Chapter 11
Appendices: Review of Topics from Foundations in Finance and Tables

A: INTRODUCTION

The expression "Time is money" certainly applies in finance. People and institutions are impatient; they want money now and are generally willing to pay (or impose a charge) for having money now (or having to wait). The time value of money is certainly among the most important concepts in finance.

Interest is a charge imposed on borrowers for the use of lenders' money. The interest cost is usually expressed as a percentage of the *principal* (the sum borrowed). When a loan matures, the principal must be repaid along with any unpaid accumulated interest.

In a free market economy, interest rates are determined jointly by the supply of and demand for money. Thus, lenders will usually attempt to impose as high an interest rate as possible on the money they lend; borrowers will attempt to obtain the use of money at the lowest interest rates available to them. Factors affecting the levels of interest rates will do so by affecting supply and demand conditions for money. Among these factors are:

1. **Inflation:** Because of diminished purchasing power, money received in the future by lenders is worth less than the money they lend now. Lenders will require a premium (interest) in addition to the principal to compensate them for this loss of purchase power. Furthermore, inflation makes current money balances more attractive to borrowers. Thus, inflation decreases the supply of and increases the demand for money. Interest rates will increase as the rate of inflation increases.
2. **Risk or Uncertainty:** Creditors naturally prefer to know with certainty that the money they loan will be repaid in its entirety. If lenders are uncertain as to whether their loans will be repaid, they will require premiums to compensate them for this risk. Higher interest rates will result from increased uncertainty.
3. **Intertemporal Monetary Preferences:** In general, consumers (and corporations) will prefer to have money now rather than be forced to wait for it. If consumers have money now, they can choose to spend it now or spend it at some later date. However, if consumers must wait for their money, they do not have the option to spend it now; they must wait for some later date to spend it. If consumers increase their desire to spend more now rather than later, interest rates will increase.
4. **Government Policy:** Governmental monetary policy will affect both supply and demand conditions for money. Through monetary policy, the government can directly control the supply of money; and through its participation in bond markets, it can influence the demand for money. Governmental fiscal policy (spending and tax programs) have a significant effect on the demand for money.
5. **Costs of Extending Credit:** Both lenders and borrowers face various negotiating and administrative costs when a loan is extended. Most of these costs can be categorized as transactions costs. Lenders will require initiation fees such as "points" or higher interest payments as compensation for these costs.

B: CALCULATION OF SIMPLE INTEREST
Interest is computed on a *simple basis* if it is paid only on the principal of the loan.

Compound interest is paid on accumulated loan interest as well as on the principal. Thus, if a sum of money (X_0) were borrowed at an annual interest rate (i) and repaid at the end of (n) years with accumulated interest, the total sum repaid (FV_n or *Future Value* at the end of Year n) is determined as follows:

(1) $$FV_n = X_0(1 + n \times i)$$

The subscripts (n) and (0) merely designate time; they do not imply any arithmetic function. The product (n · i) when multiplied by X_0 reflects the value of interest payments to be made on the loan; the value (1) accounts for the fact that the principal of the loan must be repaid. If the loan duration includes some fraction of a year, the value of (n) will be fractional; e.g., if the loan duration were one year and three months, (n) would be 1.25. The total amount paid (or, the Future Value of the loan) will be an increasing function of the length of time the loan is outstanding (n) and the interest rate (i) charged on the loan. For example, if a consumer borrowed $1000 at an interest rate of 10% for one year, his total repayment would be $1100, determined from Equation 1 as follows:

$$FV_1 = \$1000(1 + 1 \times .1) = \$1000*1.1 = \$1100$$

If the loan were to be repaid in two years, its future value would be determined as follows:

$$FV_2 = \$1000(1 + 2 \times .1) = \$1000*1.2 = \$1200$$

Continuing our example, if the loan were to be repaid in five years, its future value would be:

$$FV_5 = \$1000(1 + 5 \times .1) = \$1000*1.5 = \$1500$$

The longer the duration of a loan, the higher will be its future value. Thus, the longer lenders must wait to have their money repaid, the greater will be the total interest payments made by borrowers.

C: CALCULATION OF COMPOUND INTEREST

Interest is computed on a compound basis when a borrower must pay interest on not only the loan principal, but on accumulated interest as well. If interest must accumulate for a full year before it is compounded, the Future Value of such a loan is determined with Equation (2):

(2) $$FV_n = X_0(1 + i)^n$$

For example, if an individual were to deposit $1000 into a savings account paying annually compounded interest at a rate of 10% (here, the bank is borrowing money), the future value of the account after five years would be $1610.51, determined by Equation 2 as follows:

$$FV_5 = \$1000(1+.1)^5 = \$1000 \text{ x } 1.1^5 = \$1000 \text{ x } 1.61051 = \$1610.51$$

Notice that this sum is greater than the future value of the loan ($1500) when interest is not

compounded.

The compound interest formula can be derived intuitively from the simple interest formula. If interest must accumulate for a full year before it is compounded, then the future value of the loan after one year is $1100, exactly the same sum as if interest had been computed on a simple basis:

(3) $$FV_n = X_0(1+ni) = X_0(1+1i) = X_0(1+i)^1 = \$1000(1+.1) = \$1100$$

The future values of loans where interest is compounded annually and when interest is computed on an annual basis will be identical only when (n) equals one. Since the value of this loan is $1100 after one year and interest is to be compounded, interest and future value for the second year will be computed on the new balance of $1100:

(4) $$FV_2 = X_0(1+1i)(1+1i) = X_0(1+i)(1+i) = X_0(1+i)^2 \,,$$

$$FV_2 = \$1000 \, (1+.1)(1+.1) = \$1000 \, (1+.1)^2 = \$1210$$

This process can be continued for five years:

$$FV_5 = \$1000(1+.1)(1+.1)(1+.1)(1+.1)(1+.1) = \$1000(1+.1)^5 = \$1610.51$$

More generally, the process can be applied for a loan of any maturity. Therefore:

$$FV_n = X_0(1+i)(1+i) \cdots (1+i) = X_0(1+i)^n \,,$$

$$FV_n = \$1000 \, (1+.1)(1+.1) \cdots (1+.1) = \$1000 \, (1+.1)^n$$

D. FRACTIONAL PERIOD COMPOUNDING OF INTEREST

In the previous examples, interest is compounded annually; that is, interest must accumulate at the *stated rate* i for an entire year before it can be compounded or re-compounded. In many savings accounts and other investments, interest can be compounded semiannually, quarterly or even daily. If interest is to be compounded more than once per year (or once every fractional part of a year), the future value of such an investment will be determined as follows:

(6) $$FV_n = X_0 \, (1 + i/m)^{mn} \,,$$

where interest is compounded (m) times per year. The interpretation of this formula is fairly straightforward. For example, if (m) is 2, then interest is compounded on a semiannual basis. The semiannual interest rate is simply (i/m) or (i/2). If the investment is held for (n) periods, then it is held for (2n) semiannual periods. Thus, we compute a semiannual interest rate (i/2) and the number of semiannual periods the investment is held (2*n). If $1000 were deposited into a savings account paying interest at an annual rate of 10% compounded semiannually, its future value after five years would be $1628.89, determined as follows:

$FV_5 = \$1000(1 + .1/2)^{2x5} = \$1000(1.05)^{10} = \$1000(1.62889) = \$1628.89$

Notice that the semiannual interest rate is five percent and that the account is outstanding for ten six-month periods. This sum (\$1628.89) exceeds the future value of the account if interest is compounded only once annually (\$1610.51). In fact, the more times per year interest is compounded, the higher will be the future value of the account. For example if interest on the same account were compounded monthly (twelve times per year), the account's future value would be \$1645.31:

$$FV_5 = \$1000(1 + .1/12)^{12x5} = \$1000(1.008333)^{60} = \$1645.31$$

The monthly interest rate is .008333 and the account is open for (m*n) or 60 months. With daily compounding, the account's value would be \$1648.60:

$$FV_5 = \$1000(1 + .1/365)^{365x \cdot 5} = \$1648.60$$

Therefore, as (m) increases, future value increases. However, this rate of increase in future value becomes smaller with larger values for (m); that is, the increases in (FV_n) induced by increases in (m) eventually become quite small. Thus, the difference in the future values of two accounts where interest is compounded hourly in one and every minute in the other may actually be rather trivial.

E. CONTINUOUS COMPOUNDING OF INTEREST

If interest were to be compounded an infinite number of times per period, we would say that interest is compounded continuously. However, we cannot obtain a numerical solution for future value by merely "plugging" in infinity for m in Equation 6 - calculators have no infinity key. In the previous section, we saw that increases in (m) cause the future value of an investment to increase. As (m) approaches infinity, (FV_n) continues to increase, however at decreasing rates. More precisely, as (m) approaches infinity, the future value of an investment can be defined as follows:

(7) $$FV_n = X_0 e^{in}$$

where (e) is the natural log whose value can be approximated at 2.718 or derived from the following:

(8) $$e = \lim_{m \to \infty}(1 + \frac{1}{m})^m$$

That is, as (m) approaches infinity, the value of the limit in expression (8) approaches the number (e). Notice the similarity between Equations (6), (8) and (9). In fact, Equation (7) can be derived easily from Equations (6) and (9) which defines e^i as follows:

(9) $$e^i = \lim_{m \to \infty}(1 + \frac{i}{m})^m$$

In many calculations involving continuous compounding of interest, the value 2.718 serves as an approximation for the number (e).

If an investor were to deposit $1000 into an account paying interest at a rate of 10%, continuously compounded (or compounded an infinite number of times per year), the account's future value would be approximately $1648.72:

$$FV_5 = \$1000 \ * \ e^{.1 \cdot 5} = \$1000 * 2.718^{.5} = \$1648.72$$

The future value of this account exceeds only slightly the value of the account if interest were compounded daily. Also note that continuous compounding simply means that interest is compounded an infinite number of times per time period.

Years to maturity (n)	Future Value Simple Interest	Future Value Compounded Annually	Future Value Compounded Monthly	Future Value Compounded Daily	Future Value Compounded Continuously
1	110	110	110.47	110.52	110.52
2	120	121	122.04	122.14	122.14
3	130	133.31	134.81	134.98	134.99
4	140	146.41	148.94	149.17	149.18
5	150	161.05	164.53	164.86	164.87
10	200	259.37	270.70	271.79	271.83
20	300	672.75	732.81	738.70	738.91
30	400	1,744.94	1,983.74	2,007.73	2,008.57
50	600	11,739.09	14,536.99	14,831.16	14,841.40
Annual Percentage Yield	varies with n	.100000	.104713	.1051557	.1051709

TABLE 1: Future Values and Annual Percentage Yields of accounts with initial $100 deposits at 10% interest

QUESTIONS AND PROBLEMS

1. Why do interest rates charged by banks for the purchase of automobiles tend to exceed interest rates paid on savings accounts? Why are home loan (mortgage) interest rates usually lower than interest rates charged credit card customers?

2. The Williams Company has borrowed $10,500 at an annual interest rate of nine percent. How much will be a single lump sum repayment in eight years including both principal and interest accumulated on a simple basis? That is, what is the future value of this loan?

3. The Cobb Company has issued ten million dollars in ten percent coupon bonds maturing in five years. Interest payments on these bonds will be made semi-annually.
   a.   How much are Cobb's semi-annual interest payments?
   b.   What will be the total payment made by Cobb on the bonds in each of the first four years?
   c.   What will be the total payment made by Cobb on the bonds in the fifth year?

4. What would be the lump sum loan repayment made by the Williams Company in Problem 2 if interest were compounded:
   a. annually?
   b. Semiannually?
   c. monthly?
   d. daily?
   e. continuously?

5. The Speaker Company has the opportunity to purchase a five-year $1000 certificate of deposit (C.D.) paying interest at an annual rate of 12%, compounded annually. The company will not withdraw early any of the money in its C.D. account. Will this account have a greater future value than a five-year $1000 C.D. paying an annual interest rate of 10%, compounded daily?

6. The Waner Company needs to set aside a sum of money today for the purpose of purchasing for $10,000 a new machine in three years. Money used to finance this purchase will be placed in a savings account paying interest at a rate of eight percent. How much money must be placed in this account now to assure the Waner company $10,000 in three years if interest is compounded yearly?

7. Assuming no withdrawals or additional deposits, how much time is required for $1000 to double if placed in a savings account paying an annual interest rate of 10% if interest were:
   a. computed on a simple basis?
   b. compounded annually?
   c. compounded monthly?
   d. compounded continuously?

Solutions

2. $FV_8 = 10,500 (1 + 8 \times .09) = 10,500 \times 1.72 = 18,060$

3.   a. [10% x $10,000,000] / 2 = $500,000

         b. 10% x $10,000,000 = 2 x $500,000 = $1,000,000

         c. $10,000,000 + $1,000,000 = Principal + interest in year five = $11,000,000

4.   a. $FV_8 = 10{,}500\,(1 + .09)^8 = 10{,}500 \times 1.99256 = 20{,}921.908$

         b. $FV_8 = 10{,}500\,(1 + \dfrac{.09}{2})^{2 \times 8} = 10{,}500 \times 2.0223702 = 21{,}234.887$

         c. $FV_8 = 10{,}500\,(1 + \dfrac{.09}{12})^{12 \times 8} = 10{,}500 \times 2.0489212 = 21{,}513.673$

         d. $FV_8 = 10{,}500\,(1 + \dfrac{.09}{365})^{365 \times 8} = 10{,}500 \times 2.0542506 = 21{,}569.632$

         e. $FV_8 = 10{,}500\, e^{.09 \times 8} = 10{,}500 \times 2.0544332 = 21{,}571.549$

5.   For example, let $X_0 = \$1000$ in each case

            For CD_1 : $FV_5 = 1000(1 + .12)^5 = 1{,}762.3417$

            For CD_2 : $FV_5 = 1000(1 = \dfrac{.10}{365})^{365 \times 5} = 1{,}648.6005$

6.   Solve for X_0:

            $X_0 = \dfrac{FV_n}{(1+i)^n} = \dfrac{10{,}000}{(1+.08)^3} = 7938.322$

7.   In all cases here, $FV_n = 2X_0$. Thus, let $FV_n = 2000$ and $X_0 = 1000$

         a. $2000 = 1000\,(1 + n \times .1)$; $2 = (1 + n \times .1)$; $1 = .1n$; $n = 10$ years

         b. $2000 = 1000\,(1.1)^n$; using logs: $\log 2000 = (\log 1000) + n \cdot \log(1.1)$

            $3.30103 = 3 + n \cdot (.04139)$; $.30103 = n(.04139)$; $n = 7.2725$ years

         c. $2000 = 1000\,(1 + \dfrac{.10}{12})^{12n}$;

            $\log 2000 = (\log 1000) / [12 \cdot \log(1.008333)]$ $= n = 6.9603407$ years

         d. $2000 = 1000\, e^{.1\,n}$; use natural logs: $\ln 2000 = (\ln 1000) + .1n$; $n = 6.931478$ years

F. INTRODUCTION

Cash flows realized at the present time have a greater value to investors than cash flows realized later for the following reasons:

1. Inflation: The purchasing power of money tends to decline over time.
2. Risk: One never knows for sure whether he will actually realize the cash flow that he expects.
3. The option to either spend money now or defer spending it is likely to be worth more than being forced to defer spending the money.

The purpose of the Present Value concept is to provide a means of expressing the value of a future cash flow in terms of current cash flows. That is, the Present Value concept is used to determine how much an investor would pay now for the promise of some cash flow to be received at a later date. The present value of this cash flow would be a function of inflation, the length of wait before the cash flow is received, its risk and the time value an investor associates with money (how much he needs money now as opposed to later). Perhaps the easiest way to account for these factors when evaluating a future cash flow is to discount it in the following manner:

$$(1) \qquad PV = \frac{CF_n}{(1+k)^n}$$

where (CF_n) is the cash flow to be received in year (n), (k) is an appropriate discount rate accounting for risk, inflation, and the investor's time value associated with money, and PV is the present value of that cash flow. The discount rate enables us to evaluate a future cash flow in terms of cash flows realized today. Thus, the maximum a rational investor would be willing to pay for an investment yielding a $9000 cash flow in six years assuming a discount rate of 15% would be $3891, determined as follows:

$$PV = \frac{\$9000}{(1+\cdot15)^6} = \frac{\$9000}{2.31306} = \$3890.95$$

In the above example, we simply assumed a fifteen percent discount rate. Realistically, perhaps the easiest value to substitute for (k) is the current interest or return rate on loans or other investments of similar duration and risk. However, this market determined interest rate may not consider the individual investor's time preferences for money. Furthermore, the investor may find difficulty in locating a loan (or other investment) of similar duration and risk. For these reasons, more scientific methods for determining appropriate discount rates will be discussed later. In any case, the discount rate should account for inflation, the investment risk and the investor's time value of money.

G. DERIVING THE PRESENT VALUE FORMULA

The present value formula can be derived easily from the compound interest formula. Assume an investor wishes to deposit a sum of money into a savings account paying interest at a

rate of fifteen percent, compounded annually. If the investor wishes to withdraw from his account $9,000 in six years, how much must he deposit now? This answer can be determined by solving the compound interest formula for X_0:

$$FV_n = X_0(1+i)^n; \quad X_0 = \frac{FV_n}{(1+i)^n} = \frac{\$9000}{(1+.15)^6} = \frac{\$9000}{2.31306} = \$3890.95$$

Therefore, the investor must deposit $3890.95 now in order to withdraw $9,000 in six years at fifteen percent.

Notice that the present value formula (3.1) is almost identical to the compound interest formula where we solve for the principal (X_0):

$$PV = \frac{CF_n}{(1+k)^n}; \quad X_0 = \frac{FV_n}{(1+i)^n}$$

Mathematically, these formulas are the same; however, there are some differences in their economic interpretations. In the interest formulas, interest rates are determined by market supply and demand conditions whereas discount rates are individually determined by investors themselves (although their calculations may be influenced by market interest rates). In the present value formula, we wish to determine how much some future cash flow is worth now; in the interest formula above, we wish to determine how much money must be deposited now to attain some given future value.

H: PRESENT VALUE OF A SERIES OF CASH FLOWS

If an investor wishes to evaluate a series of cash flows, he needs only to discount each separately and then sum the present values of each of the cash flows. Thus, the present value of a series of cash flows (CF_t) received in time period (t) can be determined by the following expression:

$$(2) \quad PV = \sum_{t=1}^{n} \frac{CF_t}{(1+k)^t}$$

For example, if an investment were expected to yield annual cash flows of $200 for each of the next five years, assuming a discount rate of 5%, its present value would be $865.90:

$$PV = \frac{200}{(1+.05)^1} + \frac{200}{(1+.05)^2} + \frac{200}{(1+.05)^3} + \frac{200}{(1+.05)^4} + \frac{200}{(1+.05)^5} = 865.90$$

Therefore, the maximum price an individual should pay for this investment is $865.90 even though the cash flows yielded by the investment total $1000. Because the individual must wait up to five years before receiving the $1000, the investment is worth only $865.90. Use of the present value series formula does not require that cash flows (CF_t) in each year be identical, as does the annuity model presented in the next section.

I: ANNUITY MODELS

The expression for determining the present value of a series of cash flows can be quite cumbersome, particularly when the payments extend over a long period of time. This formula requires that (n) cash flows be discounted separately and then summed. When (n) is large, this task may be rather time-consuming. If the annual cash flows are identical and are to be discounted at the same rate, an annuity formula can be a useful time-saving device. The same problem discussed in the previous section can be solved using the following annuity formula:

$$(3) \qquad PV_A = \frac{CF}{K}\left[1 - \frac{1}{(1+k)^n}\right]$$

where (CF) is the level of the annual cash flow generated by the annuity (or series). Use of this formula does require that all of the annual cash flows be identical. Thus, the present value of the cash flows in the problem discussed in the previous section is $865.90, determined as follows:

$$PV_A = \frac{200}{(.05)}\left[1 - \frac{1}{(1+.05)^5}\right] = 4000(.216475) = \$865.90$$

As (n) becomes larger, this formula becomes more useful relative to the present value series formula discussed in the previous section. However, the annuity formula requires that all cash flows be identical and be paid at the end of each year. The present value annuity formula can be derived easily from the perpetuity formula discussed in the next section or from the geometric expansion procedure described in the derivation box.

Note that each of the above calculations assumes that cash flows are paid at the end of each period. If, instead, cash flows were realized at the beginning of each period, the annuity would be referred to as an annuity due. Each cash flow generated by the annuity due would, in effect, be received one year earlier than if cash flows were realized at the end of each year. Hence, the present value of an annuity due is determined by simply multiplying the present value annuity formula by (1+k):

$$(4) \qquad PV_{due} = \frac{CF}{K}\left[1 - \frac{1}{(1+k)^n}\right](1+k)$$

The present value of the five-year annuity due discounted at five percent is determined:

$$PV_A = \frac{200}{(.05)}\left[1 - \frac{1}{(1+.05)^5}\right](1+.05) = 4000(.2164738)(1.05) = \$909.19$$

J: BOND VALUATION

Because the present value of a series of cash flows is simply the sum of the present values of the cash flows, the annuity formula can be combined with other present value formulas to evaluate investments. Consider, for example, a 7% coupon bond making annual interest payments for 9 years. If this bond has a $1,000 face (or par) value, and its cash flows are

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discounted at 6%, its value can be determined as follows:

$$PV = \frac{70}{(.06)}\left[1 - \frac{1}{(1+.06)^9}\right] + \frac{1000}{(1+.06)^9} = \$1166.67(.4081) + \frac{\$1000}{1.689479}$$

$$= \$476.118 + 587.406 = \$1068.774$$

Thus, the value of a bond is simply the sum of the present values of the cash flow streams resulting from interest payments and from principal repayment.

Now, let us revise the above example to value another 7% coupon bond. This bond will make semiannual (twice yearly) interest payments for 9 years. If this bond has a $1,000 face (or par) value, and its cash flows are discounted at the stated annual rate of 6%, its value can be determined as follows:

$$PV = \frac{35}{(.03)}\left[1 - \frac{1}{(1+.03)^{18}}\right] + \frac{1000}{(1+.03)^{18}} = \$1166.67(.4126) + \frac{\$1000}{1.7024} = \$481.368 + 587.406 = \$1068.774$$

Again, the value of the bond is the sum of the present values of the cash flow streams resulting from interest payments and from the principal repayment. However, the semi-annual discount rate equals 3% and payments are made to bondholders in each of eighteen semi-annual periods.

K: PERPETUITY MODELS

As the value of (n) approaches infinity in the annuity formula, the value of the right hand side term in the brackets:

$$\frac{1}{(1+k)^n}$$

approaches zero. That is, the cash flows associated with the annuity are paid each year for a period approaching "forever." Therefore, as (n) approaches infinity, the value of the infinite time horizon annuity approaches:

(5) $$PV_p = \frac{CF}{k}$$

The perpetuity model is useful in the evaluation of a number of investments. Any investment with an indefinite or perpetual life expectancy can be evaluated with the perpetuity model. For example, the present value of a stock, if its dividend payments are projected to be stable, will be equal to the amount of the annual dividend (cash flow) generated by the stock divided by an appropriate discount rate. In European financial markets, a number of perpetual bonds have been traded for several centuries. In many regions in the United States, ground rents (perpetual leases on land) are traded. The proper evaluation of these and many other investments requires the use of perpetuity models.

$$PV_p = \frac{\$200}{.05} = \$4000$$

The maximum price an investor would be willing to pay for a perpetual bond generating an annual cash flow of $200, each discounted at a rate of 5% can be determined from Equation (5):

L: GROWING PERPETUITY AND ANNUITY MODELS

If the cash flow associated with an investment were expected to grow at a constant annual rate of (g), the amount of the cash flow generated by that investment in year (t) would be:

(6) $CF_t = CF_1(1+g)^{t-1}$,

where (CF_1) is the cash flow generated by the investment in year one. Thus, if a stock paying a dividend of $100 in year one were expected to increase its dividend payment by 10% each year thereafter, the dividend payment in the fourth year would be $133.10:

$$CF_4 = CF_1 (1 + .10)^{4-1}$$

Similarly, the cash flow generated by the investment in the following year (t+1) will be:

(7) $CF_{t+1} = CF_1 (1 + g)^t$

The stock's dividend in the fifth year will be $146.41:

$$CF_{4+1} = CF_1 (1+.10)^4 = \$146.41$$

If the stock had an infinite life expectancy (as most stocks might be expected to), and its dividend payments were discounted at a rate of 13%, the value of the stock would be determined by:

$$PV_{gp} = \frac{\$100}{.13 - .10} = \frac{\$100}{.03} = \$3333.33$$

This expression is called the Gordon Stock Pricing Model. It assumes that the cash flows (dividends) associated with the stock are known in the first period and will grow at a constant compound rate in subsequent periods. More generally, this growing perpetuity expression can be written as follows:

(8) $$PV_{gp} = \frac{CF_1}{k - g}$$

The growing perpetuity expression simply subtracts the growth rate from the discount rate; the growth in cash flows helps to "cover" the time value of money. This formula for evaluating growing perpetuities can be used only when (k) > (g). If (g) > (K), either the growth rate or discount rate has probably been calculated improperly. Otherwise, the investment would have an infinite value (even though the formula would generate a negative value).

(9) $$PV_{gp} = \frac{CF_1}{k - g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n} \right]$$

Cash flows generated by many investments will grow at the rate of inflation. For example, consider a project undertaken by a corporation whose cash flow in year one is expected to be $10,000. If cash flows were expected to grow at the inflation rate of six percent each year until year six, then terminate, the project's present value would be $48,320.35, assuming a discount rate of 11%:

$$PV_{gp} = \frac{\$10,000}{.11 - .06} \cdot \left[1 - \frac{(1+.06)^6}{(1+.11)^6}\right] = \$200,000(1 - .7584) = \$48320.45$$

Cash flows are generated by this investment through the end of the sixth year. No cash flow was generated in the seventh year. Verify that the amount of cash flow that would have been generated by the investment in the seventh year if it had continued to grow would have been $10,000(1.06)^6 = \$14,185$.

M: STOCK VALUATION

Consider a stock whose annual dividend next year is projected to be $50. This payment is expected to grow at an annual rate of 5% in subsequent years. An investor has determined that the appropriate discount rate for this stock is 10%. The current value of this stock is $1000, determined by the growing perpetuity model:

$$PV_{gp} = \frac{\$50}{.10 - .05} = \$1000$$

This model is often referred to as the Gordon Stock Pricing Model. It may seem that this model assumes that the stock will be held by the investor forever. But what if the investor intends to sell the stock in five years? Its value would be determined by the sum of the present values of cash flows the investor expects to receive:

$$PV_{GA} = \frac{DIV_1}{k - g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n}\right]$$

where (P_n) is the price the investor expects to receive when he sells the stock in year (n); and (DIV_1) is the dividend payment the investor expects to receive in year one. The present value of the dividends the investor expects to receive is $207.53:

$$PVGA = \frac{\$50}{.10 - .05} \cdot \left[1 - \frac{(1+.05)^5}{(1+.10)^5}\right] = 207.53$$

The selling price of the stock in year five will be a function of the dividend payments the prospective purchaser expects to receive beginning in year six. Thus, in year five, the prospective purchaser will pay $1276.28 for the stock, based on his initial dividend payment of $63.81, determined by the following equations:

$$DIV_6 = DIV_1 (1+.05)^{6-1} = \$63.81$$

$$\text{Stock value in year five} = 63.81/(.10-.05) = \$1276.28$$

The present value of the \$1276.28 the investor will receive when he sells the stock at the end of the fifth year is \$792.47:

$$PV = \frac{\$1276.28}{(1+.1)^5} = \$792.57$$

The total stock value will be the sum of the present values of the dividends received by the investor and his cash flows received from the sale of the stock. Thus, the current value of the stock is \$207.53 plus \$792.47, or \$1000. This is exactly the same sum determined by the growing perpetuity model earlier; therefore, the growing perpetuity model can be used to evaluate a stock even when the investor expects to sell it.

QUESTIONS AND PROBLEMS

1. What is the present value of a security promising to pay $10,000 in five years if its associated discount rate is:
        a. twenty percent?
        b. ten percent?
        c. one percent?
        d. zero percent?

2. What is the present value of a security to be discounted at a ten percent rate promising to pay $10,000 in:
        a. twenty years?
        b. ten years?
        c. one year?
        d. six months?
        e. seventy three days?

3. The Gehrig Company is considering an investment that will result in a $2000 cash flow in one year, a $3000 cash flow in two years and a $7000 cash flow in three years. What is the present value of this investment if all cash flows are to be discounted at an eight percent rate? Should Gehrig Company management be willing to pay $10,000 for this investment?

4. The Hornsby Company has the opportunity to pay $10,000 for an investment paying $2000 in each of the next nine years. Would this be a wise investment if the appropriate discount rate were:
        a. five percent?
        b. ten percent?
        c. twenty percent?

5. The Foxx Company is selling preferred stock which is expected to pay a fifty dollar annual dividend per share. What is the present value of dividends associated with each share of stock if the appropriate discount rate were eight percent and its life expectancy were infinite?

6. The Evers Company is considering the purchase of a machine whose output will result in a ten thousand dollar cash flow next year. This cash flow is projected to grow at the annual ten percent rate of inflation over each of the next ten years. What will be the cash flow generated by this machine in:
        a. its second year of operation?
        b. its third year of operation?
        c. its fifth year of operation?
        d. its tenth year of operation?

7. The Wagner Company is considering the purchase of an asset that will result in a $5000 cash flow in its first year of operation. Annual cash flows are projected to grow at the 10% annual rate of inflation in subsequent years. The life expectancy of this asset is seven years, and the appropriate discount rate for all cash flows is twelve percent. What is the maximum price

Wagner should be willing to pay for this asset?

8. What is the present value of a stock whose $100 dividend payment next year is projected to grow at an annual rate of five percent? Assume an infinite life expectancy and a twelve percent discount rate.

9. Which of the following series of cash flows has the highest present value at a five percent discount rate:
> a. $500,000 now
> b. $100,000 per year for eight years
> c. $60,000 per year for twenty years
> d. $30,000 each year forever

10. Which of the cash flow series in Problem 9 has the highest present value at a twenty percent discount rate?

11. What discount rate in Problem 4 will render the Hornsby Company indifferent as to its decision to invest $10,000 for the nine year series of cash flows? That is, what discount rate will result in a $10,000 present value for the series?

12. What would be the present value of $10,000 to be received in twenty years if the appropriate discount rate of 10% were compounded:
> a. annually?
> b. monthly?
> c. daily?
> d. continuously?

12.a. What would be the present value of a thirty year annuity if the $1000 periodic cash flow were paid monthly? Assume a discount rate of 10% per year.
  b. Should an investor be willing to pay $100,000 for this annuity?
  c. What would be the highest applicable discount rate for an investor to be willing to pay $100,000 for this annuity?

Solutions

1   a. $PV = \dfrac{CF_n}{(1+k)^n} = \dfrac{10{,}000}{(1+.20)^5} = \dfrac{10{,}000}{1.2^5} = \dfrac{10{,}000}{2.48832} = 4018.775$

       b. $PV = \dfrac{10{,}000}{1.10^5} = \dfrac{10{,}000}{1.61051} = 6209.213$

       c. $PV = \dfrac{10{,}000}{1.01^5} = \dfrac{10{,}000}{1.0510101} = 9514.656$

       d. $PV = \dfrac{10{,}000}{1.0^5} = \dfrac{10{,}000}{1} = 10{,}000$

2.   a. $PV = \dfrac{10{,}000}{1.1^{20}} = \dfrac{10{,}000}{6.7275} = 1486.436$

       b. $PV = \dfrac{10{,}000}{1.1^{10}} = \dfrac{10{,}000}{2.5937425} = 3855.432$

       c. $PV = \dfrac{10{,}000}{1.1^1} = \dfrac{10{,}000}{1.1} = 9090.909$

d. $PV = \dfrac{10,000}{1.1^{.5}} = \dfrac{10,000}{1.1^{.5}} = \dfrac{10,000}{1.0488088} = 9534.625;$ Note: 6 months is .5 of one year

e. $PV = \dfrac{10,000}{1.1^{.2}} = \dfrac{10,000}{1.0192449} = 9811.184;$ Note: 73 days is .2 of one year

3. $PV = \Sigma_{t=1}^{n} \dfrac{CF_t}{(1+k)^t} = \dfrac{2000}{1.08^1} + \dfrac{3000}{1.08^2} + \dfrac{7000}{1.08^3}$

   $PV = 1851.85 + 2572.02 = 5556.83 = 9980.70;$ $10,000 > 9980.70$

   Since $P_0 > PV$, the investment should not be purchased.

4. $PV_n = CF \left[\dfrac{1}{k} - \dfrac{1}{k(1+k)^n} \right]$

   a. $PV_A = 2000 \left[\dfrac{1}{.05} - \dfrac{1}{.05(1.05)^9} \right] = 2000[20 - 12.892178] = 14,215.643$

   b. $PV_A = 2000 \left[\dfrac{1}{.10} - \dfrac{1}{.10(1.10)^9} \right] = 2000[\,10 - 4.2409762\,] = 11,518.048$

   c. $PV_A = \left[\dfrac{1}{.2} - \dfrac{1}{.2(1.2)^9} \right] = 2000\,[\,5 - .9690335] = 8,061.933$

5. $PV_p = \dfrac{CF}{k} = \dfrac{50}{.08} = 625$

6. $CF_n = CF_1(1+g)^{n-1}$

   a. $CF_2 = 10,000\,(1+.1)^{2-1} = 10,000\,(1+.1) = 10,000 \times 1.1 = 11,000$

   b. $CF_3 = 10,000\,(1+.1)^{3-1} = 10,000 \times 1.21 = 12,100$

   c. $CF^5 = 10,000\,(1+.1)^{5-1} = 10,000 \times 1.4641 = 14,641$

   d. $CF_{10} = 10,000\,(1+.1)^{10-1} = 10,000 \times 2.3579477 = 23,579.477$

7. $PV_{ga} = CF_1 \times \left[\dfrac{1}{k-g} - \dfrac{(1+g)^n}{(k-g)(1+k)^n} \right]$

   $= 5000 \times \left[\dfrac{1}{.02} - \dfrac{(1+.10)^7}{(.12-.10)(1+.12)^7} \right]$

   $PV_{ga} = 5000 \times [50 - 44.075033] = 29,624.837$

8. $PV_{gp} = \dfrac{CF_1}{k-g} = \dfrac{100}{.12-.05} = 1428.5714$

9. $60,000 per year for 20 years

   a. $PV = 500,000$

   b. $PV = 100,000 \left[\dfrac{1}{.05} - \dfrac{1}{.05(1.05)^8} \right] = 646,321.27$

   c. $PV = 60,000 \left[\dfrac{1}{.05} - \dfrac{1}{.05(1.05)^{20}} \right] = 747,73262$

   d. $PV = \dfrac{30,000}{.05} = 600,000$

   Series (c) has the highest present value.

10.   a. $PV = 500,000$

   b. $PV = 100,000 \left[\dfrac{1}{.2} - \dfrac{1}{.2(1.2)^8} \right] = 383,715.98$

   c. $PV = 60,000 \left[\dfrac{1}{.2} - \dfrac{1}{.2(1.2)^{20}} \right] = 292,174.78$

   d. $PV = \dfrac{30,000}{.2} = 150,000$

11. Plug discount rates into the present value annuity function until you find one that sets PV equal to the purchase price.

Try 15%:	PV = 9543.1685 < 10,000
Try 13%:	PV = 10,803.31 > 10,000
Try 14%:	PV = 9,892.8294 < 10,000
Try 13.7%:	PV = 10,001.638 > 10,000
Try 13.71%	PV = 9,997.977 < 10,000
Try 13.704%:	PV = 10,000.174 > 10,000

Thus, K is approximately 13.704%

12. a. $PV = \dfrac{10{,}000}{1.1^{20}} = \dfrac{10{,}000}{6.7275} = 1486.436$

b. $PV = \dfrac{10{,}000}{(1+.1/12)^{12*20}} = \dfrac{10{,}000}{7.328074} = 1364.615$

c. $PV = \dfrac{10{,}000}{(1+.1/365)^{365*20}} = \dfrac{10{,}000}{7.3870321} = 1353.7236$

d. $PV = 10{,}000 * e^{-.1*20} = 1353.3528$

13. a. First, the monthly discount rate is $.1 \div 12 = .008333$

$PV = 1{,}000 * \left[\dfrac{1}{.008333} - \dfrac{1}{.008333(1+.008333)^{360}} \right]$ $= 1{,}000 * 113.95082 = \$113{,}950.82$

b. Yes, since the PV exceeds the \$100,000 price

c. $100{,}000 = 1{,}000 * \left[\dfrac{1}{(k/12)} - \dfrac{1}{(k/12)(1+k/12)^{360}} \right]$

Solve for k; by process of substitution, we find that $k = .11627$.

N: INTRODUCTION TO RETURNS

The purpose of measuring investment returns is simply to determine the economic efficiency of an investment. Thus, an investment's return will express the profits generated by an initial cash outlay relative to the amount of that outlay. There exist a number of methods for determining the return of an investment. The measures presented in this chapter are return on investment and internal rate of return. Arithmetic and geometric mean rates of return on investment will be discussed along with internal rate of return and bond return measures. These methods differ in their ease of computation and how they account for the timeliness and compounding of cash flows.

O: RETURN ON INVESTMENT: ARITHMETIC MEAN

Perhaps the easiest method to determine the economic efficiency of an investment is to add all of its profits (π_t) accruing at each time period (t) and dividing this sum by the amount of the initial cash outlay (P_0). This measure is called a holding period return. To ease comparisons between investments with different life expectancies, one can compute an arithmetic mean return on investment (ROI) by dividing the holding period return by the life expectancy of the investment (n) as follows:

$$(1) \qquad ROI_A = \frac{\sum_{t=1}^{n} \pi_t}{P_0} \div n$$

The subscript (A) after (ROI) designates that the return value expressed is an arithmetic mean return and the variable (π_t) is the profit generated by the investment in year (t). Since it is not always clear exactly what the profit on an investment is in a given year, one can compute a return based on periodic cash flows. Therefore, this arithmetic mean rate of return formula can be written:

$$(2) \quad ROI_A = \frac{\sum_{t=0}^{n} CF_t}{n \cdot P_0} = \frac{\sum_{t=1}^{n} CF_t}{n \cdot P_0} - \frac{1}{n}$$

Notice that the summation in the first expression begins at time zero, ensuring that the initial cash outlay is deducted from the numerator. (The cash flow [CF_0] associated with any initial cash outlay or investment will be negative.) The primary advantage of Equation (2) over (1) is that a profit level need not be determined each year for the investment; that is, the annual cash flows generated by an investment do not have to be classified as to whether they are profits or merely return of capital. Multiplying (P_0) by (n) in the denominator of (2) to annualize the return has the same effect as dividing the entire fraction by (n) as in (1). In the second expression, the summation begins at time one. The initial outlay is recognized by subtracting one at the end of the computation. For example, consider a stock whose purchase price three years ago was $100. This stock paid a dividend of $10 in each of the three years and was sold for $130. If time zero is the stock's date of purchase, its arithmetic mean annual return is 20%:

$$ROI_A = \frac{-100 + 10 + 10 + 10 + 130}{3 \cdot 100} = \frac{60}{300} = .20$$

Identically, the stock's annual return is determined by (3):

(3) $\qquad ROI_A = \dfrac{\sum\limits_{t=1}^{n} DIV_t}{n \cdot P_0} + \dfrac{P_n - P_0}{n \cdot P_0}$

where (DIV_t) is the dividend payment for the stock in time (t), (P_0) is the purchase price of the stock and (P_n) is the selling price of the stock. The difference $(P_n - P_0)$ is the capital gain realized from the sale of the stock.

Consider a second stock held over the same period whose purchase price was also $100. If this stock
paid no dividends and was sold for $160, its annual return would also be 20%:

$$ROI_A = 0 + \frac{160 - 100}{3 \cdot 100} = \frac{60}{300} = .20$$

Therefore, both the first and second stocks have realized arithmetic mean returns of 20%. The total cash flows generated by each, net of their original $100 investments, is $60. Yet, the first stock must be preferred to the second since its cash flows are realized sooner. The arithmetic mean return (ROI_a) does not account for the timing of these cash flows. Therefore, it evaluates the two stocks identically even though the first should be preferred to the second. Because this measure of economic efficiency does not account for the timeliness of cash flows, another measure must be developed.

P: RETURN MEASUREMENT: GEOMETRIC MEAN

The arithmetic mean return on investment does not account for any difference between dividends (intermediate cash flows) and capital gains (profits realized at the end of the investment holding period). That is, ROI_A does not account for the time value of money or the ability to re-invest cash flows received prior to the end of the investment's life. In reality, if an investor receives profits in the form of dividends, he has the option to re-invest them as they are received. If profits are received in the form of capital gains, the investor must wait until the end of his investment holding period to re-invest them. The difference between these two forms of profits can be accounted for by expressing compounded returns. That is, the geometric mean return on investment will account for the fact that any earnings that are retained by the firm will be automatically re-invested, thus compounded. If returns are realized only in the form of capital gains, the geometric mean rate of return is computed as follows:

(4) $\qquad ROI_g = \sqrt[n]{P_n / P_o} - 1$

For example, the geometric mean return on the second stock whose priced increased from $100 to $160 discussed in Section B is 16.96%, determined as follows:

$$ROI_g = \sqrt[3]{160/100} - 1 = \sqrt[3]{1.6} - 1 = .1696$$

If dividends or intermediate cash flows from the security are realized before the end of the holding period, returns r_t should be computed for each period t and then averaged as follows:

$$r_t = \frac{P_t - P_{t-1} + DIV_t}{P_{t-1}}$$

(5)
$$ROI_g = \sqrt[n]{\prod_{t=1}^{n}(1 + r_t)} - 1$$

Suppose that the stock in our previous example paid $20 in dividends in each of the three years of the holding period rather than generating a $60 capital gain over the three year period. The return r_t for each period would be 20% and the geometric mean return for the stock would be 20% computed as follows:

$$ROI_g = \sqrt[n]{\prod_{t=1}^{n}(1 + r_t)} - 1 = \sqrt[3]{\prod_{t=1}^{3}(1 + r_t)} - 1 = \sqrt[3]{(1 + .2)(1 + .2)(1 + .2)} - 1 = .20$$

Note that the geometric mean return is higher if profits can be withdrawn from the investment during the holding period.

Q: INTERNAL RATE OF RETURN

The primary strength of the internal rate of return (IRR) as a measure of the economic efficiency of an investment is that it accounts for the timeliness of all cash flows generated by that investment. The IRR of an investment is calculated by using a model similar to the present value series model discussed in Section C:

$$PV = P_0 = \sum_{t=1}^{n} \frac{CF_t}{(1+r)^t}$$

or,

(6)
$$NPV = 0 = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t}$$

where net present value (NPV) is the present value of the series net of the initial cash outlay, and (r) is the return (or discount rate) that sets the investment's NPV equal to zero. The investment's internal rate of return is that value for (r) that equates NPV with zero.

There exists no general format allowing us to solve for the internal rate of return (r) in terms of the other variables in Equation (6); therefore, we must substitute values for (r) until we

find one that works (unless a computer or calculator with a built-in algorithm for solving such problems can be accessed). Often, this substitution process is very time-consuming, but with experience calculating internal rates of return, one can find shortcuts to solutions in various types of problems. Perhaps, the most important shortcut will be to find an easy method for deriving an initial value to substitute for (r) resulting in an NPV fairly close to zero. One easy method for generating an initial value to substitute for (r) is by first calculating the investment's return on investment. If an investor wanted to calculate the internal rate of return for the first stock presented in Section B, he may wish to first substitute for (r) the stock's 20% return on investment:

$$NPV = \frac{-100}{(1+r)^0} + \frac{10}{(1+r)^1} + \frac{10}{(1+r)^2} + \frac{10+130}{(1+r)^3} = -100 + \frac{10}{(1+r)} + \frac{10}{(1+r)^2} + \frac{140}{(1.2)^3} = -3.7$$

Since this NPV is less than zero, a smaller (r) value should be substituted. A smaller (r) value will decrease the right-hand side denominators, increasing the size of the fractions and NPV. Perhaps a feasible value to substitute for (r) is 10%. The same calculations will be repeated with the new (r) value of 10%:

$$NPV = -100 + \frac{10}{1.1} + \frac{10}{(1.1)^2} + \frac{10}{(1.1)^3} = +22.54$$

Since the new NPV exceeds zero, the (r) value of 10% is too small. However, because -3.70 is closer to zero than 22.54, the next value to substitute for (r) might be closer to 20% than to 10%. Perhaps a better estimate for the IRR will be 18%. Substituting this value for (r) results in an NPV of .86:

$$NPV = -100 + \frac{10}{1.18} + \frac{10}{(1.18)^2} + \frac{140}{(1.18)^3} = -0.86$$

This NPV is quite close to zero; in fact further substitutions will indicate that the true stock internal rate of return is approximately 18.369%. These iterations have a pattern: when NPV is less than zero, decrease (r) for the next substitution; when NPV exceeds zero, increase (r) for the next substitution. This process of iterations need only be repeated until the desired accuracy of calculations is reached.

The primary advantage of the internal rate of return over return on investment is that it accounts for the timeliness of all cash flows generated by that investment. However, IRR does have three major weaknesses:

1. As we have seen, IRR takes considerably longer to calculate than does ROI. Therefore, if ease of calculation is of primary importance in a situation, the investor may prefer to use ROI as his measure of efficiency. As discussed in the appendix to this chapter, there do exist calculators and computer programs that will compute IRR very quickly.

2. Sometimes an investment will generate multiple rates of return; that is, more than one (r) value will equate NPV with zero. This will occur when that investment has associated with it more than one negative cash flow. When multiple rates are generated, there is often no method to determine which is the true IRR. In fact, none of the rates generated may make any sense. When the IRR is infeasible as a method for comparing two investments, and the investor still wishes to consider the time value of money in his calculations, he may simply compare the present values of the investments. This approach and its weaknesses will be discussed in later chapters.

3. The internal rate of return is based on the assumption that cash flows received prior to the expiration of the investment will be re-invested at the internal rate of return. That is, it is assumed that future investment rates are constant and equal to the IRR. Obviously, this assumption may not hold in reality.

R: BOND YIELDS

By convention, rates of return on bonds are often expressed in terms somewhat different from those of other investments. For example, the coupon rate of a bond is the annual interest payment associated with the bond divided by the bond's face value. Thus, a four-year $1000 corporate bond making $60 annual interest payments has a coupon rate of 6%. However, the coupon rate does not account for the actual purchase price of the bond. Corporate bonds are usually traded at prices that differ from their face values. The bond's current yield accounts for the actual purchase price of the bond:

$$(7) \ cy = \frac{INT}{P_0}$$

If the above 6% bond were purchased for $800, its current yield would be 7.5%.

The formula for current yield, while easy to work with, does not account for any capital gains
(or losses) that may be realized when the bond matures. Furthermore, current yields do not account for the timeliness of cash flows associated with bonds. The bond's yield to maturity, which is essentially its internal rate of return does account for any capital gains (or losses) that may be realized at maturity in addition to the timeliness of all associated cash flows:

$$(8) \ NPV = 0 = \sum_{t=0}^{n} \frac{CF_t}{(1+y)^t} = -P_0 \left[\sum_{t=1}^{n} \frac{INT}{(1+y)^t} \right] + \frac{F}{(1+y)^n}$$

The yield to maturity (y) of the above bond would be (6%). Thus, (y) is identical to the bond's internal rate of return. (However, in most instances, the bond's yield to maturity will not equal its coupon rate.) If the bond makes semiannual interest payments, its yield to maturity can be more accurately expressed:

$$(9) \quad NPV = 0 = \sum_{t=0}^{n} \frac{CF_t}{(1+y)^t} = -P_0 \left[\sum_{t=1}^{2n} \frac{INT/2}{(1+y/2)^t} \right] + \frac{F}{(1+y)^n}$$

Here, we are concerned with semiannual interest payments and (2 n) six-month time periods where (n) is the number of years to the bond's maturity. While yield to maturity is perhaps the most widely-used of the bond return measures, it still assumes a flat yield curve. This means that coupon payments received prior to bond maturity will be invested at the same rate as the bond's yield, an unrealistic assumption when interest rates are expected to change significantly over time.

S: INTRODUCTION TO RISK AMD EXPECTED RETURN

When a firm invests, it subjects itself to at least some degree of uncertainty regarding future cash flows. Managers cannot know with certainty what investment payoffs will be. This chapter is concerned with forecasting investment payoffs and returns and the uncertainty associated with these forecasts. We will define expected return in this chapter, focusing on it as a return forecast. This expected return will be expressed as a function of the investment's potential return outcomes and associated probabilities. The riskiness of an investment is simply the potential for deviation from the investment's expected return. The risk of an investment is defined here as the uncertainty associated with returns on that investment. Although other definitions for risk such as the probability of losing money or going bankrupt can be very useful, they are often less complete or more difficult to measure. Our definition of risk does have some drawbacks as well. For example, an investment which is certain to be a complete loss is not regarded here to be risky since its return is known to be -100% (though we note that it probably would not be regarded to be a particularly good investment).

Consider an economy with three potential states of nature in the next year and Stock A whose return is dependent on these states. If the economy performs well, state one is realized and the stock earns a return of 25%. If the economy performs only satisfactorily, state two is realized and the stock earns a return of 10%. If the economy performs poorly, state three is realized and the stock achieves a return of -10%. Assume that there is a twenty percent chance that state one will occur, a fifty percent chance that state two will occur and a thirty percent chance that state three will occur. The expected return on the stock will be 7%, determined by Equation (10):

$$(10) \quad E[R_A] = \sum_{i=1}^{n} R_{Ai} P_i$$

$$E[R] = (.25 * .20) + (.10 * .50) + (-.10 * .30) = .07 \,,$$

where (R_i) is return outcome (i) and (P_i) is the probability associated with that outcome. Therefore, our forecasted return is 7%. The expected return considers all potential returns and weights more heavily those returns that are more likely to actually occur. Although our forecasted return level is seven percent, it is obvious that there is potential for the actual return outcome to deviate from this figure. This potential for deviation (variation) will be measured in the following section.

T: VARIANCE AND STANDARD DEVIATION

The statistical concept of variance is a useful measure of risk. Variance accounts for the likelihood that the actual return outcome will vary from its expected value; furthermore, it accounts for the magnitude of the difference between potential return outcomes and the expected return. Variance can be computed with Equation (11):

(11)
$$\sigma^2 = \sum_{i=1}^{n} (R_i - E[R])^2 P_i$$

Figure 2: Expected Return, Variance and Standard Deviation of Returns for Stock A

i	R_i	P_i	R_iP_i	$R_i - E[R_a]$	$(R_i - E[R_a])^2$	$(R_i - E[R_a])^2 P_i$
1	.25	.20	.05	.18	.0324	.00648
2	.10	.50	.05	.03	.0009	.00045
3	-.10	.30	-.03	-.17	.0289	.00867

$$E[R_a] = .07 \qquad \sigma^2_a = .01560$$
$$\sigma_a = .1249$$

The variance of stock returns presented in Section G is .0156:

$$\sigma^2 = (.25 - .07)^2 *.2 + (.10 - .07)^2 *.5 + (-.10 - .07)^2 *.3 = .0156$$

The statistical concept of standard deviation is also a useful measure of risk. The standard deviation of a stock's returns is simply the square root of its variance:

$$\sigma = \sqrt{\sum_{i=1}^{n} (R_i - E[R])^2 P_i}$$

Thus, the standard deviation of returns on the stock described in Section F is 12.49%.

Consider a second security, Stock B whose return outcomes are also dependent on economy outcomes one, two and three. If outcome one is realized, Stock B attains a return of 45%; in outcomes two and three, the stock attains returns of 5% and -15%, respectively. From Figure 11, we see that the expected return on Stock B is seven percent, the same as for Stock A. However, the actual return outcome of Stock B is subject to more uncertainty. Stock B has the potential of receiving either a much higher or much lower actual return than does Stock A. For example, an investment in Stock B could lose as much as fifteen percent, whereas an equal investment in Stock A cannot lose more than ten percent. An investment in Stock B also has the potential of attaining a much higher return than an identical investment in Stock A. Therefore, returns on Stock B are subject to greater variability (or risk) than returns on Stock A. The concept of variance (or standard deviation) accounts for this increased variability. The variance of Stock B (.0436) exceeds that of Stock A (.0156), indicating that Stock B is riskier than Stock A.

Figure 3: Expected Return, Variance, and Standard Deviation of Returns for Stock B

i	R_i	P_i	R_iP_i	$R_i - E[R_b]$	$(R_i - E[R_b])^2$	$(R_i - E[R_b])^2P_i$
1	.45	.20	.09	.38	.1444	.02888
2	.05	.50	.025	.02	.0004	.00020
3	-.15	.30	-.045	.22	.0484	.01452

$$E[R_b]=.070 \qquad\qquad \sigma^2 = .04360$$
$$\sigma_b = .2088$$

With the expected return and standard deviation of returns of an investment, we can establish ranges of potential returns and probabilities that actual returns will fall within these ranges if it appears that potential returns for that investment are normally distributed. For example, consider a third stock with normally distributed returns with an expected level of 7% and a standard deviation of 10%. From Table V in the text appendix, we see that there is a 68% probability that the actual return outcome on this stock will fall between -.03 and .17:

$$E[R] - 1\sigma < R_i < E[R] + 1\sigma$$
$$(.07 - .10) < R_i < (.07 + .10)$$

A similar analysis indicates a 95% probability that the actual return outcome will fall between -.13 and .27:

$$E[R] - 2\sigma < R_i < E[R] + 2\sigma$$
$$(.07 - .20) < R_i < (.07 + .20)$$

Obviously, a smaller standard deviation of returns will lead to a narrower range of potential outcomes given any level of probability. If a security has a standard deviation of returns equal to zero, it has no risk. Such a security is referred to as the risk-free security with a return of (r_f). Therefore, the only potential return level of the risk-free security is (r_f). No such security exists in reality; however, short-term United States treasury bills are quite close. The U.S. government has proven to be an extremely reliable debtor. When investors purchase treasury bills and hold them to maturity, they do receive their expected returns. Therefore, short-term treasury bills are probably the safest of all securities. For this reason, financial analysts often use the treasury bill rate (of return) as their estimate for (r_f) in many important calculations.

U: HISTORICAL VARIANCE AND STANDARD DEVIATION

Empirical evidence suggests that historical stock return variances (standard deviations) are excellent indicators of future variances (standard deviations). That is, a stock whose previous returns have been subject to substantial variability probably will continue to realize returns of a highly volatile nature. Therefore, past riskiness is often a good indicator of future riskiness. A stock's historical return variability can be measured with a historical variance:

(12) $$\sigma_h^2 = \sum_{t=1}^{n}(R_t - \overline{R})^2 \frac{1}{n}$$

where (R_t) is the stock return in time (t) and () is the historical average return over the (n) time period sample. The stock's historical standard deviation of returns is simply the square root of its variance. If an investor determines that the historical variance is a good indicator of its future variance, he may need not to calculate potential future returns and their associated probabilities for risk estimates; he may prefer to simply measure the stock's risk with its historical variance or standard deviation given as follows:

$$\sigma_H = \sqrt{\frac{\sum_{t=1}^{n}(R_i - \overline{R})^2}{n}}$$

Figure 4 demonstrates historical variance and standard deviation computations for Stock D.

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Figure 4: Historical Variance and Standard Deviation of Returns of Stock D

t	R_t	$R_t - \overline{R}_D$	$(R_t - \overline{R}_D)^2$	$(R_t - \overline{R}_D)^2$ 1/n
1	.10	-.06	.0036	.00072
2	.15	-.01	.0001	.00002
3	.20	.04	.0016	.00032
4	.10	-.06	.0036	.00072
5	.25	.09	.0081	.00162

$_d$=.16 ^{2}=.00340;

                                                      $_d$ =.05831

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V: COVARIANCE

Standard deviation and variance provide us with measures of the absolute risk levels of securities. However, in many instances, it is useful to measure the risk of one security relative to the risk of another or relative to the market as a whole. The concept of covariance is integral to the development of relative risk measures. Covariance provides us with a measure of the relationship between the returns of two securities. That is, given that two securities returns are likely to vary, covariance indicates whether they will vary in the same direction or in opposite directions. The likelihood that two securities will covary in the same direction (or, more accurately, the strength of the relationship between returns on two securities) is measured by Equation (13):

(13) $\sigma_{k,j} = Cov[k,j] = \sum_{t=1}^{n}(R_{k,i} - E[R_k])(R_{j,i} - E[R_j])P_i$

where (R_{ki}) and (R_{ji}) are the return of stocks (k) and (j) if outcome (i) is realized and (P_i) is the probability of outcome (i). $E[R_k]$ and $E[R_j]$ are simply the expected returns of securities (k) and

(j). For example, the covariance between returns of stocks A and B is:

$$\text{cov}(A,B) = \{(.25 - .07) * (.45 - .07) * .20\}$$
$$+ \; \{(.10 - .07) * (.05 - .07) * .50\} + \{(-.10 - .07) * (-.15 - .07) * .30\}$$
$$= \{.01368\} + \{-.0003\} + \{.01122\} = .0246.$$

Since this covariance is positive, the relationship between returns on these two securities is positive. That is, the larger the positive value of covariance, the more likely one security will perform well given that the second will perform well. A negative covariance indicates that strong performance by one security implies likely poor performance by the second security. A covariance of zero implies that there is no relationship between returns on the two securities. Figure 5 details the solution method for this example.

Figure 5: Covariance between Returns on Stocks A and B

i	R_{ai}	R_{bi}	P_i	$R_{ai}-E[R_a]$	$R_{bi}-E[R_b]$	$(R_{ai}-E[R_a])(R_{bi}-E[R_b])P_i$
1	.25	.45	.20	.18	.38	.01368
2	.10	.05	.50	.03	-.02	-.00030
3	-.10	-.15	.30	-.17	-.22	.01122
						COV(A,B) = .0246

     Empirical evidence suggests that historical covariances are strong indicators of future covariance levels. Thus, if one is unable to associate probabilities with potential outcome levels, in many cases he may use historical covariance as his estimate for future covariance. Figure 6 demonstrates how to determine historical covariance for two hypothetical stocks D and E.

Figure 6: Historical Covariance between Returns on Stocks D and E

t	R_{dt}	R_{et}	$(R_{dt}-\bar{R_d})$	$(R_{et}-\bar{R_e})$	$(R_{dt}-\bar{R_d})(R_{et}-\bar{R_e})1/n$
1	.10	.15	-.06	-.05	.00060
2	.15	.18	-.01	-.02	.00004
3	.20	.25	.04	.05	.00040
4	.10	.20	-.06	0	0
5	.25	.22	.09	.02	.00036
	$R_d = .16$	$.20 = R_e$	COV(D,E) = .00140		

W: COEFFICIENT OF CORRELATION

    The coefficient of correlation provides us with a means of standardizing the covariance between returns on two securities. For example, how large must covariance be to indicate a strong relationship between returns? Covariance will be smaller given low returns on the two securities than given high security returns. The coefficient of correlation (ρ_{kj}) between returns on two securities will always fall between -1 and +1.[1] If security returns are directly related, the

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[1]Many statistics textbooks use the notation ($r_{i,j}$) to designate the correlation coefficient between variables (i) and (j). Because the letter (r) is used in this text to designate return, it will use the lower case rho (ρ_{ij}) to designate correlation coefficient.

correlation coefficient will be positive. If the two security returns always covary in the same direction by the same proportions, the coefficient of correlation will equal one. If the two security returns always covary in opposite directions by the same proportions, ($\rho_{k,j}$) will equal negative one. The stronger the inverse relationship between returns on the two securities, the closer ($\rho_{k,j}$) will be to negative one. If ($\rho_{k,j}$) equals zero, there is no relationship between returns on the two securities. The coefficient of correlation ($\rho_{k,j}$) between returns is simply the covariance between returns on the two securities divided by the product of their standard deviations:

(14) $$\rho_{kj} = \frac{COV(k, j)}{\sigma_k \cdot \sigma_j}$$

Equation (14) implies that the covariance formula can be rewritten:

(15) $COV(k,j) = \sigma_k \sigma_j \rho_{k,j}$

If an investor can access only raw data pertaining to security returns, he should first find security covariances then divide by the products of their standard deviations to find correlation coefficients. However, if for some reason the investor knows the correlation coefficients between returns on securities, he can use this value along with standard deviations to find covariances.

$$\rho_{kj} = \frac{.0246}{.1249 \cdot .21}$$

The coefficient of correlation between returns on stocks A and B is .96: This value can be squared to determine the coefficient of determination between returns of the two securities. The coefficient of determination ($\rho_{k,j}^2$) measures the proportion of variability in one security's returns that can be explained by or be associated with variability of returns on the second security. Thus, approximately 92% of the variability of stock A returns can be explained by or associated with variability of stock B returns. The concepts of covariance and correlation are crucial to the development of portfolio risk and relative risk models presented in later chapters.

Historical evidence suggests that covariances and correlations between stock returns remain relatively constant over time. Thus, an investor can use historical covariances and correlations as his forecasted values. However, it is important to realize that these historical relationships apply to standard deviations, variances, covariances and correlations, but not to the actual returns themselves. That is, we can often forecast future risk levels and relationships on the basis of historical data, but we cannot forecast returns on the basis of historical returns. Thus, last year's return for a given stock implies almost nothing about next year's return for that stock.

The historical covariance between returns on securities (i) and (j) can be found by solving Equation (16):

(16) $$\sigma_{ij} = \sum_{t=1}^{n}(R_{i,t} - \overline{R}_i)(R_{J,t} - \overline{R}_j)\frac{1}{n}$$

where the sample data is taken from (n) years. See Figure (6).

X: THE MARKET PORTFOLIO

As we shall see in the next chapter, a portfolio is simply a collection of investments. The market portfolio is the collection of all investments that are available to investors. That is, the market portfolio represents the combination or aggregation of all securities (or other assets) that are available for purchase. Investors may wish to consider the performance of this market portfolio to determine the performance of securities in general. Thus, the return on the market portfolio is representative of the return on the "typical" asset. An investor may wish to know the market portfolio return to determine the performance of a particular security or his entire investment portfolio relative to the performance of the market or a "typical" security.

Determination of the return on the market portfolio requires the calculation of returns on all of the assets available to investors. Because there are hundreds of thousands of assets available to investors (including stocks, bonds, options, bank accounts, real estate, etc.), determining the exact return of the market portfolio may be impossible. Thus, investors generally make use of indices such as the Dow Jones Industrial Average or the Standard and Poor's 500 (S&P 500) to gauge the performance of the market portfolio. These indices merely act as surrogates for the market portfolio; we assume that if the indices are increasing, then the market portfolio is performing well. For example, performance of the Dow Jones Industrials Average depends on the performance of the thirty stocks that comprise this index. Thus, if the Dow Jones market index is performing well, the thirty securities, on average are probably performing well. This strong performance may imply that the market portfolio is performing well. In any case, it is easier to measure the performance of a portfolio of thirty or five hundred stocks (for the Standard and Poor's 500) than it is to measure the performance of all of the securities that comprise the market portfolio.

Y. CONCLUSION

Risk means different things to different people. To some, it means potential for not earning profits, for others it means potential for bankruptcy. In this text, we define risk to be uncertainty. There also exist numerous measures of risk. Among the more popular measures of risk is standard deviation. Standard deviation is a particularly useful indicator of risk for several reasons:

1. Standard deviation can accommodate all potential outcomes associated with an investment. Each outcome that varies significantly from the mean or expected value increases standard deviation.
2. Potential outcomes that are more likely affect standard deviations more than outcomes which are less likely. For example, a very likely outcome that deviates substantially from expected value will increase standard deviation.
3. The normal distribution, which describes the frequency of such a large number of financial phenomena, is defined in terms of standard deviation.

Historical standard deviation is useful when the analyst believes that the historical volatility of an investment is a good indicator of its future uncertainty.

Co-movement statistics such as covariance are quite useful in determining the relationship between various financial data. The sign of covariance indicates the direction of co-movement and is useful in measurement of portfolio risk and in the computation of relative risk statistics to be discussed later. Coefficient of correlation is essentially a standardized covariance; its absolute value indicates the intensity of co-movement. The coefficient of determination indicates the proportion of variability in one data set that can be explained by variability in a second data set.

QUESTIONS AND PROBLEMS

1. An investor purchased one share of Drysdale stock for one hundred dollars in 2003 and sold it exactly one year later for $200. Calculate the investor's arithmetic mean return on investment.

2. An investor purchased one share of Wilson Company stock in 1996 for $20 and sold it in 2003 for $40. Calculate the following for this investor:
     a. arithmetic mean return on investment.
     b. geometric mean return on investment.
     c. internal rate of return.

3. An investor purchased one hundred shares of Mathewson Company stock for $75 apiece in 1995 and sold each share for $80 exactly six years later. The Mathewson Company paid annual dividends of $8 per share in each of the six years the investor held the stock. Calculate the following for the investor:
     a. arithmetic mean return on investment.
     b. internal rate of return.

4. The Paige Baking Company is considering the purchase of an oven for $100,000 whose output will yield the company $20,000 in annual after-tax cash flows for each of the next five years. At the end of the fifth year, the oven will be sold for its $40,000 salvage value. Calculate the following for the machine Paige is considering purchasing:
     a. arithmetic mean return on investment.
     b. internal rate of return.

5. What is the net present value of an investment whose internal rate of return equals its discount rate?

6. An investor purchased one hundred shares each of Grove Company stock and Dean Company stock for $10 per share. The Grove Company paid an annual dividend of one dollar per share in each of the eight years the investor held the stock. The Dean Company paid an annual dividend of $.25 per share in each of the eight years the investor held the stock. At the end of the eight-year period, the investor sold each of his shares of Grove Company stock for $11 and sold each of his shares of Dean Company stock for $18.
  a.   Calculate the sum of dividends received by the investor from each of the companies.
  b.   Calculate the capital gains realized on the sale of stock of each of the companies.
  c.   Calculate the return on investment for each of the two companies' stock using an arithmetic mean return.
  d.   Calculate the internal rate of return for each of the two stocks.
  e.   Which of the two stocks performed better during their holding periods?

7. The Lemon Company is considering the purchase of an investment for $100,000 that is expected to pay off $50,000 in two years, $75,000 in four years and $75,000 in six years. In the third year, Lemon must make an additional payment of $50,000 to sustain the investment. Calculate the following for the Lemon investment:
     a. Return on investment using an arithmetic mean return.

b. The investment internal rate of return.

c. Describe any complications you encountered in part b.

8. A $1,000 face value bond is currently selling at a premium for $1,200. The coupon rate of this bond is 12% and it matures in three years. Calculate the following for this bond assuming its interest payments are made annually:

a. Its annual interest payments.

b. Its current yield.

c. Its yield to maturity.

9. Work through each of the calculations in Problem 8 assuming interest payments are made semi-annually.

10. The Nichols Company invested $100,000 into a small business twenty years ago. Its investment generated a cash flow equal to $3,000 in its first year of operation. Each subsequent year, the business generated a cash flow that was 10% larger than in the prior year; that is, the business generated a cash flow equal to $3,300 in the second year, $3,630 in the third year, and so on for nineteen years after the first. The Nichols Company sold the business for $500,000 after its twentieth year of operation. What was the internal rate of return for this investment?

11. Suppose that a mutual fund investing on behalf of shareholders yielded the following price and price performance results:

Date	t	P_t	P_{t-1}	DIV_t
June 30	1	50	-	0
July 31	2	55	50	0
Aug. 31	3	50	55	0
Sep. 30	4	54	50	0
Oct. 31	5	47	54	2
Nov. 30	6	51	47	0

Calculate for this fund monthly returns for each of the five months July to November and compute a geometric mean return over this 5-month period.

12. Mack Products management is considering the investment in one of two projects available to the company. The returns on the two projects (A) and (B) are dependent on the sales outcome of the company. Mack management has determined three potential sales outcomes (1), (2) and (3) for the company. The highest potential sales outcome for Mack is outcome (1) or $800,000. If this sales outcome were realized, Project (A) would realize a return outcome of 30%; Project (B) would realize a return of 20%. If outcome (2) were realized, the company's sales level would be $500,000. In this case, project (A) would yield 15%, and Project (B) would yield 13%. The worst outcome (3) will result in a sales level of $400,000, and return levels for Projects (A) and (B) of 1% and 9% respectively. If each sales outcome has an equal probability of occurring, determine the following for the Mack Company:

a.   the probabilities of outcomes (1), (2) and (3).

b.   its expected sales level.

c.   the variance associated with potential sales levels.
d.   the expected return of Project (A).
e.   the variance of potential returns for Project (A).
f.   the expected return and variance for Project (B).
g.   standard deviations associated with company sales, returns on Project (A) and returns on Project (B).
h.   the covariance between company sales and returns on Project (A).
i.   the coefficient of correlation between company sales and returns on Project (A).
j.   the coefficient of correlation between company sales and returns on Project (B).
k.   the coefficient of determination between company sales and returns on Project (B).

13. Which of the projects in Problem 12 represents the better investment for Mack Products?

14. Historical <u>percentage</u> returns for the McCarthy and Alston Companies are listed in the following chart along with percentage returns on the market portfolio:

Year	McCarthy	Alston	Market
1988	4	19	15
1989	7	4	10
1990	11	-4	3
1991	4	21	12
1992	5	13	9

Calculate the following based on the preceding diagram:
a.   mean historical returns for the two companies and the market portfolio.
b.   variances associated with McCarthy Company returns and Alston Company returns as well as returns on the market portfolio.
c.   the historical covariance and coefficient of correlation between returns of the two securities.
d.   the historical covariance and coefficient of correlation between returns of the McCarthy Company and returns on the market portfolio.
e.   the historical covariance and coefficient of correlation between returns of the Alston Company and returns on the market portfolio.

15. Forecast the following for both the McCarthy and Alston Companies based on your calculations in Problem 14:
a.   variance and standard deviation of returns.
b.   coefficient of correlation between each of the companies' returns and returns on the market portfolio.

16. The following table represents outcome numbers, probabilities and associated returns for stock A:

outcome (i)	return (R_i)	Probability (P_i)
1	.05	.10
2	.15	.10
3	.05	.05

4	.15	.10
5	.15	.10
6	.10	.10
7	.15	.10
8	.05	.10
9	.15	?
10	.10	.10

Thus, there are ten possible return outcomes for Stock A.
   a.   What is the probability associated with Outcome 9?
   b.   What is the standard deviation of returns associated with Stock A?

17. The Durocher Company management projects a return level of 15% for the upcoming year. Management is uncertain as to what the actual sales level will be; therefore, it associates a standard deviation of 10% with this sales level. Managers assume that sales will be normally distributed. What is the probability that the actual return level will:
   a.   fall between 5% and 25%?
   b.   fall between 15% and 25%?
   c.   exceed 25%?
   d.   exceed 30%?

18. What would be each of the probabilities in Problem 17 if Durocher Company management were certain enough of its forecast to associate a 5% standard deviation with its sales projection?

19. Under what circumstances can the coefficient of determination between returns on two securities be negative? How would you interpret a negative coefficient of determination? If there are no circumstances where the coefficient of determination can be negative, describe why.

20. Stock A will generate a return of 10% if and only if Stock B yields a return of 15%; Stock B will generate a return of 10% if and only if Stock A yields a return of 20%. There is a 50% probability that Stock A will generate a return of 10% and a 50% probability that it will yield 20%.
   a. What is the standard deviation of returns for Stock A?
   b. What is the covariance of returns between Stocks A and B?

21. An investor has the opportunity to purchase a risk-free treasury bill yielding a return of 10%. He also has the opportunity to purchase a stock which will yield either 7% or 17%. Either outcome is equally likely to occur. Compute the following:
   a.   the variance of returns on the stock.
   b.   the coefficient of correlation between returns on the stock and returns on the treasury bill.

22. The following daily prices were collected for each of three stocks over a twelve day period.

Company X		Company Y		Company Z	
DATE	PRICE	DATE	PRICE	DATE	PRICE
1/09	50.125	1/09	20.000	1/09	60.375
1/10	50.125	1/10	20.000	1/10	60.500

1/11	50.250	1/11	20.125	1/11	60.250
1/12	50.250	1/12	20.250	1/12	60.125
1/13	50.375	1/13	20.375	1/13	60.000
1/14	50.250	1/14	20.375	1/14	60.125
1/15	52.250	1/15	21.375	1/15	62.625
1/16	52.375	1/16	21.250	1/16	60.750
1/17	52.250	1/17	21.375	1/17	60.750
1/18	52.375	1/18	21.500	1/18	60.875
1/19	52.500	1/19	21.375	1/19	60.875
1/20	52.375	1/20	21.500	2/20	60.875

Based on the data given above, calculate the following:

a. Returns for each day on each of the three stocks. There should be a total of ten returns for each stock - beginning with the date 1/10.
b. Average daily returns for each of the three stocks.
c. Daily return standard deviations for each of the three stocks.

Solutions

1 a. $ROI = \sum_{t=0}^{n} CF_t \div nP_0$

$= (-100 + 200) /(1 \cdot(100)) \quad = 1.00 \text{ or } 100\%$

2 a. $ROI = (40 - 20)/(7(20))$

$= .1428 \text{ or } 14.28\%$

b. $ROI = (40/20)^{1/7} - 1 \quad = (2)^{1/7} - 1 \quad = .1041 \text{ or } 10.41\%$

c. $IRR = 10.41\%$; Note that $ROI_{AG} = IRR$ when there is only a capital gain profit.

3 a. $ROI = (500 + 4,800)/6(7,500) = .1178 \text{ or } 11.78\%$

b. $IRR = 11.49\%$

4 a. $ROI = (-100,000 + 20,000 + 20,000 + 20,000 + 20,000 + 60,000)/5(100,000)$

$= 40,000/500,000 \quad = .08 \text{ or } 8\%$

b. $IRR = 10.21\%$

5 $NPV = 0$, by definition of IRR.

6 a. Dividends: Grove = $800   Dean = $200

b. Capital Gains: Grove = $1,100 - $1,000 = $100   Dean = $1,800 - $1,000 = $800

c. Arithmetic Mean Capital Gain Return:

Grove = (100 + 800)/8(1,000)   = .1125 or 11.25%

Dean = (800 + 200)/8(1,000)   = .125 or 12.5%

d. IRR: Grove = 11.0165%   Dean = 9.5%

SUMMARY OF RESULTS

Company	Dividends	Cap Gains	ROI$_a$	IRR
Grove	800	100	11.25%	10.0%
Dean	200	800	12.50%	9.5%

e. Under ROI$_a$ = Dean   Under IRR = Grove

The performance evaluation depends on the measure used. Depends on the investor's time value of money. Higher time value indicates that IRR is more useful.

7
    a. $ROI_A = \sum_{t=0}^{n} CF_t \div nP_0$

        = (-100,000 + 50,000 - 50,000 + 75,000 + 75,000)/6(100,000) = .083 or 8.3%

    b. $100,000 = 50,000/(1+r)^2 - 50,000/(1+r)^3 + 75,000/(1+r)^4 + 75,000/(1+r)^6$

       IRR = 9.32487405%, IRR = -227.776188859%,

    c. There are actually two internal rates of return for this problem. However, 9.32487% seems to be a reasonable rate.

8
    a. Its annual interest payments:

        $i_y = Int/F_0$

        $Int = i_y(F_0)$ = (.12)(1000)   = \$120

    b. Its current yield:

        $cy = Int/P_0$ = 120/1,200   = .10

    c. With Equation 8, yield to maturity is found to be .04697429 or 4.697429%

9
    a. Its annual interest payments: \$120, or \$60 every six months.

    b. 120 ÷ 1200 = Its current yield = .10 or 10%.

    c. Its yield to maturity:

       $-1200 + 60/[1+(r/2)]^1 + 60/[1+(r/2)]^2 + ... + 60/[1+(r/2)]^5 + 1060/[1+(r/2)]^6$

       r = .0476634

10. $PV_{ga} = CF_1 \times \dfrac{1}{r-g} - \dfrac{(1+g)^n}{(r-g)(1+r)^n} + \dfrac{CF_n}{(1+r)^n}$

       $CF_1 = \$3,000$, n = 20 , g = .10   Solve for r above to obtain IRR = .11794166365

11.   First, compute 5 monthly returns as follows:

Date	t	P_t	P_{t-1}	DIV_t	r_t	NOTES
June 30	1	50	-	0	-	First Month
July 31	2	55	50	0	.100	(55÷50) - 1 = .10
Aug. 31	3	50	55	0	-.091	(50÷55) - 1 = -.091
Sep. 30	4	54	50	0	.080	(54÷50) - 1 = .08
Oct. 31	5	47	54	2	-.092	ex-\$2 dividend; [(47+2)÷54)]-1= -.092
Nov. 30	6	51	47	0	.081	(51÷47) - 1 = .085

Next, compute the 5-month geometric mean return for the fund:

$$ROI_g = \sqrt[n]{\prod_{t=1}^{n}(1+r_t)} - 1 = \sqrt[3]{\prod_{t=1}^{5}(1+r_t)} - 1$$

$$= \sqrt[5]{(1+.10)(1-.091)(1+.08)(1-.092)(1+.081)} - 1 = .01246$$

12.

  a. Each outcome has a one-third or .333 probability of being realized since the probabilities are equal and must sum to one.

b. E[SALES] = (800,000 * .333)+(500,000 * .333)+(400,000 * .333)
  E[SALES] = 566,667

c. var[sales] = [$(800,000 - 566,667)^2$ * .333 + $(500,000 - 566,667)^2$ * .333 + (400,000 - 566,667) *.333] = 28,888,000,000 = σ^2_{SALES}

d. Expected return of Project A = (.3*.333)+(.15*.333)+(.01*.333)   = .15333

e. Variance of A's Returns = [$(.3-.1533)^2$ *.333 + $(.15-.1533)^2$ *.333 + $(.01-.1533)^2$ *.333] = .0140222 = σ^2_A

f. Expected Return of Project B = (.2*.333)+(.13*.333)+(.09*.333) = .14 .

  Variance of B's Returns = [$(.2-.14)^2$ *.333 + $(.13-.14)^2$ *.333 + $(.09-.14)^2$ *.333] = .0020666 = σ^2_B

g. Standard deviations are square roots of variances.

$\sigma_{SALES} = 169,964 \quad \sigma_A = .1184154 \quad \sigma_B = .0454606$

h. $COV[SALES,A] = \sum_{i=1}^{n} (SALES_i - E[SALES]) \quad (R_{Ai} - E[R_A]) \quad P_i$

$COV[SALES,A] = (800,000 - 566,667) * (.3 - .1533) * .333$
$\qquad + (500,000 - 566,667) * (.15 - .1533)* .333$
$\qquad + (400,000 - 566,667) * (.01 - .1533)* .333$
$\qquad = 19,444 = \sigma_{SALES, A}$

i. $\rho_{s,A} = \dfrac{\sigma_{SALES, A}}{\sigma_{SALES}*\sigma_A} = \dfrac{19,444}{169,964 * .118} = .97$

j. First, find the covariance between sales and returns on B.
$COV[SALES,B] = (800,000 - 566,667) * (.20 - .14) * .333$
$\qquad + (500,000 - 566,667) * (.13 - .14) * .333$
$\qquad + (400,000 - 566,667) * (.09 - .14) * .333 \qquad = 7666.67 = \sigma_{SALES, B}$

$\sigma_{SALES, B} = \dfrac{\sigma_{SALES, B}}{\sigma_{SALES} \sigma_B} = \dfrac{7666}{169,964*.0454} = .993$

k. Coefficient of Determination is simply Coefficient of Correlation squared: $.993 * .993 = .986$

13. Project A has a higher expected return; however, it is riskier. Therefore, it does not clearly dominate Project B. Similarly, B does not dominate A. Therefore, we have insufficient evidence to determine which of the projects are better.

14
a. $\bar{R}_{Mc} = .062$

$\bar{R}_A = .106$

$\bar{R}_M = .098$

b. $\sigma^2_{Mc} = .000696$ (Remember to convert returns to percentages.)

$\sigma^2_A = .008824$ (Square roots of these variances are standard

$\sigma^2_M = .001576$ deviations.)

c. $COV[Mc,A] = [(.04-.062)*(.19-.106) + (.07-.062)*(.04-.106) + (.11-.062)*(-.04-.106) + (.04-.062)*(.21-.106) + (.05-.062) *(.13-.106)] / 5 = -.001624$

$\sigma_{Mc,A} = \dfrac{COV[Mc,A]}{\sigma_{Mc} \sigma_A} = \dfrac{-.001624}{.0264 * .094} = -.6544$

d. $COV[Mc,A] = [(.04-.062)*(.15-.098) + (.07-.062)*(.10-.098) + (.11-.062)*(.03-.098) + (.04-.062)*(.12-.098) + (.05-.062) *(.09-.098)] / 5 = -.000956$

$\sigma_{Mc,M} = \dfrac{COV[Mc,M]}{\sigma_L \sigma_M} = \dfrac{-.000956}{.0264 * .039} = -.912$

e. $COV[M,A] = [(.15-.098)*(.19-.106) + (.10-.098)*(.04-.106)$
$\qquad + (.03-.098)*(-.04-.106) + (.12-.098)*(.21-.106) + (.09-.098)*(.13-.106)] / 5 = .003252$

$\sigma_{M,A} = \dfrac{COV[M,A]}{\sigma_M \sigma_A} = \dfrac{.003252}{.039 * .094} = .872$

15. Assuming variance and correlation stability, the forecasted values would be the same as the historical values in Problem (14).

16.a. Since probabilities must sum to one, the probability must equal .15.

b. First, note that there is a .25 probability that the return will be .05 (.10+.05+.10) and .20 and .55 probabilities that the return will be .15. Thus, the expected return is $.05*.25 + .10*.20 + .15*.55 = .115$. The variance is $.25*(.05-.115)^2 + .20*(.10-.115)^2 + .55*(.15-.115)^2 = .001775$ which implies a standard deviation equal to .04213.

17. Standardize returns by standard deviations and consult "z" tables: $\dfrac{R_i- E[R]}{Standard\ Deviation} = z$. Only use positive values for z.

a. $\dfrac{.05 - .15}{} = z(low) = 1 \qquad \dfrac{.25 - .15}{} = z(high) = 1$

.10   .10

From the "z" table, we see that the probability that the security's return will fall between .05 and .15 is .34. .34   is also the probability that the security's return will fall between .15 and .25. Therefore, the probability that the security's return will fall between .05 and .25 is .68.

b. From (16.a.), we see that the probability is .34.

c. .16

d. .0668

18.   Simply reduce the standard deviations in the z scores in Problem (17) to .05.

a. .95

b. .47

c. .0228

d. .0013

19.   Never because coe4fficient of determination is always a positive squared value.

20.a. var = .0025; std.dev. = .05

b. -.00125

21.

a. VAR = .0025

b. 0 : The coefficient of correlation between returns on any asset and returns on a riskless asset must be zero. Riskless asset returns do not vary.

22.

a.

Date	Company X Return	Company Y Return	Company Z Return
1/09	-	-	-
1/10	0	0	.00207
1/11	.00249	.00625	-.00413
1/12	0	.00621	-.00207
1/13	.00248	.00617	-.00207
1/14	-.00248	0	.00208
1/15	.03980	.04907	.04158
1/16	.00239	-.00584	-.02994
1/17	-.00238	.00588	0
1/18	.00239	.00584	.00205
1/19	.00238	-.00581	0
1/20	-.00238	.00584	0

b., c.

Stock	Average Return	Standard Deviation
X	.004064	.011479
Y	.006693	.014150
Z	.000869	.015537

Table 1:
The z-Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0159	.0199	.0239	.0279	.0319	.0358
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0909	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2356	.2389	.2421	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2793	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3437	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3906	.3925	.3943	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.492	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4986	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Table 2:
The t-Distribution

df	Right-tail area, "				
	0.100	0.050	0.025	0.010	0.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.695	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
4	1.282	1.645	1.960	2.326	2.576

Example: The t value for 4 degrees of freedom that bounds a right-tail area of 0.025 is 1.960.