero, the numerical value of the function that we have added to our original function to be optimized will be zero. Although the numerical value of our original function is unchanged by the supplement representing the constraint, its derivatives will be affected by the constraint.

We solve our optimization problem by setting partial derivatives equal to zero. That is, we set partial derivatives of function L with respect to each of our variables x, z, and λ equal to zero. This process is known as finding first order conditions:

$$\frac{\partial L}{\partial x} = 10x + 6z - 3\lambda = 0,$$

$$\frac{\partial L}{\partial z} = 24z + 6x - 2\lambda = 0,$$

$$\frac{\partial L}{\partial \lambda} = 50 - 3x - 2z = 0.$$

We rewrite this system as follows:

$$\begin{aligned} 10x + 6z - 3\lambda &= 0, \\ 6x + 24z - 2\lambda &= 0, \\ -3x - 2z + 0\lambda &= -50. \end{aligned}$$

This system is solved in matrix format as follows:

$$\begin{bmatrix} 10 & 6 & -3 \\ 6 & 24 & -2 \\ -3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -50 \end{bmatrix},$$

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{s}.$$

We find the inverse for the coefficients matrix then solve for security weights and our Lagrange multiplier as follows:

$$\begin{bmatrix} 0.021739 & -0.03261 & -0.32609 \\ -0.03261 & 0.048913 & -0.01087 \\ -0.32609 & -0.01087 & -1.1087 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -50 \end{bmatrix} = \begin{bmatrix} x \\ z \\ \lambda \end{bmatrix},$$

$$\mathbf{C}^{-1} \cdot \mathbf{s} = \mathbf{x}.$$

We find that $x = 16.30435$, $z = 0.543478$, and $\lambda = 55.43478$. Thus, the minimum value for y subject to the constraint is 1,279.537. This constraint is binding since λ is nonzero. Note that $3x + 2z = 50$.

The term λ may be considered to be a sensitivity coefficient. This sensitivity coefficient indicates the change in y that would result from a change in the constraint on $3x + 2z$. If, for example, we were to increase the constraint by 1 from 50 to 51, the value of y would increase by approximately 55.43478, since this is the value of λ. The accuracy of this approximation declines as the change in the constraint increases.

APPLICATION 8.7: OPTIMAL PORTFOLIO SELECTION (Background reading: sections 6.3, 6.4, and 8.7)

One important application of Lagrange optimization in finance is in specifying weights of most efficient portfolios. Suppose that an investor has the opportunity to construct a portfolio consisting of three assets whose characteristics are specified as follows:

Asset	E[R]	σ
A	0.06	0.20
B	0.09	0.30
C	0.12	0.40
$\rho_{A,B} = \rho_{A,C} = \rho_{B,C} = 0.4$		

Suppose that the investor intends to invest \$1,000,000 into a portfolio that enables him to minimize his risk level. The investor requires an expected portfolio return of at least 10% and the weights of his portfolio should sum to 1. We need to determine how much the investor should place into each of the three securities. First, we use the data given above to compute relevant variances and covariances for the individual securities and pairs. The portfolio objective function and constraints are given as follows:

$$OBJ\text{: } Min\ \sigma_p^2 = 0.04w_A^2 + 0.09w_B^2 + 0.16w_C^2 + 0.048w_Aw_B + 0.064w_Aw_C + 0.096w_Bw_C$$

$$s.t.\text{: } 0.06w_A + 0.09w_B + 0.12w_C = 0.10,$$

$$w_A + w_B + w_C = 1.$$

The Lagrange function is constructed as follows:

$$\begin{aligned} L = {} & 0.04w_A^2 + 0.09w_B^2 + 0.16w_C^2 + 0.048w_Aw_B + 0.064w_Aw_C + 0.096w_Bw_C \\ & + \lambda_1(0.10 - 0.06w_A - 0.09w_B - 0.12w_C) + \lambda_2(1 - w_A - w_B - w_C). \end{aligned}$$

Our first order conditions are given by the following:

$$\begin{aligned} \frac{\delta L}{\delta w_A} &= 0.08w_A + 0.048w_B + 0.064w_C - 0.06\lambda_1 - \lambda_2 = 0, \\ \frac{\delta L}{\delta w_B} &= 0.18w_B + 0.048w_A + 0.096w_C - 0.09\lambda_1 - \lambda_2 = 0, \\ \frac{\delta L}{\delta w_C} &= 0.32w_C + 0.064w_A + 0.096w_B - 0.12\lambda_1 - \lambda_2 = 0, \\ \frac{\delta L}{\delta \lambda_1} &= 0.10 - 0.06w_A - 0.09w_B - 0.12w_C = 0, \\ \frac{\delta L}{\delta \lambda_2} &= 1 - 1w_A - 1w_B - 1w_C = 0. \end{aligned}$$

This system is written in matrix format as follows:

$$\begin{bmatrix} 0.08 & 0.048 & 0.064 & -0.06 & -1 \\ 0.048 & 0.18 & 0.096 & -0.09 & -1 \\ 0.064 & 0.096 & 0.32 & -0.12 & -1 \\ -0.06 & -0.09 & -0.12 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_A \\ w_B \\ w_C \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.10 \\ -1 \end{bmatrix},$$

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{S}.$$

We now invert matrix **C** and then solve for vector **x**:

$$\begin{bmatrix} 1.488095 & -2.97619 & 1.488095 & 20.2381 & -2.14286 \\ -2.97619 & 5.952381 & -2.97619 & -7.14286 & 0.285714 \\ 1.488095 & -2.97619 & 1.488095 & -13.0952 & 0.875143 \\ 20.2381 & -7.14286 & -13.0952 & -66.9841 & 4.457143 \\ -2.14286 & 0.285714 & 0.857143 & 4.457143 & -0.37029 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.10 \\ -1 \end{bmatrix} = \begin{bmatrix} w_A \\ w_B \\ w_C \\ \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

We determine the following weights: $w_A = 0.119$, $w_B = 0.429$, and $w_C = 0.452$. The two Lagrange multipliers are $\lambda_1 = 2.24$ and $\lambda_2 = -0.075$. The expected portfolio return and standard deviation are 0.10 and 0.273, respectively. The portfolio variance is 0.074. Since the first Lagrange multiplier is 2.24, an increase by 0.01 in the return constraint would change portfolio weights, leading to an increase in portfolio variance of approximately 0.0224. We would find the actual new portfolio variance by inserting 0.11 into the return constraint of the Lagrange function. In this case, we see that the actual revised portfolio variance increases to 0.100.

EXERCISES

8.1. Solve the following for y':

$$y' = \lim_{h\to 0} \frac{7(x+h)^2 - 7x^2}{h}.$$

8.2. Find derivatives of y with respect to x for each of the following:

(a) $y = 5$;
(b) $y = 7x^3$;
(c) $y = 2x^4 + 5x^3$;
(d) $y = 10x^{0.5} - 11x^3$;
(e) $y = 5x^{1/5}$;
(f) $y = 2/x^2 + 2/3x^{1/2} + 3x^{-1/5} + 1/x$.

8.3. Find second derivatives of y with respect to x for each function in problem 8.2.

8.4. Identify those functions that have finite maximum values for y. For these functions, what values for x maximize y?

(a) $y = 15x^2 + 12$;
(b) $y = 20x$;
(c) $y = -3x^2 + 6x$;
(d) $y = 2x^3 - 6x^2 + x - 12$;
(e) $y = 12x^3$;
(f) $y = 3 + x^2 + 10x$.

8.5. Identify those functions that have finite minimum values for y. For these functions, what values for x minimize y?

(a) $y = 15x^2 + 12$;
(b) $y = 6x$;
(c) $y = 3x^2 + 6x$;
(d) $y = x^3 - 3x^2 + 2x - 21$;
(e) $y = 12x^3$;
(f) $y = 13 - x^2 + 8x$.

8.6. Find the durations for the following pure discount (zero coupon) bonds:

(a) A \$1,000-face-value bond maturing in one year. The bond is currently selling for \$900.
(b) A \$1,000-face-value bond maturing in two years. The bond is currently selling for \$800.
(c) A \$2,000-face-value bond maturing in three years. The bond is currently selling for \$1,400.

(d) A portfolio consisting of one of each of the three bonds listed in parts (a), (b), and (c) of this problem.
(e) A portfolio consisting of \$100,800 in each of the three bonds listed in parts (a), (b), and (c) of this problem.

8.7. Find the duration of each of the following \$1,000-face-value coupon bonds assuming that coupon payments are made annually:

(a) a-three year 7% bond currently selling for \$950;
(b) a-three year 12% bond currently selling for \$1,040;
(c) a-four year 10% bond currently selling for \$900;
(d) a-four year 10% bond currently selling for \$800.

8.8. Based on duration computations, what would happen to the prices of each of the bonds in problem 8.7 if market interest rates ($r = ytm$) were to increase by 1%?

8.9. What is the duration of a portfolio containing one of each of the bonds listed in problem 8.7?

8.10. Consider each of the following functions:

(a) $y = 5x$;
(b) $y = 5x^2 + 10z$;
(c) $y = 2x^7 + 8q^5$;
(d) $y = 5x^3 + 10z^2 + 7xz$;
(e) $y = 12x^3z^5 + 3xz^2$;
(f) $y = \sum_{i=1}^{n} nx^i z^2$.

(i) For each of the functions (a)–(f) above, find partial derivatives for y with respect to x.
(ii) For each of the functions (a)–(f) above, find partial derivatives for y with respect to z.

8.11. Find derivatives for y with respect to x for each of the following:

(a) $y = (4x + 2)^3$;
(b) $y = (3x^2 + 8)^{1/2}$;
(c) $y = 6x(4x^3 + 5x^2 + 3)$;
(d) $y = (1.5x - 4)^3(2.5x - 3.5)^4$;
(e) $y = 25/x^2$;
(f) $y = (6x - 16) \div (10x - 14)$.

8.12. Investors have the opportunity to invest in any combination of the 5% riskless asset and the two risky securities given in the table below:

i	$E[R_i]$	σ_i	$\sigma_{1,i}$	$\sigma_{2,i}$	$\sigma_{3,i}$
1	0.15	0.50	0.25	0.05	0
2	0.08	0.40	0.05	0.16	0

(a) What are the security weights for the optimal (market) portfolio of risky assets?
(b) What are the market portfolio expected return and standard deviation levels?

8.13. Investors have the opportunity to invest in any combination of the 5% riskless asset and the three risky securities given in the table below:

i	$E[R_i]$	σ_i	$\sigma_{1,i}$	$\sigma_{2,i}$	$\sigma_{3,i}$
1	0.15	0.50	0.25	0.05	0.04
2	0.08	0.40	0.05	0.16	0.03
3	0.06	0.30	0.04	0.16	0.09

(a) What are the security weights for the optimal (market) portfolio of risky assets?
(b) What are the market portfolio expected return and standard deviation levels?

8.14.* An investor has the opportunity to invest in a portfolio combining the following two risky stocks:

Security	Expected return	Standard deviation	
A	0.08	0.30	COV(A,B) = 0
B	0.12	0.60	

The investor can borrow money at a rate of 6%, lend money at 4%, and has $5,000,000 to invest. The investor intends to minimize the risk of her portfolio, but requires an expected return of 18%. How much money should she borrow or lend? How much should she invest in each of the two stocks?

8.15. Differentiate each of the following with respect to x:

(a) $y = e^{0.05x}$; (c) $y = 5 \ln(x)$;
(b) $y = (e^x)/x$; (d) $y = e^x \ln(x)$.

8.16. Find durations and convexities for each of the following bonds:

(a) a 6% three-year bond selling for $1,020;
(b) a 9% four-year bond selling for $1,100.

8.17. For each of the bonds listed in problem 8.16 above, complete the following, assuming that all interest rates (yields) change to 8%:

(a) Use the duration (first order) approximation models to estimate bond value changes induced by changes in interest rates (yields) to 8%.
(b) Use the convexity (second order) approximation models to estimate bond value changes induced by changes in interest rates (yields) to 8%.
(c) Find the present values of each of the bonds after yields (cash flow discount rates) change to 8%.

8.18. Our objective is to find the value for x that enables us to maximize the function $y = 50x^2 - 10x$ subject to the constraint that $0.1x = 100$. Set up and solve a Lagrange function for this problem.

8.19. Solve the following: MAX $y = 25 + 3x + 10x^2$ s.t.: $5x = 10$.

8.20. An investor intends to create a portfolio of two assets with the following expected return and standard deviation levels:

Asset	E[R]	σ
A	0.10	0.20
B	0.20	0.40
		$\sigma_{AB} = 0.04$

Determine the following:

(a) Optimal portfolio weights given each of the following expected return constraints:

(i) $E(R_p) = 0.15$;
(ii) $E(R_p) = 0.12$;
(iii) $E(R_p) = 0.18$.

(b) Optimal portfolio weights given each of the same expected return constraints in part (a) above, securities A and B from above and assuming the existence of a riskless asset with a 9% expected return. Use the Lagrange optimization procedure.

8.21. Securities A, B, and C have expected standard deviations of returns equal to 0, 0.50, and 0.90, respectively. Securities A, B, and C have expected returns equal to 0.05, 0.07, and 0.11, respectively. The covariance between returns on B and C is zero. What are the security weights of the optimal portfolio with an expected return of 0.10?

APPENDIX 8.A DERIVATIVES OF POLYNOMIALS

(Background reading: sections 2.12 and 8.3)

The derivative of the polynomial $y = cx^n$ with respect to x is determined by

$$f'(x) = \lim_{h\to 0} \frac{c(x+h)^n - cx^n}{h}$$

$$= \lim_{h\to 0} \frac{cx^n + cnx^{n-1}h + c\binom{n}{1}x^{n-2}h^2 + \ldots + cnxh^{n-1} + ch^n - cx^n}{h}. \quad (8.A.1)$$

The binomial theorem enables us to obtain the second part of equation (8.A.1) from the first. Recall from section 2.3 that the term $\binom{n}{1}$ reads "n choose one". Generally, the function $\binom{n}{j}$ can be used to determine the number of ways in which a subset of size j can be taken from a set of size n. Its value is determined as follows:

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \frac{n(n-1)(n-2)\cdot \ldots \cdot 2\cdot 1}{j(j-1)(j-2)\cdot \ldots \cdot 1(n-j)(n-j-1)\cdot \ldots \cdot 1}. \quad (8.A.2)$$

For example, $\binom{5}{2}$ reads "5 choose 2" and has a value equal to $5! \div [2!(5-2)!] = 120 \div [2(3 \cdot 2)] = 10$. Thus, there are 10 subsets of 2 outcomes from a set of 5.

To simplify the right-hand side of equation (8.A.1), we first note that the cx^n terms cancel out. Next, we note that h is divided into each of the remaining terms, leaving us with

$$f'(x) = \lim_{h\to 0}\left[cnx^{n-1} + c\binom{n}{1}x^{n-2}h + \ldots + cnxh^{n-2} + ch^{n-1}\right]. \quad (8.A.3)$$

As h approaches zero, all terms multiplied by h, or by h raised to any positive integer power, will approach zero. This leaves us with

$$f'(x) = \lim_{h\to 0}[cnx^{n-1}] = cnx^{n-1}. \quad (8.A.4)$$

Thus, the derivative of cx^n with respect to x is cnx^{n-1}. If we were to have a polynomial of the form

$$y = \sum_{j=1}^{m} c_j \cdot x^{n_j}, \quad (8.A.5)$$

the derivative of such a function y with respect to x would be given by

$$\frac{dy}{dx} = \sum_{j=1}^{m} c_j \cdot n_j \cdot x^{n_j - 1}. \quad (8.A.6)$$

APPENDIX 8.B A TABLE OF RULES FOR FINDING DERIVATIVES

Function	Derivative	Example	Derivative
$f(x) = c$	$f'(x) = 0$	$f(x) = 3$	$f'(x) = 0$
$f(x) = cx$	$f'(x) = c$	$f(x) = 3x$	$f'(x) = 3$
$f(x) = cx^n$	$f'(x) = cnx^{n-1}$	$f(x) = 3x^5$	$f'(x) = 15x^4$
$f(x) = g(x) + h(x)$	$f'(x) = g'(x) + h'(x)$	$f(x) = 3x^5 + 3x$	$f'(x) = 15x^4 + 3$
$f(x) = g(x) \cdot h(x)$	$f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$	$f(x) = (2 + 7x)(3x^4 + 11x)$	$f'(x) = 7(3x^4 + 11x) + (12x^3 + 11) \cdot 7$
$f(x) = g(x) \div h(x)$	$f'(x) = [g'(x) \cdot h(x) - h'(x) \cdot g(x)]/[h(x)]^2$	$f(x) = (2 + 7x) \div (3x^4 + 11x)$	$f'(x) = [7(3x^4 + 11x) - 7(12x^3 + 11)]/(3x^4 + 11x)^2$
$f(x) = g(h(x))$	$f'(x) = g'(h(x)) \cdot h'(x)$	$f(x) = (10 + 4x^2)^7$	$f'(x) = 7 \cdot 4 \cdot 2(10 + 4x^2)^6$
$f(x) = \ln(x)$	$f'(x) = 1/x$	$f(x) = \ln(x)$	$f'(x) = 1/x$
$f(x) = e^x$	$f'(x) = e^x$	$f(x) = e^x$	$f'(x) = e^x$
$f(x) = e^{g(x)}$	$f'(x) = g'(x) \cdot e^{g(x)}$	$f(x) = e^{0.1x}$	$f'(x) = 0.1e^{0.1x}$
$f(x) = c^x$	$f'(x) = c^x \ln(c)$	$f(x) = 3^x$	$f'(x) = 3^x \ln(3)$

Table 8.C.1 Portfolio optimization problem (spreadsheet routine). Problem: minimize the portfolio variance given the inputs in cells A1:B2, D1:D2, and E1:F1

	A	B	C	D	E	F		A	B	C	D	E	F
1	0.1	−0.03		0.05	0.07	1		σsA	σsA,B		E[RA]	rp	1
2	−0.03	0.5		0.15				+b1	σsB		E[RB]		
3													
4	0.02	−0.05	−0.05	−1				=2*a1^2	=2*b1	=-D1	=-F1		
5	−0.05	0.5	−0.15	−1				=2*b1	=2*b2^2	=-D2	=-F1		
6	−0.05	−0.15	0	0				=-D1	=-D2	0	0		
7	−1	−1	0	0				=-F1	=-F1	0	0		
8													
9	0	0	10	−1.5	0			Cell range A9:D12 is the inverted				0	
10	0	0	−10	0.5	0			coefficients matrix. Highlight this range,				0	
11	10	−10	−62	3.8	−0.07			and use the MINVERSE function under				=-E1	
12	−1.5	0.5	3.8	−0.245	−1			the Paste function. Don't forget to enter				=-F1	
13								by pressing Ctrl., Shft & Enter together.					
14	0.8							A14 is wA	Cell range A14:A17 represents the				
15	0.2							A15 is wB	final solutions. This range is obtained by				
16	0.54							A16 is λ1	multiplying range A9:D12 by E9:E12				
17	−0.02							A17 is λ2	using MMULT under the Paste function *(fx)*.				
18													
19													

APPENDIX 8.C PORTFOLIO RISK MINIMIZATION ON A SPREADSHEET

(Background reading: section 8.8, application 8.8, and appendix 7.C)

This appendix employs the Lagrange optimization routine (application 8.8) on an Excel™ spreadsheet to determine optimal portfolio weights. The routine minimizes portfolio standard deviation subject to an expected portfolio return. Consider an investor who intends to select weights for an *n*-security portfolio whose characteristics may be inferred from the following system:

$$\begin{bmatrix} 2\sigma_1^2 & 2\sigma_{1,2} & \cdots & 2\sigma_{1,n} & -\mathrm{E}[R_1] & -1 \\ 2\sigma_{2,1} & 2\sigma_2^2 & \cdots & 2\sigma_{2,n} & -\mathrm{E}[R_2] & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\sigma_{n,1} & 2\sigma_{n,2} & \cdots & 2\sigma_n^2 & -\mathrm{E}[R_n] & -1 \\ -\mathrm{E}[R_1] & \mathrm{E}[R_2] & \cdots & \mathrm{E}[R_n] & 0 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -r_{\mathrm{p}} \\ -1 \end{bmatrix}.$$

This system may be applied regardless of the size of the portfolio. Where the number of securities under consideration for the portfolio equals n, the square coefficients matrix will have $n + 2$ rows and $n + 2$ columns.

This system may be input to a spreadsheet to solve for the system of n weights and two Lagrange multipliers. Table 8.C.1 represents an Excel™ spreadsheet printout of a system used to solve for optimal weights in a two-security portfolio. The target return for this portfolio is 0.07 and the expected security returns are 0.05 for security A and 0.15 for security B. Security standard deviations are expected to be 0.1 for A and 0.5 for B. The covariance between returns on A and B is expected to be −0.025. These optimization inputs are represented in the first two rows of the spreadsheet.

The left part of the table represents numerical values displayed by the spreadsheet; the right part represents actual cell entries. Rows 1 and 2 are numerical inputs for the file from the problem to be solved; Rows 4–7 are the rows of the coefficients matrix to be inverted. See appendix 7.A for details on how to invert this matrix. We then multiply the solutions vector in range E9:E12 by the inverse of the coefficients matrix in range A9:D12, using the procedure discussed in appendix 7.A. The solutions to the problem are given in range A14:A17. This procedure is easily extended to accommodate as many securities as is necessary.