THE AGGREGATION OF INVESTOR'S DIVERSE JUDGMENTS AND PREFERENCES IN PURELY COMPETITIVE SECURITY MARKETS

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The vector of equilibrium aggregate market values (or per share prices) of a given set of risk assets trading in purely competitive markets of individually risk-averse investors has been derived in earlier work under certain simplifying assumptions, including the absence of taxes and transactions costs and a single (uniform) holding period for the assessment of uncertain outcomes (See [8], [9], [10], [15], and [12]).[1] The other critical assumptions in these studies were (a) the existence of a riskless asset available for holding or borrowing at a fixed, exogenously determined interest rate, (b) an assumption that all investors act in terms of identical joint probability distributions over end-of-period outcomes, and (c) the acceptance of a mean-variance criterion for portfolio decisions.[2]

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[1]For comment on another study falling outside the framework of the present paper, see footnote 3.

[2]Other authors have justified the last assumption, following Markowitz, by implicitly or explicitly assuming that all investor's preference functions were quadratic, but apart from being very restrictive, the form of a utility function has seriously undesirable properties ([8,fn.20],[2],[13],[15]). In my own earlier papers ([8],[9]), on the other hand, portfolio selections and stock prices based on means and variances resulted from the assumption that investor's probability assessments are normally distributed (Gaussian) and the use of Tobin's Separation Theorem [17]. The Separation Theorem asserts that in any given stochastic situation the investor's choice of the optimal portfolio mix of all his investments in risk assets will be independent of the scale of his total investment in risky assets (and hence of the amount of his borrowing or dollar holding of the riskless asset). Hakansson [5] and Leland [7] have shown that this separation property holds (for any probability distribution) whenever investor's utilities are any power or exponential or logarithmic function of wealth (or a variable linear in wealth). In [10], as in this paper, it was convenient to compute direct solutions without recourse to the separation property. See pages 349-351 below.

Under these conditions, each investor maximizes the expected utility of his end-of-period wealth by holding the same portfolio mix of risk assets as every other investor regardless of differences in their respective degrees of risk aversion. Each investor also holds the same fraction of the total outstanding supply of each security -- and this fraction is equal to the ratio of his risk-tolerance (the reciprocal of his risk-aversion coefficient) to the sum of the risk tolerance of all investors in the market. Moreover, if each investor has a given (constant) risk aversion, we can hypothesize a composite market utility function, also with a certain constant risk aversion, and show that the whole market, acting as a single "price taker" on the basis of the agreed probability assessments, will have a demand for each security which is equal to the amounts outstanding if (but only if) the prices for each security are identical to those given by a purely competitive process in equilibrium. The market's wealth is the sum of the investible funds of its component investors, and the market's risk aversion parameter is "the market price of risk," which in turn is equal to the harmonic mean of its several investor's risk aversion parameters divided by the number of investors in the markets. These results -- together with their implications of the level and distribution of aggregate wealth for the market price of risk and hence for the general level of stock prices -- were established in [10]. It was also shown that the conclusions hold both when (a) investors during the process of market equilibration act in terms of fixed assessments of the joint distribution of rates of return independent of current market prices (as suggested by various versions of the "random walk" hypothesis), and (b) when they maintain fixed assessments of the joint distribution of end-of-period prices or values while the market finds its equilibrium vector of current prices (as suggested by the observed behavior of various large institutions with professional management supported by a sizeable staff).

With a riskless asset and identical Gaussian probability distributions throughout the market, the aggregation of individual investor's parameters into market parameters is thus relatively simple and straightforward. This article will derive the appropriate specification of the market's composite parameters and probability assessments when individual investors differ in their probability judgments as well as in their personal risk-aversion

(Section I), when there is no riskless asset available for holding or borrowing (Section II), and when short sales are limited and when most investors are simply ignorant (i.e., have literally no judgments) concerning the prospects of most available securities (Section III). In addition, at each stage, explicit formulae for the Pareto-optimal distribution of the holding of any risk asset among investors will be developed in terms of each investor's parameters and assessments relative to those of the market.

In any study of this kind, the specific results obtained obviously depend upon the particular mathematical assumptions made. Unhappily, it usually turns out that analysis designed to be very general in some respects has to sacrifice structural detail which is crucial to the investigation of other important problems.[3] The fruitful use of the normal distribution in developing the theory of experimental design and stratified sampling provides some precedent for using this distribution to explore the structural issues indicated above. Although variance is not a sufficient or precise measure of risk in general, and the normal is only a loose approximation to empirical distributions of stock prices, rigorous work on the properties of models of markets in which means and variances are sufficient statistics by assumption has surely added greatly to our understanding of important features of security market equilibrium under more general conditions of uncertainty. In the same spirit, we maintain an assumption that all distributions are Gaussian in order to highlight the essential elements of the inherently complex interactions and structural phenomena of immediate interest

[3]A notable instance in point is Fama's recent study of the "Risk, Return and Equilibrium in a Stable-Paretian Market [4]"; but to take advantage of the well known greater generality of these Paretian distributions, Fama had to restrict his analysis to symmetric forms and assume that all investors acted in terms of a distribution with the same characteristic exponent (essentially the shape parameter) and in addition assume that all investors' assessments of risks and return were identical and made in terms of "the market model." (The market model assumes that all estimates of security returns are made from regressions on common market factors. See Sharpe [16], Lintner [9], Fama [3], and Jensen [6].)

in the clearest and simplest possible context.[4] It is correspondingly convenient to assume that each investor acts in terms of his own unique but constant degree of risk aversion in the Pratt-Arrow sense [2, 13], since this implies that all investor's utility functions are negative exponential in form and enable us to index each investor's preferences by a single number -- his measure of risk aversion a_k. The combination of Gaussian distributions with this form of utility function also provides an analysis which is exact for "large" as well as small risks[5] and leads to explicit closed-form solutions.

Important concepts and conclusions are underscored in the text. A companion paper [11] will present the general equilibrium conditions for a securities market in which all assessments of ending prices are lognormal rather than Gaussian, and all investors differ both in their assessments and in their elasticity of utility of real wealth (or level of constant proportional risk aversion). To a perhaps surprisingly close approximation, conclusions regarding the aggregation of individual investor's risk aversions and differing assessments, the distribution of securities among investors, and the equilibrium structure of current prices in these purely lognormal competitive markets for risk assets, will be shown to parallel those derived here.

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[4]For any readers who prefer to avoid our Gaussian assumption by recourse to quadratic utilities, we note that the properties of investor and market equilibrium presented below correspond precisely to those which would have been derived had we assumed instead that all investors act in terms of quadratic utility functions (on the basis of any arbitrary form of probability distributions). See footnote 11.

[5]Limited liability is not introduced in this paper. Our reliance on Gaussian distributions could also, of course, be justified in considerable measure by reliance on the so-called Law of Large Numbers. Indeed, if we merely assume that (a) investor's utility functions all satisfy the condition for the Separation Theorem (Cf. footnote 2) and (b) that all investors act in terms of the same probability distribution of whatever shape, it follows that all investors will have the same portfolio mix of risk assets; consequently if there is a substantial number of different securities available in the market, the Gaussian distribution will be a good approximation to the distribution of portfolio returns when expectations are "homogeneous." But we do not rely on this justification of our assumption of normal distributions because the purpose of this paper is precisely to examine the consequences of different probability assessments; also in Section III below, there would be no assurance that a sufficient number of securities will be held in most portfolios to justify the assumption of Gaussian distributions on these grounds.

We should also note that although our analysis is formally confined to a "one-period" model, the conclusions reached (with one scalar adjustment)[6] are valid with respect to the first period of rather general dynamic settings -- specifically, those in which each investor seeks to maximize the expected utility of his n^{th} period wealth when he expects to revise his portfolio at the end of each interim period *en route* on the basis of some given Markovian set of probability assessments.[7]

SECTION I

Equilibrium in Gaussian Markets With a Riskless Asset

I.1 The Setting

In this section we analyze the structure of equilibrium prices in a purely competitive "single period" securities market in which M investors

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[6]See next footnote.

[7]Hakansson [5] and Leland [7] have shown that in this dynamic setting, the percentage mix of risk assets in any investor's optimal portfolio (conditional on any set of market prices) in the first period will be the same as that derived in a one-period model *regardless of the functional form of his probability assessments* so long as his utility function satisfies the conditions of the *Separation Theorem* (see footnote 2, p. 347). The negative exponential used here, and the quadratic and constant elasticity utility functions are all members of this admissible set, and these conditionally optimal portfolios of individual investors are the crucial building blocks of our entire analysis. Leland has also extended these results to models in which utility is defined over consumption in each period rather than "wealth."

A scalar adjustment is required (when utility functions are negative exponential, as assumed here) to reflect the fact that the *scale* of investment in the first period of the dynamic sequence will not generally be the same as in the one period case (even though the best risk-asset mix is the same). This is readily handled by interpreting each investor's risk aversion parameter a_k in our one period model as being proportional to the investor's true risk aversion; this one element of reinterpretation changes the *size* of the market price of risk, but all the structural analysis carries through as before.

indexed by $k(1 \ldots k \ldots M)$ trade in N different risk assets indexed by i and j $(1 \ldots i \ldots j \ldots N)$. We let n_i and n_j be the fixed number of shares of the i^{th} and j^{th} security outstanding in the market, and n_{ik} and n_{jk} be the number of shares respectively bought and held by the k^{th} investor. Short sales are permitted, but even though some n_{ik} may be negative the condition $\Sigma_k n_{ik} = n_i$ is always satisfied in the market. W_{ok} is the total investible wealth of the k^{th} investor and $\tilde{W}_{ik}$ is the random value of k's end-of-period wealth.

$\tilde{P}_{1i}$ is the random end-of-period price of the i^{th} security, and $\bar{P}_{1ik}$, S_{iik}, and S_{ijk} are respectively its expected value, variance and co-variances (with $\tilde{P}_{ij}$) as assessed by the k^{th} investor. Each investor assesses the $\tilde{P}_{1i}$ to have a joint-normal Gaussian distribution with finite first and second moments. P_{oi} is the current market price per share of the i^{th} stock. In addition to the N risk assets, each investor has a riskless asset available which he can borrow or lend (invest in) in unlimited amounts at a fixed, exogenously determined rate of return (or interest cost of r^*).

Each investor allocates his available funds W_{ok} over the N available risk assets and the riskless asset. His objective is to maximize the expected value of his utility of end-of-period wealth. We assume that each investor treats his Pratt-Arrow [13, 2] measure of risk-aversion $a_k = -U''/U'$ as a constant during the process of portfolio selection. Each investor consequently acts in terms of a negative exponential utility function

$$U^k(\tilde{W}_{1k}) = -\exp(-a_k \tilde{W}_{1k}), \qquad a_k > 0, \tag{1}$$

where the a_k have a different value for each investor in the market, $k = 1 \ldots M$.

I.2 The Individual Investor's Portfolio Equilibrium, Conditional on Any Set of Possible Market Prices

In accordance with accepted theory, each investor chooses that attainable portfolio of risk assets (and that holding of the riskless asset or borrowing) which will maximize his expected utility of end-of-period wealth $E[U^k(\tilde{W}_{1k})]$. But $\tilde{W}_1$ has a normal (Gaussian) distribution, (because $\tilde{W}_{1k}$ is a linear mixture of the random end-of-period prices $\tilde{P}_{1i}$ for each risk asset held), and consequently[8]

$$E[U^k(\tilde{W})_{1k}] = -\exp\{a_k[(\bar{W}_{1k} - a_k \check{W}_{1k})/2]\}. \tag{2}$$

The investor's optimal investment position is consequently the one which maximizes the certainty-equivalent of his end-of-period wealth

$$Q_k = \bar{W}_{1k} - a_k \check{W}_{1k}/2. \tag{3}$$

for any possible vector $\{n_{ik}\}$ of the number of shares held of each security at any possible set of market prices $\{P_{oi}\}$. His budget constraint requires that he simultaneously invest an amount[9] $(W_{ok} - \Sigma_i |n_{ik}| P_{oi})$ in the riskless asset if this entire expression is positive, or borrow this net amount if the expression is negative, and the rate of return on <u>this</u> portion of his overall investment position will be r^*. The expected value of his ending wealth will <u>also</u> include the expected value of his holdings of risk assets, which equals

$$\Sigma_i n_{ik}[\bar{P}_{1i}] = \Sigma_i[n_{ik}\bar{P}_{1ik} - n_{ik}P_{oi}(1+r^*) + |n_{ik}|P_{oi}(1+r^*)].$$

whether or not some n_{ik} are negative.[10] After satisfying the budget

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[8]This expectation is the value of the integral for the moment generating function of the normal distribution. See, for instance [1, p. 37]. Note that with negative exponential utility functions and normal probability distributions, investment decisions depend only on means and variances and the ensuing analysis is exact for "large" as well as small risks.

[9]This expression assumes that 100 percent margin is required against the short sale, but this is nevertheless perfectly general because it merely shifts any borrowing possible against the short sale as such into the total net debt position of the investor.

[10]The formula is clearly correct for all $n_{ik} > 0$. When $|n_{ik}|P_{oi}$ is invested in a <u>short</u> position $(n_{ik} < 0)$, the gross investment equals (a) the proceeds of the sale of the stock which must be placed in escrow, and (b) an amount equal to margin requirements on the sales proceeds (assumed to be 100%) which must be remitted or loaned to the actual owner of the stock borrowed for the short sale, and both components may be assumed to draw interest at a rate r^*. When $n_{ik} < 0$, therefore, the expected gain on the short sale is $|n_{ik}|[2P_{oi}(1+r^*) - \bar{P}_{1i}]$ which reduces exactly to the bracket

constraint, therefore, the investor's expected end-of-period wealth $\bar{W}_{ik}$ will be

$$\bar{W}_{ik} = (W_{ok} - \Sigma_i |n_{ik}| P_{oi})(1+r^*)$$

(4a)
$$+ \Sigma_i [n_{ik}\bar{P}_{1ik} - n_{ik}P_{oi}(1+r^*) + |n_{ik}| P_{oi}(1+r^*)]$$

$$= W_{ok}(1+r^*) + \Sigma_i n_{ik}\bar{P}_{1ik} - \Sigma_i n_{ik} P_{oi}(1+r^*),$$

and the variance of his ending wealth position will clearly be

(4b)
$$\overset{v}{W}_{1k} = \Sigma_i \Sigma_j n_{ik} n_{jk} S_{ijk}$$

whether or not any short sales are included in his portfolio.

The investor's preferred set of holdings will be the one which maximizes (3) after the substitution of (4a) and (4b).[11] By differentiation, we find that, conditional on any possible set of market prices $\{P_{oi}\}$, and his risk aversion a_k and assessments of $\{\bar{P}_{1ik}\}$ and $[S_{ijk}]$, the investor's most preferred (i.e., Pareto-optimal) portfolio will be given by the set of values which simultaneously satisfy[12]

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on the right side of the expression in the text. In situations where the investor does not in fact also receive interest at r^* on the cash he remits to the lender of the stock, use of this formula assumes that the investor has adjusted his estimate of the ending expected price $\bar{P}_{1ik}$ to allow for this circumstance.

[11]Recall that the wealth constraint was substituted into the objective function.

[12]This statement of equilibrium conditions for an individual investor's best conditional risk asset portfolio are all we need for purposes of this section where we assume there is a riskless asset. His additional decision on the optimal scale of his holding of the riskless asset itself is determined by substituting the solution values for n^o_{ik} from (5a) into his wealth constraint. The optimality of this procedure is confirmed below, pages 374 and 375.

We should perhaps also note that if the investor acts in terms of the quadratic utility $U_k = \tilde{W}_{1k} - b_k\tilde{W}^2_{1k}/2$ (with any probability distribution), his optimizing equations will still be given by (5a) after substituting $b_k/(1 - b_k\bar{W}_{1k})$ for a_k. The resulting equations are nonlinear but may be solved by successive approximation since everything is convex.

(5a) $$a_k \Sigma_j n_{jk} S_{ijk} = \bar{P}_{1ik} - (1+r^*)P_{oi} \qquad i,j \ldots N,$$

or

(5b) $$a_k v_{ik} = \bar{P}_{1ik} - (1+r^*)P_{oi} = \bar{X}_{ik} \qquad i,j = 1 \ldots N,$$

where $v_{ik} = \Sigma_j n_{jk} S_{ijk}$ represents the aggregate dollar variance of his entire portfolio which is attributable to each[13] (and therefore to the marginal[14] share of the i^{th} stock. The particular set of market prices which will be found in a purely competitive equilibrium will simply be those subject to which the summation of the holdings of each stock by all M investors together will equal the number available in the market, all of which have to be held by someone -- i.e., the set $\{P_{oi}\}$ satisfying the market clearing conditions.

Certain further properties of each investor's conditional portfolio optimum should be noted before proceeding to the equilibrium of the market as a whole. The righthand side of equation (5b) measures the k^{th} investor's expected net gain (above riskless return r^*P_{oi}) in ending wealth on each share of the i^{th} stock, which we write $\bar{X}_{ik}$. The investor's risk aversion coefficient a_k thus measures the net expected gain he requires per unit of marginal portfolio variance measured in dollars. An investor's equilibrium (relative to any possible set of market prices) requires that his portfolio be adjusted so that the ratio of expected net gain per share to the marginal portfolio variance be the same for all the different securities in his portfolio (and equal to a_k). If we write $\bar{X}_{ik} = \bar{P}_{1ik} - (1+r^*)P_{oi}$, we have

(5c) $$\bar{X}_{ik}/v_{ik} = a_k, \qquad \begin{cases} \text{all } i = 1 \ldots N, \\ \text{all } k = 1 \ldots M. \end{cases}$$

By rearranging (5b), we also see that when faced with any possible set of market prices, the investor will reallocate his funds in the light of his

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[13] Note that $\Sigma_i n_{ik} v_{ik}$ exactly equals $\check{W}_{ik}$ in (4b).

[14] The derivative of $\check{W}_k$ with respect to n_{ik} is exactly v_{ik}.

own risk aversion a_k and assessments $\{\bar{P}_{1ik}\}$ and $[S_{ijk}]$ until the ratio of his risk adjusted expected return per share on any pair of securities equals the ratio of their prices in the market:

$$\text{(5d)} \qquad \frac{\bar{P}_{1ik} - P_{oi} - a_k v_{ik}}{\bar{P}_{1jk} - P_{oj} - a_k v_{jk}} = \frac{P_{oi}}{P_{oj}}, \qquad \begin{cases} \text{all } i, j = 1 \dots N \\ \text{all } k = 1 \dots M. \end{cases}$$

Moreover, since this is a condition for the portfolio equilibrium of each investor and all investors face the same prices $\{P_{oi}\}$ in the market, this ratio of expected risk adjusted returns per share between any given pair of stocks must be the same for all investors in the market -- just as the marginal rate of substitution between any pair of commodities is the same for all consumers in a perfectly competitive goods market[15] -- and this will be true regardless of all differences in the various investor's assessments of expected prices and/or risks, and of differences in their risk aversion parameters a_k. Note that this is a result of each investor's own optimal adjustment to any set of market prices, and does not depend on market prices themselves being in equilibrium.

Finally, if we rearrange (5b) in a similar way for any i^{th} security and divide by P_{oi} to move from a per share to a per dollar basis, we see from

$$\text{(5e)} \qquad [\bar{P}_{1ik} - P_{oi} - a_k v_{ik}]/P_{oi} = r^*, \qquad \begin{cases} \text{all } i = 1 \dots N, \\ \text{all } k = 1 \dots M. \end{cases}$$

that the investor will reallocate his funds in the face of any given set of market prices $\{P_{oi}\}$, until the risk-adjusted expected rate of return per dollar of investment is (a) the same on every security in his portfolio, and (b) equal to the riskless rate r^* (since at the margin, funds are added to or withdrawn from the riskless asset or debt). Like the preceding

[15]Since expected utility is a positive linear function of certainty-equivalent, the correspondence with consumer theory is exact because the numerator and denominator on the left of (5d) are precisely the marginal certainty-equivalents of wealth with respect to variations in n_{ik} and n_{jk} respectively.

observation, this property of each investor's individual investment equilibrium holds regardless of differences in assessments or risk aversion even when markets as a whole are not in equilibrium.

I.3 Portfolio Selection by the Composite Market, Regarded as a Single "Price-Taking" Investor

Now suppose that the market as a whole invests as a single composite "price-taker" on the basis of its own Gaussian assessments $\{P_{1im}\}$ and $[S_{ijm}]$ in terms of a market utility function having the same form as equation (1) with its own index m, and specifically with its own constant risk-aversion parameter $a_m > 0$. The total amount of funds which the market as a whole must invest in the available set of risk assets or in the riskless asset is, of course, the sum of such funds in the hands of its component investors, $W_{om} = \Sigma_k W_{ok}$. Like any individual investor, it would seek to maximize (3) after the substitution of (4a) and (4b), with the result that its demand n_{im} for any security conditional on any possible vector of current prices $\{P_{oi}\}$ will be the given i^{th} term in the set of values $\{n^o_{im}\}$ which simultaneously satisfy

$$\text{(5m)} \qquad a_m \Sigma_j n_{ijm} = \bar{P}_{1im} - (1+r^*)P_{oi} \qquad i, j = 1 \ldots N$$

or

$$\text{(5m)(i)} \qquad a_m v_{im} = \bar{P}_{1im} - (1+r^*)P_{oi} = \bar{X}_{im}$$

where v_{im} now represents the aggregate dollar variance attributable to each (and therefore the marginal) share of the i^{th} stock for the market as a whole. Since $\bar{X}_{im}$ is the net expected gain over the riskless return per share of the i^{th} stock in the markets' aggregate portfolio, its risk-aversion parameter a_m measures the net expected gain required by the market per dollar of marginal market portfolio variance. But the marginal rate of substitution between net expected gain and variance in the market as a whole is also precisely the market price of risk γ and we can hereafter use $a_m = \gamma$ interchangeably as the context indicates.

All the properties of the individual investor's equilibrium noted in I.2 above clearly apply to the market's demand functions (5m) and (5m'), even when the market's demands $\{n_{im}\}$ conditional on $\{P_{oi}\}$, do not "clear the market" in the sense that all n_{im}^o do not equal the outstanding supplies n_i^o. In order to exhibit explicitly the vector of market prices for which all $n_{im} = n_i^o$, it is convenient to let Z_m represent the matrix $[S_{ijm}]$ and let $\bar{\underline{P}}_{im}$, P_o and $\underline{n}_m$ be column vectors representing the obviously corresponding elements in (5m), which may be written

(5m)(ii) $$a_m Z_m \underline{n}_m = \bar{\underline{P}}_{1m} - (1+r^*)\underline{P}_o = \bar{\underline{X}}_m.$$

The market's demand as a price taken conditional on any vector of current prices is thus

(5m') $$\underline{n}_m^o = a_m^{-1} Z_m^{-1} [\underline{P}_{1m} - (1+r^*)\underline{P}_o] = a_m^{-1} Z_m^{-1} \bar{\underline{X}}_m.$$

Simple rearrangement then shows that the only vector of possible prices $\underline{P}_o$ which will satisfy the condition that the market's composite demand $\underline{n}_m^o$ equals the number of shares outstanding $\underline{n}^o$ is given by

(5m")(i) $$(1+r^*)\underline{P}_o = \bar{\underline{P}}_{im} - a_m Z_m \underline{n}^o, \qquad a_m = \gamma.$$

Alternatively, if we let $\underline{v}_m^o = Z_m \underline{n}^o$ represent the vector $\{v_{im}\}$, these equilibrium conditions can be written

(5m")(ii) $$(1+r^*)\underline{P}_o = \bar{\underline{P}}_{1m} - a_m \underline{v}_m^o.$$

Specifically for the i^{th} stock, we have

(5m")(iii) $$(1+r^*)P_{oi} = \bar{P}_{1im} - a_m v_{im}^o$$

where v_{im}^o represents the market's assessment of the contribution of each share (and also the contribution of the marginal share) of the i^{th} stock to the variance of the whole market's portfolio when that portfolio consists of the entire outstanding supplies $\{n_i^o\} = \underline{n}^o$ of all stocks.

I.4 Aggregation of Assessments and Preferences in Securities Market Equilibrium

I.4.1 Properties Based on Fractional Holdings of Outstanding Supplies of Securities

The principal purpose of this whole paper is to derive the explicit relations which must hold between the market's assessments $\{P_{1im}\}$ and $\{S_{ijm}\}$ and its risk aversion a_m (or equivalently, the market price of risk γ) on the one hand, and the corresponding assessments $\{\bar{P}_{1ik}\}$ and $\{S_{ijk}\}$ and risk aversion a_k of each of its M individual investors on the other, when (a) each investor is in personal equilibrium and (b) the market clearing conditions are satisfied. One important property of such a Pareto-optimal market equilibrium follows very directly from the individual investor's equilibrium if we impose the market clearing condition in the following way. Let $f_{ik} = n_{ik}^o/n_i^o$ represent the fraction of the outstanding supply n_i^o of each i^{th} security which the k^{th} investor holds when he is in personal equilibrium.[16] The market clearing equations will be satisfied if $\Sigma_k f_{ik} = 1$, which we assume since we want to characterize market equilibrium prices. After we multiply all terms in the investor's equilibrium equations (5b) by f_{ik} and sum over all investors, we have

$$(1+r^*)P_{oi} = \Sigma_k f_{ik}\bar{P}_{1ik} - \Sigma_k f_{ik} a_k v_{ik}. \tag{6}$$

When we compare this equation with (5m") and equate corresponding terms, we see that in equilibrium both the product of the market's risk aversion parameter and its variance assessment, and the market's assessment of expected end-of-period prices, can validly be interpreted as the same weighted average of the corresponding elements for its component investors, with the investor's proportionate holdings of the i^{th} stock itself serving as

[16] Specifically we assume that equations (5b) are solved for the optimal conditional holdings n_{ik}^o of each stock, based on his own assessments, and that these values are then substituted back in (5b) before the latter is summed over investors.

weights. In interpreting this property it should be emphasized that the v_{ik} measures the contribution of each (and also the marginal share) of the i^{th} stock to the aggregate variance of the investor's entire portfolio, and that the corresponding marginal aggregate portfolio variance for the market as a whole is not a simple sum or weighted average of investor's marginal risk assessments. Only the products of variance assessments and their risk aversion parameters can be combined in a simple weighted average to equal the corresponding product for the market as a whole.

From our market equilibrium conditions (6), we can derive further properties of the aggregation of subjectively assessed risks and riskbearing by multiplying through by the aggregate number of shares of the i^{th} stock outstanding n_i^o, which gives

$$(7) \qquad (1+r^*)V_{oi} = \Sigma_k n_{ik}^o \bar{P}_{1ik} - \Sigma_k a_k n_{ik}^o v_{ik}$$

since $n_i^o P_{oi} = V_{oi}$ the aggregate market value of the i^{th} stock. For comparison, similarly multiplying the equilibrium equations (5m") based on the market's own assessments through by n_i^o gives[17]

$$(7m) \qquad (1+r^*)V_{oi} = \bar{V}_{1im} - a_m n_i^o v_{im},$$

where $\bar{V}_{1im} = n_i^o \bar{P}_{1im}$. The market's composite assessment of the end-of-period aggregate value of the i^{th} stock is thus the weighted sum of each investor's judgment of expected ending price per share weighted by the number of shares he holds. Apart from riskless discounting, the current aggregate value V_{oi} also depends on an aggregate "risk discount" $a_m n_i^o v_{im}$ which equals the sum of each individual investor's own (subjective and diverse) estimates of per share variance after weighting by both his risk aversion a_k and the number of shares he holds n_{ik}. Now the product $n_{ik} v_{ik}$ equals the dollar

[17] It should be noted that this is precisely the same equation developed for the aggregate market value of the i^{th} security in [8, pp. 26-37], since v_{im} is defined as the i^{th} row sum of the aggregate dollar variance-covariance matrix $[R_{ij}]$ in the notation of the earlier paper.

variance of the investor's total portfolio contributed by the i^{th} stock,[18] and the product $n_i^o v_{im}$ has the same meaning for the markets' portfolio of all i^{th} stock outstanding. We may, consequently, also say that the market's estimate of aggregate dollar risk attributable to the i^{th} stock, weighted by its risk aversion a_m (or the market price of risk γ), equals the sum (using individual risk aversions a_k as weights) of all individual investor's assessments of the risks they are individually bearing from their holdings of the i^{th} stock. But as I have shown elsewhere,[19] $n_i^o v_{im} >> \Sigma_k n_{ik} v_{ik}$, because a large purely competitive securities market is not only a very effective risk-sharing mechanism: it is also (and consequently) a very effective risk-eliminating mechanism.

Equations (6) and (7) bring out significant balances which must be satisfied by a purely competitive securities market in equilibrium when expectations and risk assessments differ among investors. But it will be observed that in both equations, the number of shares $\{n_{ik}\}$ an investor holds forms one element in the weight given to his assessments in forming the market composite. While this is perfectly legitimate in characterizing equilibrium properties of the market, it is clear that each n_{ik} is itself dependent on what the market price may be and we consequently need to derive an explicit solution in which only the "givens" of the problem appear on the righthand side of any equation.

I.4.2 Aggregations Based on Explicit Solutions for Market-Clearing Prices

If we let Z_k represent the matrices $[S_{ijk}]$ and $\bar{\underline{P}}_{1k}$, $\underline{P}_o$ and $\underline{n}_k$ be column vectors representing the obviously corresponding elements in (5a), the latter equation for the investor's conditional personal portfolio equilibrium may be written

$$a_k Z_k \underline{n}_k = \underline{P}_{1k} - (1+r^*)\underline{P}_o = \bar{X}_{ik} \qquad (5a')$$

[18] The dollar variance on an investor's entire portfolio is

$$\Sigma_i \Sigma_j n_{ik} n_{jk} S_{ijk} = \Sigma_i n_{ik} \Sigma n_{jk} S_{ijk} = \Sigma_i n_{ik} v_{ik}.$$

[19] See [10], especially the text following Theorem I.

which may be solved to yield his demand functions conditional on market prices

$$(8) \qquad \underline{n}_k = a_k^{-1} Z_k^{-1} [\bar{\underline{P}}_{1k} - (1+r^*)\bar{\underline{P}}_o] = a_k^{-1} Z_k^{-1} \bar{\underline{X}}_k.$$

The market clearing condition in turn requires that the sum of the shares of each stock voluntarily held be equal to the total available. This equilibrium condition gives us

$$(9) \qquad \underline{n}^o = \Sigma_k \underline{n}_k = \Sigma_k a_k^{-1} Z_k^{-1} [\bar{\underline{P}}_{1k} - (1+r^*)\underline{P}_o] = \Sigma_k a_k^{-1} Z_k^{-1} \bar{\underline{X}}_k$$

which may be solved for the equilibrium vector of market prices:

$$(10) \qquad (1+r^*)\underline{P}_o = \left(\Sigma_k a_k^{-1} Z_k^{-1} \right)^{-1} \Sigma_k a_k^{-1} Z_k^{-1} \bar{\underline{P}}_{1k} - \left(\Sigma_k a_k^{-1} Z_k^{-1} \right)^{-1} \underline{n}^o.$$

It will be noted that (apart from numbers of shares outstanding) only the set of risk aversion parameters and the probability assessments of all stocks by all investors appear on the righthand side of this equation, and these are the fundamental determinants of market values in this model. (The number of shares $\underline{n}^o$ has to enter explicitly into this equation for prices _per share_ because _ceteris paribus_ the dollar variance contributed to an investor's portfolio by a given holding of any company's stock must be independent of stock splits.)[20]

Once again in equation (10), as in (6), and (7), current prices are weighted averages of expected ending prices less a risk adjustment. The average of future expectations is not only different but much more complex than those in previous equations, and _the weighted average of investors' respective assessments of expected end-of-period prices differs from the weighting in the market's average of their risk aversions and variance assessments_. Specifically, the first term on the right side of (10)

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[20]See footnote 18. Since $\Sigma_j n_{ik} n_{jk} S_{ijk}$ is invariant to stock splits, all assessments of the variances and covariances involving the i^{th} stock (consequently all elements of the i^{th} row of the inverse) are adjusted in inverse proportion to the ratio of new shares to old when a split occurs.

equates with the market's price assessment $\bar{P}_{1m}$ in (5m"), and the "risk adjustment matrix" $(\Sigma_k a_k^{-1} Z_k^{-1})^{-1}$ in (10) equates with the product of the market's risk-aversion and covariance matrix $a_m Z_n$ (5m").

When dependencies and equivalences get this complex, it is desirable to bring out the underlying logic of the structure by considering some intermediate special cases whose interpretation is a little more transparent. These special cases also enable us to trace through the different elements which determine the k^{th} investor's "sharing ratio" f_{ik} -- his fractional holdings of the i^{th} stock expressed as a fraction of the total outstanding -- under different simplifying assumptions.

I.4.2(i) Homogeneous Expectations of P's and Z's

When all $\bar{P}_{ik}$ and Z_k are identical, the market's assessments will be the same as each investor's. In this case (10) reduces[21] to

$$(1+r^*)\underline{P}_o = \bar{\underline{P}}_1 - (\Sigma_k a_k^{-1})^{-1} Z\underline{n}^o. \tag{10i}$$

We established earlier that the market price of risk $\gamma = a_m$, the market's risk aversion. We now see that the coefficients of $Z\underline{n}^o$ in (10i) and (5m") must be equal, so we have

$$\gamma = a_m = (\Sigma_k a_k^{-1})^{-1} = H_a/M. \tag{11}$$

The market's risk aversion a_m (and the market price of risk γ) are both equal to the harmonic mean of its constituent investor's risk aversions divided by the number of investors; equivalently, the market's "risk tolerance" (a_m^{-1}) is equal to the sum of the risk tolerances of the set of investors in the market.[22] We will see later that this identification of the market

[21] This is seen most simply by dropping the subscripts on Z and $\underline{P}_1$ in (5a'); then multiply both sides by a_k^{-1} and sum over all investors, noting that $\Sigma_k \underline{n}_k = \underline{n}^o$ when markets clear.

[22] Raiffa [14] and Wilson [18] have shown that this relation of risk tolerances holds in small group bargaining situations when allocations are Pareto-optimal. In [10], I have also shown that the above theorem is valid for purely competitive markets in which investors hold their assessments of

price of risk in terms of investor's risk aversions continues to be valid in spite of divergent expectations and when there is no risk asset.[23]

To determine investor's "sharing ratio" for the i^{th} security in this special case, we divide each side of the corresponding rows of (8) by (5m'), and find that

$$f_{ik} = n^o_{ik}/n^o_i = a_k^{-1}/\Sigma_k a_k^{-1} = w_{ak}. \tag{12}$$

Since w_{ak} is independent of the parameters of any stock, <u>when there is a riskless asset and all investors use the same probability distributions, the k^{th} investor holds the same fraction of all outstanding shares of each stock and this fraction equals the ratio w_{ak} of his risk tolerance to the summation of the risk tolerance of all investors in the market</u>.

It should also be noted that even with common expectations $\bar{P}_1$ and Z through the market -- and in particular with all per-share covariance matrices $Z_k = Z_m = Z$ <u>identical</u> -- it does <u>not</u> follow that the marginal (or per share) risk attributable to the i^{th} share of stock in the market's inclusive portfolio will be the same as (<u>indeed, it must be very much less than</u>) the marginal (or per share) risk of the same stock in the portfolio of any individual investor. The market's assessment of the contribution of <u>each share</u> of the i^{th} stock to the variance of the whole "market" portfolio is $v^o_{im} = \Sigma_j n^o_j S_{ij}$, the inner product of the number of shares of all stocks with the i^{th} row of the (common) covariance matrix Z, while the corresponding marginal or per-share risk contribution of the i^{th} stock to the individual investor's portfolio is $v_{ik} = \Sigma_j n_{jk} S_{ij}$, an inner product involving <u>only the number of shares he holds</u>. Clearly, $v^o_{im} > v^o_{ik}$ because $n^o_i > n^o_{ik}$ for all i.

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distributions of rates of return during the process of market equilibration, rather than their assessments of ending prices (as assumed here).

Note also that pre-multiplication of either (10i) or (5m") by $\underline{n}$ exactly reproduces the equation for the aggregate market values $V_{oi} = n^o_i P_{oi}$ of risk assets originally given in [8, pp. 26-27] under the assumption of homogeneous expectations.

[23]But restrictions on short selling and ignorance require a potentially sizeable but quite straightforward adjustment. See page 389 below.

I.4.2(ii) Homogeneous Z's but Different $\bar{P}_{1k}$

When all $Z_k = Z_m$, but each investor has a different vector of expected ending prices $\bar{P}_{1k}$, (10) reduces to

$$(10ii) \qquad (1+r^*)\underline{P}_o = \Sigma_k w_{ak}\bar{\underline{P}}_{1k} - \gamma Z\underline{n}, \qquad \gamma = a_m.$$

The market risk discount per share $\gamma Z\underline{n}$ is unchanged by the differences in price expectations. When all investors assess the same covariance matrix Z_k even though their price expectations differ, the market's assessment of expected future prices $\bar{\underline{P}}_{1m} = \Sigma_k w_{ak}\bar{\underline{P}}_{1k}$ says that $\bar{P}_{1im} = \Sigma_k w_{ak}\bar{P}_{1ik}$ for each separate i^{th} stock. Common covariance matrices thus make the market's future price assessment a simple weighted average of the individual investor's assessments for the given stock alone, with the weights being their respective fractions of the market's total risk tolerance. But while this average is a simple one, particular emphasis should be given to the fact that it is an average of the assessments of all investors. Even in this simplified situation, it is not true that market price depends only on the assessment (or risk aversion) of the so-called "marginal" investor. The assessments and risk preferences of all investors are explicitly involved.

Now consider the k^{th} investor's fractional holding of the i^{th} stock in this still simplified situation. With all $Z_k = Z_m$, we see from corresponding rows of (8) and (5m') that

$$(12ii) \qquad f_{ik} = n_{ik}/n_i^o = \Sigma a_k^{-1}\bar{X}_{ik}/\Sigma_k a_k^{-1}\bar{X}_{ik} = w_{ak}\bar{X}_{ik}/\bar{X}_{im}.$$

When expected future price assessments differ, the k^{th} investor's fractional holding of all outstanding i^{th} company shares in equilibrium depends on the ratio of the product of his risk tolerance and expected excess return on the stock to the corresponding product summed over all investors in the market. Alternatively,[24] his fractional holding is equal to the

[24] In (10ii) (b) we have $a_k^{-1}\bar{X}_{ik}/\Sigma_k a_k^{-1}\bar{X}_{ik} = w_{ak}\bar{X}_{ik}/\Sigma_k w_{ak}\bar{X}_{ik}$, after dividing numerator and denominator by $\Sigma_k a_k^{-1}$. Also, $\Sigma_k w_{ak}\bar{X}_{ik} = \bar{X}_{im}$ since $\Sigma_k w_{ak}\bar{P}_{ik} = \bar{P}_{im}$.

product of (a) the ratio of his risk tolerance to that of the entire market and (b) the ratio of his assessment of expected dollar return per share to that of the entire market.

Note that $f_{ik} \neq f_{jk}$: differences in expected future prices among investors are sufficient to prevent any investor from holding the same fraction of the outstanding supply of each of the different stocks in his portfolio. Moreover, when investors hold different judgments of future prices they will hold different optimal portfolios, as would be expected.

I.4.2(iii) Homogeneous $\bar{P}_{1k}$ but Different Z_k

When all $\bar{P}_{1k} = \bar{P}_{1m} = \bar{P}_1$ but each investor has a different assessment of the relevant covariance matrix, (10) reduces to

$$\text{(10iii)}\quad (1+r^*)P_o = \bar{P}_{1m} - (\Sigma_k a_k^{-1} Z_k^{-1})^{-1} n^o = \bar{P}_{1m} - \gamma(\Sigma_k w_{ak} Z_k^{-1})^{-1} n^o,$$

since $a_k^{-1}/\Sigma_k a_k^{-1} = w_{ak}$ and $\gamma = a_m = (\Sigma_k a_k^{-1})^{-1}$. When we compare this equation with (10i), it is thus apparent that

$$\text{(13)}\qquad Z_m = (\Sigma_k w_{ak} Z_k^{-1})^{-1},$$

i.e., that the market's covariance matrix is equal to the inverse of a weighted average of the inverses of the covariance matrices of its constituent investors, where each investor's inverse risk assessment Z_k^{-1} receives a weight equal to the ratio of his risk tolerance to the total risk tolerance of the market. Moreover, the market's assessment of the variance v_{im} contributed by each share of the i^{th} stock outstanding to the market's whole portfolio is still equal to the inner product over all stocks of the i^{th} row of Z_m with the number of shares of all stock outstanding -- but the market's complex weighted average variance-covariance assessments Z_m must be substituted for the simple common assessment Z which was previously adequate.

As would be expected from the previous case, differences among investors in their variance assessments Z_k also lead to different optimal portfolio mixes across investors, and lead every investor to hold different fractions of the outstanding supply of different stocks. Specifically, with all $\bar{P}_{1k} = \bar{P}_{1m}$ dividing the i^{th} scalar product of (8) by the corresponding element in (9) yields

$$(12iii) \qquad f_{ik} = n^o_{ik}/n^o_i = a_k^{-1} e_i' Z_k^{-1} / \Sigma_k a_k e_i' Z_k^{-1} = w_{ak} e_i' Z_k^{-1} / e_i' Z_m^{-1}$$

where e_i is the i^{th} unit vector. Each investor holds a fraction of all the i^{th} stock outstanding equal to the ratio of the weighted i^{th} row-sum of the inverse of his covariance matrix to the corresponding row-sum of the inverse of the market's composite covariance matrix (which equals the sum of this row-sum over all investors), with the same weights w_{ak} as before.

I.4.2(iv) Both $\bar{P}_{1k}$ and Z_k Differ among Investors

Equation (10) gave the equilibrium price solution for this case. Comparing (10iii) with our original (10) shows that the second terms on the right, and hence the market's risk discount, are the same as in the previous case. Consequently, differences in price assessments among investors do not change the market's assessment of risks per share nor the discount it imposes below expected values to compensate for these risks in this model when there is a riskless asset. But comparisons of the first terms on the right side of (10ii) and the original (10) shows that differences in risk assessments Z_k do change the market's expected ending price assessments.

Equating the first terms on the right side of (10) and (5m") shows that when covariance matrices Z_k differ among investors, we have

$$(14a) \qquad \bar{P}_{1m} = (\Sigma_k a_k^{-1} Z_k^{-1})^{-1} \cdot \Sigma_k a_k^{-1} Z_k^{-1} \bar{P}_{1k}$$

$$(14b) \qquad = (\Sigma_k w_{ak} Z_k^{-1})^{-1} \cdot \Sigma_k w_{ak} Z_k^{-1} \bar{P}_{1k}$$

$$(14c) \qquad = Z_m \Sigma_k w_{ak} Z_k^{-1} \bar{P}_{1k},$$

after using (13), or

$$(14d) \qquad Z_m^{-1} \bar{P}_{1m} = \Sigma_k w_{ak} Z_k^{-1} \bar{P}_{1k}.$$

Equations (14a) and (14b) show that the vector of the market's assessments of expected ending prices $\bar{P}$ is a complex weighted average of products of each investor's "precision matrix" Z_k with the entire vector of his assessments of ending prices $\bar{P}$. Alternatively, equations (14d) shows that the market's vector of expected price assessments weighted by the market's precision matrix is a simple w_{ak}-weighted average of all the individual investor's correspondingly precision-weighted vector-assessments of expected ending prices. (Note that all the individual investor's precision-weighted vector assessments $Z_k^{-1}\bar{P}_{1k}$ bear the same relation to the market's composite $Z_m^{-1}\bar{P}_{1m}$ in equation (14d) as their respective assessments of individual prices $\bar{P}_{1ik}$ bore to the market's $\bar{P}_{1im}$ in (10ii) when all covariance matrices were identical.)

It is apparent from a comparison of (10ii) with (14a) or (14b), that any differences in investor's covariance matrices Z_k will alter the market's assessments of expected ending prices in two significant ways: First, when investor's covariance matrices are the same, a simple weighted average of the various investor's assessments of expected ending values of the individual stock sufficed for any stock. But when investors assess different covariance matrices the market's expectation of ending price for any single i^{th} stock is a compound weighted average of the variously assessed ending price vectors of all investors for all stocks. Second, the weights in the market's ending price expectations were determined simply by investor's risk tolerances when all Z_k are the same. But when investor's covariance assessments differ, the required weights explicitly include all the elements of the i^{th} row of the inverse of every investor's variance-covariance matrix as well as every investor's risk tolerances a_k^{-1}.[25]

The basic reason why differences in covariance matrices necessarily affect the market's ending price assessments is most clearly indicated by briefly considering the following simplified case: Let each investor act

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[25]Equation (14c) shows that for any i^{th} stock, the market's assessment of expected ending price $\bar{P}_{1im}$ is equal to the sum of the inner products of (a) the elements in the i^{th} row of the market's composite variance-covariance matrix Z_m, with (b) the w_{ak}-weighted sum over all investors of the inner products of the i^{th} row of the inverse of their respective Z_k with the entire vector of their respective assessments of expected ending prices on all stocks.

in terms of a diagonal matrix $[\sigma_{ijk}]$ with variance terms $\sigma_{iik} > 0$, but all $\sigma_{ijk} = 0$ for all $i \neq j$ and all k. In these circumstances, we would have the market's variances $\sigma_{iim}^{-1} = \Sigma_k w_{ak} \sigma_{iik}^{-1}$ from (13), and its price assessments would have the simpler form

$$\bar{P}_{1im} = (\Sigma_k a_k^{-1} \sigma_{iik}^{-1})^{-1} \Sigma_k a_k^{-1} \sigma_{iik}^{-1} \bar{P}_{1ik}, \tag{14i}$$

$$= \sigma_{iim} \Sigma_k w_{ak} \sigma_{iik}^{-1} \bar{P}_{1ik} \tag{14ii}$$

or

$$\bar{P}_{1im}/\sigma_{iim} = \Sigma_k w_{ak} (\bar{P}_{1ik}/\sigma_{iik}). \tag{14iii}$$

When all covariances are zero, the market's assessment of any single ending price $\bar{P}_{1im}$ depends only on investor's assessments with respect to the single stock; but when the variances of the different investor's assessments differ, it is necessary to weight each of these assessments by their "reliability" (i.e., weight them inversely to their respective variances) before combining them into the market's composite assessment.[26]

Similarly, when all the covariances σ_{ijk} are not zero and each investor assesses a full covariance matrix Z_k, assessments of the expected ending prices of other stocks $\bar{P}_{1jk}$ must enter explicitly into the composite market assessment of any given stock's expected ending price $\bar{P}_{1im}$ for essentially the same reason that covariances themselves are fundamental determinants of optimal portfolios for any separate investor. (Recall from equation (6) above that one of the valid weighting schemes in general market equilibrium gives each investor's assessments a weight in the market's assessment equal to the fraction $f_{ik} = n_{ik}^o/n_i^o$ of the outstanding shares of any given i^{th} security which he holds -- and any investor's best value of $\underline{n}_{ik}^o$ fundamentally depends on the entire vector of his assessments of $\underline{\bar{P}}_{1k}$ on all stocks as well as on the covariances he assesses between stocks.)

[26]Note that in subsection I.4.2(ii) above the w_{ak} were the only weights involved in the formula for $\underline{\bar{P}}_{1m}$. With identical $Z_k = Z_m = Z$ the "precisions" of the estimates of all investors cancelled out because they were all the same; and for this same reason the separate estimates of $\bar{P}_{1ik}$ for each stock could be used directly in forming market estimates.

A covariance matrix can properly be regarded as the variance of a vector and its inverse is the precision of a vector. The market effectively makes a simultaneous composite assessment of the vector of expected ending prices $\bar{\underline{P}}_{1m}$ by combining every investor's vector of assessments $\bar{\underline{P}}_{1k}$ after weighing each investor's assessments by its precision.

The reader will recall that good statistical practice always requires that the evidence of each "cell" in any segmented or stratified sampling design be weighted inversely to its variance (i.e., by its "precision") in constructing the global estimate of the statistic of interest. Each investor's assessments of expected endings prices $\bar{\underline{P}}_{1k}$ and the covariance matrix Z_k represent his best judgment of the mean and variance of the random vector $\tilde{\underline{P}}_1$ whose actual value will only be known one period later. Although every investor seeks to make his assessments of the mean and variance of this vector equal to the "true" values in some sense of the "actual" distribution of $\tilde{\underline{P}}_1$, every investor's (necessarily subjective) assessments of $\bar{\underline{P}}_{1k}$ and Z_k in fact differ from every other investor's assessments. We now see that purely competitive security markets, in effect, form their composite assessments $\bar{\underline{P}}_{1m}$ of the expectations of the random vector $\tilde{\underline{P}}_1$ by using a stratified sample of investor's judgments in which each investor is included and is treated as a separate stratum. Moreover, a purely competitive security market necessarily and automatically uses the ideal weights[27] of sampling theory in combining the separate vector assessments $\bar{\underline{P}}_{1k}$ of each stratum (investor) into its composite market estimate $\bar{\underline{P}}_{1m}$, as shown by equation (14c). The rest of our analysis then shows that these "best composite market estimates" $\bar{\underline{P}}_{1m}$, together with the market's composite risk aversion a_m and its composite covariance assessment Z_m -- which is also a weighted average of the Z_k by equation (13) -- determine current market prices $\bar{\underline{P}}_{io}$ by way of equation (5m").

Even in the simple case with common covariance assessments, we saw that the market's assessment of an expected ending price depended on the risk tolerances and price assessments (of the one stock) of all investors,

[27]The variance of the "cell mean" in our case, where we only have a sample of one observation (assessment) from the investor's distribution is, of course, the same as the variance of the "parent population" (the investor's own distribution).

not just the marginal investor. (This incidentally is $M(N+1)$ items of data.)[28] We now see that in the (undoubtedly more realistic) case with different assessments of covariance matrices, the market's assessment of the expected ending price $\bar{P}_{1im}$ for any security depends on every investor's assessment of the expected ending price for every security and every element in every investor's assessment of his NxN covariance matrix Z_k, as well as the risk tolerance of every investor -- a total[29] of $M(1+N)(N+2)/2$ elements. If 100 stocks are traded by 100 investors, the market's assessment of $\bar{P}_{1im}$ for any one stock would be an explicit function of 515,100 assessments and preferences.[30] And since current market price P_{oi} monotonically varies with the market's assessment of expected ending price $\bar{P}_{1im}$, the market price of any security will in principle change as a result of any change in any one of these assessments or preferences (degree of risk aversion), more than $MN^2/2$ in number.

Any carryover of earlier Marshallian or Ricardian notions of "marginal" buyers setting prices[31] in purely competitive markets is utterly unjustified and misleading when dealing with security markets under uncertainty. Every investor is a marginal holder with respect to his last share (or his last dime's worth) of each security he holds. Even more to the point, every current price $\underline{P}_o$ is exactly equal to the value given by a single composite price-taker's demand functions when this single price taker ("the market")

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[28]Each of the M investors has a risk aversion and each makes an assessment of $\bar{P}_{1ik}$ for each of N stocks.

[29]This number is larger than the previous by the $N(N+1)/2$ variance and covariance assessments of each of the M investors.

[30]Correspondingly, in this model any one $\bar{P}_{1im}$ in a market of 25 million investors trading 1,600 stocks would reflect more than 3.10^{13} assessments! Even if a single factor diagonalization of the Z_k were adopted following Markowitz and Sharpe (but each investor uses his own estimates of all parameters), the number in the text is reduced to $3M(N+1)$, which in the case just cited is still 120 billion. (Each of M investors estimates the mean and variance of the index and has his risk aversion; and for each of N stocks he estimates a constant and slope and residual variance from the regression in which $\bar{P}_{1i}$ is the dependent variable).

[31]These dogmas of marginal buyers setting prices have appeared frequently in the literature on corporation finance and related papers on stock prices. But it should be recognized that such ideas represented a misreading of Marshall and Ricardo who explicitly argued that in competitive markets the "intensive" and "extensive" margins were equalized; both represented optimizing adjustments in the same equilibrating process and neither determined the other.

is required to hold neither more nor less than the number of shares $\underline{n}^o$ of each security outstanding. This aggregated or composite price-taker's demand functions depend on its composite assessments of ending prices $\bar{\underline{P}}_{1m}$, and its covariance matrix Z_m and its risk aversion a_m. Its risk aversion is the inverse of a harmonic sum of <u>every</u> investor's risk aversion; and <u>both of its assessments are complex weighted averages of all investor's assessments</u>.

This more general case also extends previous results regarding the fraction of any company's stock which will be held by any investor. We have seen that differences in either $\bar{P}_{1ik}$ or in Z_k lead to different optimal portfolio mixes among investors, and either type of diversity in assessments among investors means that <u>any</u> investor will hold a different fraction of the total market supply of the different stocks which are in his portfolio. Both effects are compounded when neither the assessments of $\bar{\underline{P}}_{1k}$ or of Z_k are the same for all investors. In this situation, dividing the i^{th} scalar of (8) by the corresponding element in (5m') gives us

$$\text{(12iv)(a)} \qquad f_{ik} = n_{ik}/n_i = a_k^{-1}\underline{e}_i'Z_k^{-1}\bar{\underline{X}}_k/\Sigma_k a_k^{-1}\underline{e}_i'Z_k^{-1}\bar{\underline{X}}_k$$

$$= w_{ak}\underline{e}_i'Z_k^{-1}\bar{\underline{X}}_k/\Sigma_k w_{ak}\underline{e}_i'Z_k^{-1}\bar{\underline{X}}_k$$

$$\text{(12iv)(b)} \qquad = w_{ak}\underline{e}_i'Z_k^{-1}\bar{\underline{X}}_k/\underline{e}_i'Z_m^{-1}\bar{\underline{X}}_m$$

since from (10) and (13) we have

$$\text{(12iv)(c)} \qquad \bar{\underline{X}}_m = Z_m\Sigma_k w_{ak}Z_k^{-1}\bar{\underline{P}}_{1k} - (1+r^*)P_{oi} = Z_m\Sigma_k w_{ak}Z_k^{-1}\bar{\underline{X}}_k.$$

The numerator of f_{ik} is equal to the investor's risk tolerance multiplied by the <u>sum of the cross products</u> of all the elements in the i^{th} row of the inverse of his Z_k with his assessment of the expected excess returns per share on every security; he holds the fraction of all i^{th} stock equal to the ratio of this scalar number to the summation of all such numbers in the market. Alternatively, as shown in (12iv)(c), this is equivalent to the ratio of his i^{th} inner product $Z_k^{-1}\bar{\underline{X}}_k$ across all stocks, weighted by his share w_{ak} of the market's aggregate risk tolerance, to the market's inner product $Z_m^{-1}\bar{X}_m$.

Note in particular that any investor holds a <u>different fraction of</u> the outstanding supply of each pair of securities, $f_{ik} \neq f_{jk}$, although an increase in his risk tolerance or reduction in his risk aversion would raise his holdings of all securities. His holding of any security will also be greater (a) the more confidence he has in his estimates of its ending price, conditional on all other prices (the lower its conditional variance) and (b) the less interaction (covariance) he attributes to this stock relative to other stocks -- i.e., the greater the subjective "precision" of his estimates of ending prices -- and it will also be greater the more bullish he is on its expected ending price $\bar{P}_{1ik}$ if he is long (or the more bearish, if he is short). In <u>purely competitive</u> (<u>and Pareto-optimal</u>) securities markets in which a fixed list of different securities are traded, the fractional holdings of the total outstanding supply of <u>each</u> security held by <u>each</u> investor behave in this very reasonable and sensible way <u>in response to each element of his subjective assessments</u>, given those of everyone else.[32]

SECTION II

Equilibrium When There Is No Riskless Asset

II.1 The Investor's Conditional Equilibrium

To this point we have assumed that all investors have access to a riskless security and can borrow unlimited amounts if they wish at the same rate r*. But in fact savings deposits or governments with a maturity at the end of the period also have an uncertain real return when general

[32] This is true even though in an Arrow-Debreu world (more distinct issues of securities than the inclusive set of all possible states of the world conceived with non-zero probability by any investor) or in a bargaining situation with unlimited side-betting, the "sharing rule" would be the simple one we found above in I.4(i) with homogeneous, identical expectations. See Wilson [18].

commodity and consumer goods price levels may change, and the real costs of borrowing are similarly uncertain. We now examine the model when all returns are uncertain, when each investor's utility function is defined over the real purchasing power of outcomes, and when each investor has a different assessment of changes in the purchasing power of nominal dollars and its covariance with all other securities.

Specifically, savings deposits (or debt) are just another risk asset and are included with non-zero covariances with all other securities in each investor's covariance matrix Z^*_k. For generality, we continue to assume that each investor assesses a different vector of the (N+1) expected real outcomes $\underline{\bar{P}}^*_{ik}$ and variance-covariance matrix Z^*_k, which has rank $N + 1$, where asterisks indicate variables expressed in units of purchasing power. We let the subscript $N + 1$ denote the asset with known dollar returns or costs, and let its current market price $P_{o(N+1)} = 1$. The number of "shares" held n_{N+1} then represents the number of dollars invested in the savings deposit (or, if negative, the amount of borrowing to lever the purchases of other assets), and $\bar{P}_{1(N+1)k}$ is the expected <u>real</u> interest return per dollar (or real borrowing cost per dollar of debt). Just as in previous sections we assumed the distributions of nominal returns to be normal, we now assume that each investor assesses a jointly normal distribution of the $N + 1$ <u>real</u> outcomes.

When the investor's utility function (1) is assessed in terms of real wealth W^*_{ik}, his optimal investment position will maximize

$$Q^*_k = \bar{W}^*_{ik} - a_k \check{W}^*_{ik}/2 \tag{3*}$$

subject to his wealth constraint. From his assessments of real outcomes, the expected real-value of ending wealth will be

$$W^*_{ik} = \underline{n}'_k \underline{\bar{P}}^*_{ik}, \tag{4a*}$$

and its variance will be

$$\check{W}^*_{ik} = \underline{n}'_k Z^*_k \underline{n}_k \tag{4b*}$$

while his wealth constraint becomes

$$W_{ok} = n_{(N+1)k} + \Sigma_1^N |n_{ik}| P_{oi}. \tag{4c*}$$

From these identifications, we form the Lagrangian expression

$$G_k = \underline{n}_k' \bar{\underline{P}}_{1k}^* - a_k \underline{n}_k' Z_k^* \underline{n}_k + \zeta_k [W_{ok} - n_{(N+1)k} - \Sigma_1 |n_{ik}| P_{oi}].$$

Differentiation then shows that the $N + 2$ necessary first order conditions[33] required to determine the $N + 1$ optimal holdings n_{ik}^o and the shadow-price ζ_k of the wealth constraint are

$$\delta G_k / \delta \underline{n}_k = \bar{\underline{P}}_{1k}^* - a_k Z_k^* \underline{n}_k - \zeta_k \underline{P}_o = 0 \tag{15a}$$

and

$$\delta G_k / \delta \zeta_k = W_{ok} - n_{(N+1)k} - \Sigma_1^N |n_{ik}| P_{oi} = 0. \tag{15b}$$

Conditional on any given set of market prices $\bar{\underline{P}}_o$, there will be a different set of $N + 1$ values $\{n_{ik}^o | \zeta_k\}$ for each possible value of ζ_k, and the only acceptable value of ζ_k is the value ζ_k^o for which equation (15b) is satisfied by the conditional solution-values of $\{n_{ik}^o | \zeta_k^o\}$ given by equation (15a) for all (N + 1) "securities."

The shadow-price of the investor's wealth constraint is thus in general a function of his risk aversion and his assessments of $\bar{\underline{P}}_{1k}^*$ and of risks Z_k^*, as well as of market prices. But with these other elements given and fixed, as we are assuming, the ζ_k^o in equation (15a) are simply a function of the vector of market prices $\underline{P}_o$, and the vector of equilibrium holdings $\underline{n}_k^o$ can then be validly expressed as a linear function of the product $\zeta_k^o \underline{P}_o$. If we write $\underline{n}_k^o$ for the vector of values $\{n_{ik}^o | \zeta_k^o\}$, the equations for any investor's individual equilibrium, given his assessments and any set of market prices $\underline{P}_o$, will consequently be

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[33]The second order conditions for a maximum will necessarily be satisfied because the covariance matrix Z_k^* is positive definite.

(16a) $$a_k Z^*_k \underline{n}^o_k = \underline{\bar{P}}^*_{ik} - \zeta^o_k \underline{P}_o \qquad \text{(N+1 equations)},$$

or

(16b) $$a_k \underline{v}^{*o}_k = \underline{\bar{P}}_{1k} - \zeta^o_k \underline{P} = \underline{\bar{X}}^*_k .$$

It is worth noting that these more general optimizing equations include the analysis in Section I as a special case. If the (N+1) security (or debt) were universally regarded as having a riskless return or cost $(1+r^*)$, then the last equation in (16a) or (16b) would be $P_{(N+1)k} = \zeta^o_k P_{o(N+1)} = \zeta^o_k$ for every investor, and with an agreed riskless rate r^* we have $\zeta^o_k = \zeta^o = 1 + r^*$ for everyone in the market. In this special case, equations (16) are identical to (5) and all the analysis of Section I would follow as before.

But even when there is no riskless asset, it is clear that the shadow-price ζ^o_k of each investor's wealth constraint measures the marginal real (riskless) certainty-equivalent of his end-of-period wealth Q^*_k, and the value of ζ^o_k will be different for every investor (because of his different probability assessments and his different risk aversion a_k). Moreover, the investor's equations for the optimization of his personal investment position, given any possible set of market prices $\underline{P}_o$, now require that he adjust the number of shares of each of his risk assets and his debt or holdings of the nominally riskless asset until the ratio $\bar{X}^*_{ik}/v^{*o}_{ik} = a_k$ is the same for each of the N + 1 assets he can hold. This property is the same as that found in (5c) above, but this more general case emphasizes that the balance involves $\bar{X}^*_{ik}$ the conditionally expected excess real return per share over the marginal real riskless return on the funds, with $v^{*o}_{ik} = \underline{e}'_i Z^*_k \underline{n}^o_k = \Sigma_j n^o_{jk} S^*_{ijk}$, the aggregate variance in real dollar terms which is attributable to each (and therefore the marginal share) of his holding of the i^{th} security or debt, and this marginal real portfolio variance on any stock now includes its row-sum covariances with the nominally riskless asset (or debt).

Similarly, as in (5d), the ratios of the risk-adjusted expected real return per share $[\bar{P}^*_{1ik} - P_{oi} - a_k v^{*o}_{ik}]$ on any pair of securities must equal the ratio of their respective prices in the market for any investor in personal equilibrium. Moreover, as in (5e), an investor's individual equilibrium with respect to any set of market prices (whether or not the market as a whole is in equilibrium) requires that his available funds be reallocated until the marginal risk-adjusted real return per dollar of investment is (a) the same for every security in his portfolio, including the investment with a nominally riskless return, and (b) equal to ζ_k^o, his marginal real certainty-equivalent of ending wealth, and (c) that he increase the funds available for other investments if and to the extent necessary to bring about the equalities required in (a) and (b). Note further that the amount of his optimal investment in the nominally riskless asset (or his optimal amount of borrowing) is a function of the entire variance-covariance matrix Z_k^*, whose $(N+1)^{th}$ row includes specifically the covariances of the real returns on (or real costs of) this asset with those on the other N assets.

II.2 Equilibrium Conditions for the Entire Market

An explicit expression for the optimal portfolio of investments (including the nominally riskless asset or debt) for any investor, conditional on any possible set of market prices $\underline{P}_o$, is given by the following solutions to equations (16a):

$$n_k^o = a_k^{-1} Z_k^{-1} [\underline{\bar{P}}^*_{1k} - \zeta_k^o \underline{P}_o] = a_k^{-1} Z_k^{-1} \underline{\bar{X}}^*_k, \tag{16c}$$

where $\underline{\bar{X}}^*_k = \underline{\bar{P}}^*_{1k} - \zeta_k^o \underline{P}_o$ is introduced for convenience just below. The usual market clearing conditions for the market as a whole to be in equilibrium now require

$$\underline{n}^o = \Sigma_k \underline{n}_k^o = \Sigma_k a_k^{-1} Z_k^{-1} [\underline{\bar{P}}_{1k} - \zeta_k^o \underline{P}_o] = \Sigma_k a_k^{-1} Z_k^{-1} \underline{\bar{X}}^*_k, \tag{17}$$

and these equations may be solved for the equilibrium vector of market prices:

$$P_o = (\Sigma_k a_k^{-1} \zeta_k^o Z_k^{-1})^{-1} \Sigma_k a_k^{-1} Z_k^{-1} \underline{\bar{P}}_{1k} - (\Sigma_k a_k^{-1} \zeta_k^o Z_k^{-1})^{-1} \underline{n}^o. \tag{18}$$

But in interpreting these equations, the reader will recall that the usual market clearing condition is inappropriate for the $(N+1)^{th}$ asset which we have assumed to be available in unrestricted supply at an exogenously determined nominal rate $\bar{P}_{1(N+1)}/P_{o(N+1)}$. However, our condition that $P_{o(N+1)} = 1$ takes the place of the usual market clearing condition $n^o_{(N+1)} = \Sigma_k n_{(N+1)k}$ for this one distinctive asset, and we do require that all excess demands be simultaneously zero $[n^o_i = \Sigma_k n^o_{ik}$, all $i \neq (N+1)]$ for everyone of the first N securities which are in fixed supply and for which nominal returns (as well as real returns) are uncertain. With this understanding, equations (18) set forth the strictly determinate conditions for equilibrium in securities markets when there is no riskless asset and when all investors individually and collectively may hold (or borrow) as much of the asset with a nominally fixed return as they may wish. By derivation, every investor is in a Pareto-optimal investment position with respect to all $(N+1)$ assets. It is apparent, incidentally, that if all investor's $\zeta^o_k = 1 + r^*$ as in Section I, the first N equations in (18) reduce to the simpler form (10).

II.3 Equilibrium Prices Based on Composite Market Demand

The significance of these equilibrium conditions in terms of the aggregation of investor's preferences and assessments can be best explained by noting the corresponding equilibrium conditions for "the market" viewed as a single "price-taking" entity. Proceeding as in Section I.3, we ascribe to the market a utility function and assessments having the same form as each individual investor's, but with their own distinctive parameter values by m. By a derivation identical to that just above for any other single investor, we find that the market's conditional demands $\underset{\sim}{n}_m$ for each of the $N + 1$ securities must satisfy

(16m) $$a_m \underset{\sim}{Z}^*_m \underset{\sim}{n}^o_m = \underset{\sim}{\bar{P}}^*_{1m} - \zeta^o_m \underset{\sim}{P}_o \ ,$$

or

(16m') $$a_m \underset{\sim}{v}^o_m = \underset{\sim}{\bar{P}}^*_{1m} - \zeta^o_m \underset{\sim}{P}_o = \bar{X}^*_m \ ,$$

where ζ^o_m is the shadow-price of the markets' given stock of investible wealth $W_{om} = \Sigma_k W_{ok}$, which is equivalent to the marginal riskless real certainty-equivalent of this constraint for the market as a whole; and each of $\underline{v}^o_m$ has the same identification of marginal real portfolio risk for the market as a whole as the corresponding elements in (16b) for any individual investor's portfolio.

The only set of values for ζ^o_m and $\underline{P}_o$ for which the market's demand n^o_m will equal the supplies outstanding $\underline{n}^o$ is given by

$$\zeta^o_m \underline{P}_o = \bar{P}^*_{1m} - a_m Z^*_m \underline{n}^o .$$

After dividing by the scalar term ζ^o_m, we thus have vector of equilibrium prices based on the composite market assessments given by

$$\text{(16m'')} \qquad \underline{P}_o = \zeta^{o-1}_m \underline{\bar{P}}^*_m - a_m \zeta^{o-1}_m Z^*_m \underline{n}^o .$$

The aggregation properties of the market when there is no riskless asset follow from the fact that the first and second terms on the righthand side of (18) must be equal to the corresponding terms in (16m"). This content, however, is most easily identified if we consider some special cases, as we did before in section I. For later reference, however, we note here that the inverse relations between $\underline{n}^o$, the supplies of securities held, and the assessments of the composite market are given by

$$\text{(16n)} \qquad \underline{n}^o = a^{-1}_m Z^{*-1}_m [\bar{P}^*_{1m} - \zeta^o_m \underline{P}_o] = a^{-1}_m Z^{*-1}_m \underline{\bar{X}}^*_m$$

II.4 Market Aggregation With No Riskless Asset

II.4(i) Homogeneous Assessments of $\underline{P}$*'s and Z*'s

In order to relate the marginal real certainty-equivalent of the market's composite wealth (the shadow-price of "the market's" wealth constraint ζ^o_m) to the vector of corresponding values $\{\zeta^o_k\}$ for the individual investors in the market, we revert briefly to the simplified context in which all investors' price and variance assessments are identical so that all $\underline{\bar{P}}^*_{1k} = \underline{\bar{P}}^*_{1m} = \underline{\bar{P}}^*_1 = \underline{\bar{P}}^*_1$ and all $Z^*_k = Z^*_m = Z^*$. Under these conditions

(18) reduces to

$$\text{(18i)} \qquad \underline{P}_o = [(\Sigma_k a_k^{-1})/(\Sigma_k a_k^{-1}\zeta_k^o)]\bar{\underline{P}}_1 - (\Sigma_k a_k^{-1}\zeta_k^o)^{-1} Z\underline{n}^o$$

while the vector of expected prices in (16m") is simply $\zeta_m^{o-1}\bar{\underline{P}}^*_1$ in this case.[34] Equating the scalar coefficients of prices in these two equations indicates that[35]

$$\text{(19)} \qquad \zeta_m^o = \Sigma_k w_{ak}\zeta_k^o,$$

where $w_{ak} = a_k^{-1}/\Sigma_k a_k^{-1}$ as before. Moreover, we find exactly the same relation between ζ_m^o and $\{\zeta_k^o\}$ from equating the variance coefficients in these two equations,[36] when we also use the same identification of the market price of risk and the market's risk aversion $\gamma = a_m = (\Sigma_k a_k^{-1})^{-1}$ which we derived in equation (11). The market *price of risk and the market's risk aversion thus bear exactly the same relation to the risk aversions of investors in the market when there is no riskless asset available for investment (or for borrowing)*. Moreover, in the absence of a riskless asset, *the marginal real shadow-price of the market's wealth constraint is a simple weighted average of* the corresponding shadow-prices for each individual investor, where the weights are the ratios of each investor's risk tolerance to the simple sum of the risk tolerances of all investors in the market.

II.4(ii) Homogeneous Z^* but Differing $\underline{P}^*_{1k}$

When all $Z_k^* = Z_m^* = Z$, but $\bar{\underline{P}}_{1k}$ differ among investors, we have

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[34] To see this most simply, note that under these conditions equation (16a) becomes $a_k Z^* \underline{n}_k = \bar{\underline{P}}^*_1 - \zeta_k^o \underline{P}_o$. Now divide both sides by a_k and sum over k to get $Z^*\underline{n} = (\Sigma_k a_k^{-1})\bar{\underline{P}}_1 - (\Sigma_k a_k^{-1}\zeta_k^o)\underline{P}_o$ from which (18i) follows immediately.

[35] From the price coefficients we have $\zeta_m^{-1} = [(\Sigma_k a_k^{-1})/(\Sigma_k a_k^{-1}\zeta_k)] = [(\Sigma_k w_{ak})/\Sigma_k w_{ak}\zeta_k] = (\Sigma_k w_{ak}\zeta_k)^{-1}$, since by construction $\Sigma_k w_{ak} = 1$.

[36] From the variance coefficients in (16m") and (18i), we have $a_m\zeta_m^{-1} = (\Sigma_k a_k^{-1}\zeta_k)^{-1} = [(\Sigma_k a_k^{-1})(\Sigma_k w_{ak}\zeta_k)]^{-1} = a_m(\Sigma_k w_{ak}\zeta_k)^{-1}$.

(18ii) $$\underset{\sim}{P}_o = (\Sigma_k a_k^{-1}\zeta^o)^{-1}(\Sigma_k a_k^{-1}\underset{\sim}{P}_{1k}) - a_m\zeta_m^{-1}Z^*\underline{n}^o$$

after using (11) and (19). Equating the first terms on the right of (18ii) and (16m") we have

(20) $$\underset{\sim}{\bar{P}}_{1m} = \zeta_m^{-1}(\Sigma_k w_{ak}\underset{\sim}{\bar{P}}_{1k}),$$

which is the same as implied by (10ii) except for the substitution of the scalar ζ_m for $(1+r^*)$. We thus conclude that the <u>absence of a riskless asset does not alter the market's average of differing price expectations so long as variance assessments are identical.</u>[37]

II.4(iii) <u>Identical</u> $\underset{\sim}{\bar{P}}{}^*_{1k}$, <u>but Differing</u> Z^*_k

The market's assessment of its covariance matrix is made more complex by the absence of a riskless asset when variance assessments differ among investors. To isolate the effect of differing ζ_k upon the market's average of investors' different Z_k's, we assume that all $\underset{\sim}{\bar{P}}{}^*_{1k} = \underset{\sim}{\bar{P}}{}^*_{1m} = \underset{\sim}{\bar{P}}{}^*_1$. In this situation, (18) would reduce to

(18iii) $$\underset{\sim}{P}_o = \zeta_m^{o-1}\underset{\sim}{\bar{P}}{}^*_1 - (\Sigma_k a_k^{-1}\zeta_k^o Z_k^{*-1})^{-1}\underline{n}^o,$$

which is the same as (16m") with $\underset{\sim}{\bar{P}}{}^*_{1m} = \underset{\sim}{\bar{P}}{}^*_1$ when we equate[38]

(21) $$Z^*_m = (\Sigma_k w_{ak} w_{\zeta k} Z_k^{*-1})^{-1},$$

where $w_{\zeta k} = \zeta_k^o/\zeta_m^o$. When there is no riskless asset and variance assessments are not homogeneous, the market's covariance matrix is still the

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[37]We see below, however, that differing ζ_k do modify the fractions f_{ik} of outstanding shares held.

[38]From (18iii) and (16m") we have $a_m\zeta_m^{o-1}Z_m^{*-1}\underline{n}^o = (\Sigma_k a_k^{-1}\zeta_k^o Z_k^{*-1})^{-1}\underline{n}^o$ but $(\Sigma_k a_k^{-1}\zeta_k^o Z_k^{*-1})^{-1} = a_m(\Sigma_k w_{ak}\zeta_k^o Z^{*-1})^{-1} = a_m\zeta_m^{*-1}(\Sigma_k w_{ak} w_{\zeta k} Z_k^{-1})^{-1}$, using (11) and (19).

inverse of the weighted sum of the inverses of the investors' covariance assessments, but each investor's Z_k^{*-1} is now weighted by the scalar product $w_{ak}w_{\zeta k}$ -- i.e., by the product of (i) the ratio of his marginal real shadow-price of wealth ζ_k^o to that of the market ζ_m^o (which itself is a weighted average of the ζ_k^o), and (ii) the ratio of his risk tolerance a_k^{-1} to the sum of all the risk tolerances in the market.

II.4(iv) Differing Assessments of $\bar{\underline{P}}^*_{1k}$ and of Z^*_k

The basic equations for the equilibrium vector of current market prices $\underline{P}_o$ in this more general case were given above in equation (18). It will be recalled we found in subsection I.4.2(iv) above, that, when there is a riskless asset, the market's assessment of the risks per share on any stock and the discount it imposes below expected values to compensate for these risks are independent of the price expectations held by investors in the market. Since the second terms of (18iii) and our general equation (18) are identical, it might appear that this is still the case in the absence of a riskless asset. Nevertheless, without a riskless asset this is no longer true. From equation (16b), we know that the market's assessment of marginal real risk per share on the market's aggregate portfolio is $\underline{v}^{*o}_m = Z^*_m \underline{n}^o$, and with no riskless asset available, we know from (21) that Z^*_m is now a specific function of the various investor's marginal certainty-equivalents of wealth ζ_k^o (as well as their risk aversions and their (real) covariance assessments, a_k^{-1} and Z^*_k). Moreover, from equation (16a), it is clear that each investor's ζ_k^o is a function of his assessed $\bar{\underline{P}}^*_{1k}$ (as well as his a_k and Z^*_k). Consequently, with the returns on all assets (and the real costs of borrowing) uncertain, price expectations inherently affect the assessments of risks in the market.

When there is no riskless asset, it of course continues to be true that differences in risk assessments Z_k directly alter the market's assessments of expected prices. Indeed, the market's weighted average price expectation is essentially the same[39] as it was when there was a riskless

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[39] Equating the first terms of (16m") and (18) we have $\zeta_m^{-1}\bar{\underline{P}}_{1m} = (\Sigma_k a_k^{-1}\zeta_k^o Z_k^{*-1})^{-1}\Sigma_k a_k^{-1} Z_k \bar{\underline{P}}_{1k} = \zeta_m^{-1}(\Sigma_k w_{ak} w_{\zeta k} Z_k^{*-1})^{-1}\ \Sigma_k w_{ak} Z_k^{*-1}\bar{\underline{P}}_{1k} = \zeta_m^{-1} Z^*_m \Sigma_k w_{ak} Z_k^{*-1}\bar{\underline{P}}_{1k}$. After cancelling ζ_m^{-1} on both sides, this is the same as (14c) above except for the use of (21) instead of (13) to identify Z^*_m.

asset, except for the now familiar substitution of Z^*_m for Z_m and an explicit allowance for the investors' respective ζ^o_k in the market's real shadow-price of wealth ζ^o_m.

II.5 Distribution of Stock Among Investors and Composition of Each Investor's Portfolio

As would be expected, when there is no riskless asset, differences in the investor's ζ^o_k significantly affect the fraction of each issue of outstanding stock which each investor will hold. This is a fundamental consequence of the absence of a riskless asset even when all probability assessments are identical for all investors. It will be recalled from section I.4.2(i) above that when there is a riskless asset and all investor's assessments of all expected prices and covariance matrices are assumed to be the same, we had $f_{ik} = n_{ik}/n^o_i = a^{-1}_k/\Sigma_k a^{-1}_k = w_{ak}$ for all stocks in any k^{th} investors portfolio and $f_{ik} = f_{ij}$ for all k. But even in the corresponding special case where all investor's assessments of expected real ending prices and covariances are identical [so that all $\underline{\bar{P}}^*_{1k} = \underline{\bar{P}}^*_{1m} = \underline{\bar{P}}_1$ and all $Z^*_k = Z^*_m = Z^*$] but there is no riskless asset, each f_{ik} will be a function not only of the investor's relative risk tolerance w_{ak}, but also of (a) his shadow-price of real wealth ζ^o_k, and (b) that of the market ζ^o_m, and (c) of the absolute level of the (common) price expectation $\bar{P}_{1i}$ -- and, indeed, it will implicitly also be a function of the absolute level and composition of the (commonly assessed) covariance matrix Z (since this will, other things equal, affect ζ^o_k). From equations (16c) and (16n), after allowing for the identity of assessments we have

$$f_{ik} = n_{ik}/n^o_i = a^{-1}_k[\bar{P}^*_{1i} - \zeta^o_k P_{oi}]/\Sigma_k a^{-1}_k[\bar{P}^*_{1i} - \zeta^o_k P_{oi}],$$

$$= w_{ak}[\bar{P}^*_{1i} - \zeta^o_k P_{oi}]/[\bar{P}^*_{1i} - \zeta^o_m P_{oi}]. \qquad (12a)$$

Clearly $f_{ik} \neq w_{ak}$ unless all $\zeta^o_k = \zeta^o_m$ contrary to assumption. Moreover, even when price expectations are common to all investors, we have $\bar{P}^*_{1i}/P_{oi} \neq \bar{P}^*_{1j}/P_{oj}$ so that $f_{ik} \neq f_{jk}$ for any investor; and different portfolios for different investors follows from the fact that $\Sigma_k f_{ik} = 1$ for all stocks.

Since these results hold even in the special case where investors' assessments of $\bar{\underset{\sim}{P}}_{1k}$ and Z_k are identical (Knight's "pure risk" situation), we conclude that the absence of a riskless asset is a sufficient condition for concluding that (a) any investor will hold a different fraction of the total market supply of the different stocks in his portfolio, and that (b) there will be different optimal portfolio mixes for different investors. Both conclusions are, of course, substantially strengthened in their numerical significance when we also allow for all the differences which are to be expected in practice in the assessments which different investors in the market will make of the vector of expected ending prices $\bar{\underset{\sim}{P}}_{1k}$ and of their associated covariance matrix Z_k^*. With all $\zeta_k^o \neq \zeta_m^o$, the equations for f_{ik} in this more general case are formally identical with (12iv)(b) above after substituting $\bar{\underset{\sim}{X}}_k^* = \bar{\underset{\sim}{P}}_{1k}^* - \zeta_k^o \underset{\sim}{P}_o$ for $\bar{\underset{\sim}{X}}_k$ in the latter expressions. Note that $\bar{\underset{\sim}{X}}_k^*$ subsumes both the differences among investors in their assessments of $\bar{\underset{\sim}{P}}_{1k}$ and their differing ζ_k^o.

In these more general circumstances, the distribution of the assessments $\bar{\underset{\sim}{P}}_{1k}$ and Z_k^* of different investors in the market, as well as the distribution of their risk tolerances a_k^{-1} and of their marginal real valuation of their respective wealth constraints, will all affect the distribution of any single security among investors in the market. The market distribution of each security will differ from that of any other security. No investor will hold the same fraction of the outstanding supply of any two securities. No two investors will hold the same percentage mix or portfolio of securities (even when the security with a nominally riskless return is excluded from consideration). And, in particular, no investor will hold the same portfolio mix as the market's average holdings. Very specifically, no investor holds "the market portfolio."

SECTION III

The Effects of Restrictions on Short-Selling and of Sheer Ignorance

III.1 Equilibrium for Individual Investors

To this point we have not introduced any restrictions on short selling. We have permitted any investor to go short as heavily as he wished on any security in the market. The effects of realistic restrictions on negative holdings can most easily be developed by assuming that short selling of any security (as distinct from borrowing) is prohibited, and that the amount of borrowing at any given nominal rate of interest is restricted to be less than some specified amount $N^*_{(N+1)k}$ for each investor. This involves adding the additional constraints

$$(22) \qquad \begin{cases} n_{ik} \geqslant 0; & \text{all } k, \text{ and all } i = 1 \ldots N \\ n_{(N+1)k} - n^*_{(N+1)k} \geqslant 0; & \text{all } k. \end{cases}$$

to our previous derivations.

For greater generality we continue to assume there is no riskless asset and that investors' probability assessments are heterogeneous. Each investor maximizes his expected utility denominated in real end-of-period wealth, subject both to his wealth constraint and the constraints on short selling. Drawing upon the analysis of section II, this involves the maximization[40] of

$$Q^*_k = \underline{n}'_k \underline{\bar{P}}^*_{1k} - a_k \underline{n}'_k Z^*_k \underline{n}_k / 2,$$

subject to (4c*) and also to (22), and we specify that $P_{o(N+1)} = 1$. After forming the appropriate Lagrangian expression and using the Kuhn-Tucker theorem, we find that, conditional on any possible vector of current market prices $\underline{P}_o$, the conditions for the k'^{th} investor's optimal investment position $\underline{n}^o_k$ (including his holding of the nominally riskless asset or borrowing) are

$$(23a) \qquad a_k Z^*_k \underline{n}^o_k = \underline{\bar{P}}_{1k} - \zeta^o_k \underline{P}_o + \underline{u}_k, \qquad \text{(N+1 equations)}$$

when the further conditions

[40] See pages 373 and 374.

(23b) $$n_{ik} \geq 0, \qquad i = 1 \ldots N,$$

(23c) $$n_{(N+1)k} \geq n^*_{(N+1)k},$$

(23d) $$u_{ik} \geq 0, \qquad i = 1 \ldots N + 1,$$

(23e) $$\begin{cases} n_{ik}u_{ik} = 0, \qquad i = 1 \ldots N, \\ [n_{(N+1)k} - n^*_{(N+1)k}]u_{(N+1)k} = 0, \end{cases}$$

are simultaneously satisfied, and u_{ik} are the shadow prices of the corresponding constraints (23b) and (23c) for each of the (N + 1) investments available. The reader will note that we have "solved out" an additional equation for the shadow-price of the investor's wealth constraint as on page 375 above. This shadow-price ζ^o_k still measures the marginal certainty-equivalent of his real end-of-period wealth, but its value now depends upon the constraints (23b–e) as well as his assessments of $\bar{\underline{P}}^*_{1k}$ and Z^*_k and his risk aversion a_k as well as upon the particular set of current prices facing him in the market. We also observe that ζ^o_k will be larger than its value in the case without short selling constraints.[41]

Conditional upon any set of market prices $\underline{P}_o$, and (23b–e), the investor's optimal schedule of holdings of each security is given by

(23a') $$\underline{n}^o_k = a^{-1}_k Z^{*-1}_k [\bar{\underline{P}}^*_{1k} + \underline{u}^o_k - \zeta^o_k \underline{P}_o].$$

Whenever any investor's demand for any one or more securities would be negative without considering the constraint on short-selling, the effect of imposing the non-negativity constraints $\underline{n}_k > 0$ is to force some of

[41]The latter conclusion reflects the fact that the imposition of an effective constraint will in general lower the maximum certainty equivalent of ending real wealth Q^*_k attainable by any investor, thereby raising the marginal value ζ^o_k.

the shadow-prices in $\underline{u}_k \geqslant 0$ to sufficiently large positive values as to make the corresponding subsets of inner products

$$\text{(23i)} \qquad \underline{e}_j' a_k^{-1} Z_k^{*-1} [\underline{\bar{P}}_{1k}^* + \underline{u}_k^o - \zeta_k^o \underline{P}_o] \geqslant 0,$$

and this inner product will equal zero with respect to the subset of stocks which do not appear in the investor's equilibrium portfolio. Moreover, if we now let $\hat{\hat{Z}}_k$ represent the original (N+1) x (N+1) matrix Z_k^* after the elimination of all rows and columns representing stocks for which his equilibrium demand (given the equilibrium set of market prices) is zero, and let a double carat over a vector have the corresponding interpretation, each investor's demand for the securities in his portfolio after solving (23a-e) is accurately summarized by the solutions of the "reduced form"

$$\text{(23f)} \qquad a_k \hat{\hat{Z}}_k^* \underline{n}_k = [\hat{\hat{P}}_{1k}^* - \zeta_k^o \hat{\hat{\underline{P}}}_o]$$

since $\underline{u}_k \equiv 0$ for all stocks positively held in his equilibrium portfolio; for convenience later, these positive demands for securities will also be written explicitly

$$\text{(23f')} \qquad \hat{\hat{n}}_k = a_k^{-1} \hat{\hat{Z}}^{*-1} [\hat{\hat{\underline{P}}}_{1k} - \zeta_k^o \hat{\hat{\underline{P}}}_o].$$

Finally, it will be recalled (last two paragraphs in section II.1, pages 376-377) that important properties of the investor's personal equilibrium involved adjustments in the aggregate variance in real dollar terms which is attributable to each (and to the marginal) share of his holding of the i^{th} security or debt, measured by $\underline{v}_{1k}^{*o} = \underline{e}_i' Z_k^* \underline{n}_k^o$. It is apparent that all these further conditions of personal equilibrium stated above hold exactly for all stocks included positively in the investor's portfolio and represented in $\hat{\hat{Z}}_k^*$ and $\hat{\hat{\underline{P}}}_{1k}$. The only stocks for which these same conditions do not hold exactly are those which do not appear at all in the investor's portfolio. Moreover, we see that

$$\text{(24)} \qquad \hat{\hat{v}}_{ik}^{*o} = \underline{e}_i' \hat{\hat{Z}}_k^* \hat{\hat{\underline{n}}}_k^o = \underline{e}_i' Z_k^* \underline{n}_k = v_{1k}^{*o}$$

The corresponding vectors $\hat{\underline{v}}^{*o}_k$ and $\underline{v}^{*o}_k$ differ only by zeros in the latter for elements representing stock not held in equilibrium portfolios. From (23a) and (23f) we consequently have two further equivalent statements using (24) of the conditions of personal equilibrium for the individual investor:

(23g) $$a_k \underline{v}^{*o}_k = \bar{\underline{P}}^*_{1k} + \underline{u}^o_k - \zeta^o_k \underline{P}_o ,$$

and

(23h) $$a_k \hat{\underline{v}}^{*o}_k = \hat{\bar{\underline{P}}}_{1k} - {}^o_k\hat{\underline{P}}_o .$$

Note in particular that the variance assessments relevant to an investor's personal equilibrium are strictly independent of his ending price and variance assessments with respect to all stocks which do not enter into his equilibrium portfolio, and specifically they are strictly independent of all covariance assessments involving stocks he does not hold positively in equilibrium. Although an obvious point, this observation has critically important consequences brought out below.

III.2 Conditions for Short Selling Constraints to Be Ineffective

In two special cases, the formal addition of short-selling constraints would have no effect on any investor's portfolio or on the market prices derived without reference to any constraints. Under these circumstances $\underline{u}_{ik} = 0$ for all stocks and all investors, and the vectors in (23f') will be identical to those in (23a'). The sufficient conditions for non-negativity constraints to be irrelevant are (a) the presence of a riskless asset with an agreed return r^*, and (b) identical price assessment throughout the market [all $\bar{P}_{1ik} = \bar{P}_{1im} = \bar{P}_{1i}$], with (c1) identical covariance matrices[42] [all $Z_k = Z_m = Z$] or (c2) all covariance assessments $S_{ijk} = 0$

[42] See section I.4.2(i), pages 363 and 364 above. Under these conditions, $n_{ik} > 0$ for all i and k without consideration of any short-selling constraints.

throughout the market[43] [all Z_k diagonal]. Note that short selling constraints have no effects on market prices even when variance assessments differ among investors, provided all covariances are zero and there is a riskless asset.

III.3 Sufficient Conditions for Short Selling Constraints to Be Effective

Short selling constraints will in principle affect investor's portfolios and equilibrium prices in the market if (a) there is no riskless asset so that ζ_k^o differ among investors, or (b) if all investors do not assess identical vectors of expected end-of-period prices $\bar{P}_{1k}$. Even when conditions (a) and (b) are absent, any differences in assessments of non-diagonal covariance matrices Z_k which would otherwise lead to any short selling will of course also activate the constraints. Even if all covariances are ignored and all expected future price assessments are identical, diversity in ζ_k^o due to the absence of a riskless asset would lead to short sales of some securities by some investors in the absence of a short selling constraint,[44] and condition (a) above is therefore sufficient. Condition (b) is seen to be sufficient to activate the short-selling constraints by noting that even if there is a riskless asset (so that all $\zeta_k^o = \zeta_m^o = 1 + r^*$) and covariances are zero, different investors will generally be relatively

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[43] With (a), (b) and (c2), the number of shares of the i^{th} stock held by the k^{th} investor will be $n_{ik} = [\bar{P}_{1i} - (1+r^*)P_{oi}]/a_k S_{iik}$. The numerator must be strictly positive for any investor to hold any of the stock, and it must obviously be positive for the market to clear; but with all $\bar{P}_{1ik} = \bar{P}_{1i}$, if the numerator is positive for one investor, it will be positive for all; and consequently $n_{ik} > 0$ for all stocks and all investors, since all a_k and $S_{iik} > 0$ necessarily. Since $n_i = \Sigma_k n_{ik}$, we can sum over all investors and see that when there is a riskless asset and expected values are assessed the same by different investors, the fraction of the i^{th} security held by the k^{th} investor will be $f_{ik} = n_{ik}/n_i = a_k^{-1}/\Sigma_k a_k^{-1}$ if variance assessments are the same and $f_{ik} = a_k^{-1} S_{iik}/\Sigma_k a_k^{-1} S_{iik}^{-1}$ if they differ.

[44] With no covariances but all $\bar{P}_{1ik} = \bar{P}_{1i}$, whether a stock is held (see previous footnote) depends upon the sign of the numerator in $[\bar{P}_{1i} - \zeta_k^o P_{oi}]_o / a_k S_{iik}$. We observe that one or more investors (those with the smallest ζ_k) will have positive holdings of every one of the N securities in their individual portfolios; but as one moves along the spectrum to investors with larger ζ_k, there will be a progressively smaller number of securities appearing in their respective portfolios if short selling is restricted.

more optimistic or pessimistic on different sets of stocks; with no constraints, those stocks on which an investor is relatively pessimistic would be sold short; but with the constraint, each investor will have no short positions and will simply hold that subset of stocks for which his expectations are most optimistic relative to those of other investors. Incidentally, note also that diversity in the assessment of expected ending prices $\bar{P}_{1ik}$ among investors is sufficient to remove the implication that any one or more of the N securities must be held by any one investor in the market.

III.4 Equilibrium Market Prices and Their Aggregation Properties

As in section II,[45] the first N elements in the vector $\underline{n}^o$ represent the given and fixed outstanding supplies of each of these securities n_i^o. The usual equations (23a) and (23f) for each investor's personal equilibrium imply the following market clearing conditions

$$(25) \qquad \underline{n}^o = \Sigma_k \underline{n}_k^o = \Sigma_k a_k^{-1} Z_k^{*-1} [\bar{\underline{P}}_{1k} + \underline{u}_k^o - \zeta_k^o \underline{P}_o].$$

These in turn can be solved for the equilibrium vector of current market prices:

$$(26) \quad \underline{P}_o = (\Sigma_k a_k^{-1} \zeta_k^o Z_k^{*-1})^{-1} \Sigma_k a_k^{-1} Z_k^{*-1} (\bar{\underline{P}}_{1k}^* + \underline{u}_k^o) - (\Sigma_k a_k^{-1} \zeta_k^o Z_k^{*-1})^{-1} \underline{n}^o,$$

provided that conditions (23b,c,d,e) are simultaneously satisfied for all M investors with the equilibrium conditions stated in (25) and hence in (26).

Since the outstanding supply of each of the first N assets is necessarily positive, our equations for equilibrium prices based on "the

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[45]See page 378 above. Note that as above, the condition $P_{o(N+1)} = 1$ takes the place of the usual market clearing requirement for the $(N + 1)^{th}$ asset since it is still assumed in unrestricted supply. We thus retain $N + 1$ conditions for the solution of the $N + 1$ simultaneous market relations in (24) and (25). Note also that the market's aggregate net holding of (or net debt position in) the nominally riskless asset is still $N_{(N+1)m} = \Sigma_k n_{(N+1)k}$.

market" as a single price-taking entity are still identical to those given in section II.3 above (pages 378 and 379). It is consequently tempting to equate corresponding terms in (16m") and (26) and to conclude that all the identifications and conclusions reached in section II continue to hold in this more general case after (a) the sum $(\bar{\underline{P}}^*_{1k} + \underline{u}_k)$ is substituted whenever $\bar{P}^*_{1k}$ appears alone in the earlier equations and text, and (b) allowance is made for the larger values[46] of ζ^o_m in this constrained case.[47] Such a conclusion, however, would be incorrect. Among other things, it would imply that when purely competitive security markets are in equilibrium after the imposition of effective constraints on short selling, the market price of risk is the same for all securities and it is not. Such an identification of (26) with (16m") would also imply that the shadow price of market wealth relevant to the pricing of all securities is the same for all securities, and it is not. This identification would also imply that the market's covariance matrix Z^*_m is the same weighted average of the inverses of its participating investor's full (N+1)-rank covariance matrices as given by (21), even when there are effective short-selling constraints, and it is not.

In the course of proving these assertions, we will also show that -- although (26) is a perfectly valid equation for the vector of market clearing when all investors are in personal current prices equilibrium and, indeed, must be used for various other purposes -- this statement of market equilibrium prices involves redundancies which obscure important properties of any particular market equilibrium position. Specifically, the general equilibrium equations (26) do not bring out -- and actually seem to be misleading with respect to -- the partial equilibrium conditions for any one security when all markets and all investors are in full equilibrium.[48]

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[46]See footnote 41.

[47]In the special case with a riskless asset, the analysis of section II showed that all $\zeta^o_k = \zeta^o_m = 1 + r^*$. With this specification, the corresponding spurious inference would be that all the conclusions reached in section I also continue to hold even with restraints on short selling, after the indicated substitution of $\bar{\underline{P}}_{ik} + \underline{u}_k$ for the symbol $\bar{\underline{P}}_{ik}$ is made.

[48]The reader will note that, in contrast to excessively common practice in discussions of conditions of "partial equilibrium," we will explicitly derive our conclusions regarding the "partial" conditions for an individual

To establish these propositions, we assume that each investor 1 ... M has solved the full set of equations (23a,b,c,d,e) representing his personal equilibrium conditional on the equilibrium market clearing vector of prices $\underline{P}_o$. We then renumber the investors in the market so that the inner product for the i^{th} stock in equation (24) is zero for the investors $k = 1, \ldots,$ $k' - 1$. These investors do not hold the stock -- their $n^o_{ik} = 0$ -- in their preferred personal portfolio at market equilibrium prices $\underline{P}_o$. Correspondingly, this inner product for the i^{th} stock will be strictly > 0 for investors numbered $k = k', \ldots, \ldots, M$. If Σ_k indicates summation over $k = 1 \ldots M$, and Σ^i_k indicates summation only over $k = k' \ldots M$, the market clearing equations (25) for the i^{th} stock may now be written

$$\text{(25a)} \qquad n^o_i = \Sigma_k n^o_{ik} = \Sigma^i_k n^o_{ik} = \Sigma^i_k a_k^{-1} e_i' Z_k^{*-1} [\bar{\underline{P}}^*_{1k} + \underline{u}^o_k - \zeta^o_k \underline{P}_o].$$

Each of the investors numbered $k = k' \ldots M$ has $n^o_{ik} > 0$, a strictly positive holding of the i^{th} stock at these prices, but the non-negativity constraints will have eliminated certain other stocks from each of their portfolios. If we now substitute the equivalent[49] "reduced form" equations (23f) or (23f') for each investor, market clearing equation (25a) for the i^{th} stock can consequently be further simplified to

$$\text{(25b)} \qquad n^o_i = \Sigma_k n^o_{ik} = \Sigma^i_k n^o_{ik} = \Sigma^i_k a_k^{-1} \underline{e}_i' \hat{\hat{Z}}_k^{*-1} [\hat{\hat{\underline{P}}}^*_{1k} - \zeta^o_k \hat{\underline{P}}_o].$$

This equation, of course, cannot be directly solved for the equilibrium price of any stock because $\hat{\hat{Z}}^*_k$ and $\hat{\hat{\underline{P}}}^*_o$ have a different rank (and covers a different subset of stocks) for each investor,[50] but because we are only

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security to be in Pareto-optimal, market-clearing equilibrium from the general equilibrium conditions for the entire market given by equations (23a-e) and (25).

[49]By construction, the solutions for n_k of the reduced form equations (23f) for any k^{th} investor are identical to those given by the solutions to his full set of equilibrium conditions (23a-e).

[50]In general, both the particular subset of stocks in each Z^*_k and the subset of investors included in the summation for any one stock will depend upon the particular vector of current market prices $\underline{P}_o$ which satisfy the full set of equilibrium conditions in the market given by (23a-e) for every investor and by (24) for the market as a whole. This, however, clearly creates no problem for present purposes.

seeking a valid statement of the equilibrium price P_{oi} of any one stock when all investors and all other stocks are in full equilibrium, we can proceed as follows. We define a (row) vector of numbers $\underline{\sigma}^*_k$ such that

$$(27) \qquad \hat{\hat{Z}}^{*-1}_k[\hat{\hat{\underline{P}}}^*_{1k} - \zeta^o_k\hat{\hat{\underline{P}}}_o] \equiv \hat{\hat{\underline{\sigma}}}^{-1}_k[\hat{\hat{\underline{P}}}^*_{1k} - \zeta^o_k\hat{\hat{\underline{P}}}_o],$$

when all current market prices in $\hat{\hat{\underline{P}}}_o$ are equilibrium market prices. The economic interpretation of $\hat{\hat{\underline{\sigma}}}_k$ is clearly the set of variance assessments which (with all covariances zero) would have led the k^{th} investor to hold the identical portfolio of securities at market equilibrium prices as the set he actually chose to hold on the basis of his "full" covariance matrix $\hat{\hat{Z}}_k$. (The "as if" numbers $\hat{\hat{\underline{\sigma}}}_k$ depend upon all his assessments and upon the vector of market prices, but they do fully incorporate his covariance assessments given the equilibrium vector of market prices). After substituting the right side of (27) in (25b) we have

$$(25c) \qquad n^o_i = \Sigma_k n^o_{ik} = \Sigma^i_k n^o_{ik} = \Sigma^i_k a^{-1}_k \sigma^{-1}_{iik}[\bar{P}^*_{1ik} - \zeta^o_k P_{oi}],$$

which in turn shows that the equilibrium price[51] of the i^{th} stock when the whole market is in full equilibrium is related to individual investor's parameters and assessments by

$$(26a) \quad P_{oi} = (\Sigma^i_k a^{-1}_k \zeta^o_k \sigma^{-1}_{iik})^{-1}(\Sigma^i_k a^{-1}_k \sigma^{-1}_{iik}\bar{P}_{1ik}) - (\Sigma^i_k a^{-1}_k \zeta^o_k \sigma^{-1}_{iik})^{-1} n^o_i.$$

Given equilibrium prices for other stocks P_{oj}, $j \neq i$, the equilibrium price P_{oi} for the i^{th} stock in equation (26a) must be identical to that given by the i^{th} row in equation (26). As an immediate corollary, we have the following critically important propositions: apart from the (equilibrium) prices of other stocks, the equilibrium price of any stock depends

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[51]Note that (25c) is specifically not the partial equilibrium demand function for the i^{th} security, given equilibrium prices on other assets; but it is an equilibrium condition for P_{oi}, given equilibrium elsewhere, and that is all we need for present purposes.

only on the a_k and ζ_k^{*o} of those who own it, together with the assessments $\hat{\hat{Z}}_k$ and $\hat{\hat{P}}_k$ of this subset of investors with respect to the various subsets of stocks which appear with positive holdings in their respective equilibrium portfolio. In particular, the equilibrium price of any i^{th} stock is independent of all the assessments and the risk aversion and the marginal real wealth certainty equivalents ζ_k^o of all investors who do not hold that stock in general equilibrium. It is also independent of all "potential" covariances assessed by those who do own it with respect to stocks they do not hold in their equilibrium portfolios.

These propositions in turn have further important corollaries. The last term in (26a) obviously represents the composite risk adjustment for the i^{th} stock to the weighted average expectation of ending price given by the preceding term, and this risk adjustment can be expressed in terms of the composite assessments and parameters of the subset of investors who hold it. Specifically, we have

(28) $$(\Sigma_k^i a_k^i \zeta_k^o \sigma_{iik}^{-1})^{-1} = \gamma_i \zeta_m^{oi} \sigma_{iim},$$

where

(28a) $$\gamma_i = a_{m(i)} = (\Sigma_k^i a_k^{-1})^{-1},$$

and

(28b) $$\zeta_m^{oi} = \Sigma_k^i w_{ak}^i \zeta_k^o, \qquad w_{ak}^i = a_k^{-1}/\Sigma_k^i a_k^{-1},$$

and

(28c) $$\sigma_{iim} = (\Sigma_k^i w_{ak}^i w_{\zeta k}^i \sigma_{iik}^{-1})^{-1}, \qquad w_{\zeta k}^i = \zeta_k^{oi}/\zeta_m^{oi}.$$

Similarly, the first term on the right side of (26a) represents the market's weighted average assessment of the expected ending price of the i^{th} stock,

(26d) $$\bar{P}_{1im} = \sigma_{iim} \Sigma_k^i w_{ak}^i \sigma_{iik}^{-1} \bar{P}_{1ik}.$$

The derivation of these relations exactly parallels that on pages 379-382 above[52] with respect to the subset of investors who hold the i^{th} stock. Beyond emphasizing that the parameters and assessments of all other investors are irrelevant to the equilibrium price and "the markets" assessments regarding the i^{th} stock, nothing more need be said of the market's composite weighted average assessment of the expected ending price $\bar{P}_{1im}$ and "as if" variance σ_{iim} of any stock.

With respect to the shadow price of the market's wealth constraint ζ_m^{oi} relevant to the equilibrium price of the i^{th} stock when there is no riskless asset, however, more does need to be said. We know that in general different subsets of investors will hold each different stock in equilibrium when short selling constraints are effective. Consequently, since ζ_m^{oi} is a weighted average of the ζ_k^o over the subset of investors who hold this stock, ζ_m^{oi} will in general have a different value for every different stock in the market when all investors and all stocks are in equilibrium -- i.e., $\zeta_m^{oi} \neq \zeta_m^{oj}$ unless identical subsets of investors hold stocks i and j in equilibrium. Moreover, ζ_m^{oi} may be either larger or smaller than the unique value ζ_m^o (common to all stock) derived in section II in the absence of short selling constraints.

In (28a), γ_i is the market price of risk relevant to the equilibrium price of the i^{th} stock, and $a_{m(i)}$ is the corresponding "market risk aversion" relevant to the i^{th} stock in equilibrium. These measures are equal for the usual reasons for any i^{th} stock. But since γ_i in equilibrium for any stock depends only on the risk tolerances of those investors who actually[53] hold it, whenever short selling constraints are effective, the market price of risk relevant to the partial equilibrium of any i^{th} stock is necessarily larger than when such constraints are absent or ineffective. Since $a_k^{-1} > 0$, $(\Sigma_k^i a_k^{-1})^{-1} > (\Sigma_k a_k^{-1})^{-1}$. The economic rationale

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[52]For (26d), compare pages 367-369.

[53]These holdings must be positive (long) when short selling is prohibited, as we have been assuming for expositional convenience. Obviously, if short selling were merely restricted, the a_k of all those who affirmatively hold the stock long or short would affect its γ_i, but those with corner solutions on the stock would be irrelevant to its market price of risk.

is that in the absence of constraints the (smaller) market price of risk is "pricing out" all the added risks of all the variances and covariances of the i^{th} stock with other stocks in the portfolios of all the investors who would hold it (short) in the absence of the short-selling constraint, but who do not hold it at all when short selling constraints are effective. In effect, the imposition of short-selling constraints reduces the effective size of the market for the i^{th} stock, and I have shown elsewhere [10] that the market price of risk varies inversely with the size of the market.

In spite of its reduced size due to effective constraints on short selling, the market for each i^{th} stock continues to satisfy all the conditions of pure competition, and all the covariances between the different stocks appearing in each investor's equilibrium portfolio are fully incorporated in his risk assessment for the i^{th} stock (and hence in the market's "as if" assessment). Nevertheless, when short-selling constraints are effective, the market price of risk relevant to the equilibrium price of the i^{th} stock in general differs from that of every other stock in the market — once again, because different subsets of investors hold different stocks when short selling is restricted. Indeed, $\gamma_i = \gamma_j$ only if the sum of the risk tolerances of the subsets of investors who hold the two stocks is the same.

We observe that γ had a unique value applicable to the equilibrium of all stocks under the conditions of sections I and II above only because those assumptions in the absence of a short selling constraint implied that every one of the M investors in the market would hold some positive or negative amount of every one of the N securities available in the market (as well as of the nominally riskless asset).

III.5 Aggregation of Assessments in Market Equilibrium Using Fractional Holdings of the Outstanding Supply of Each Security

We may briefly observe that the alternative aggregation of investor's preferences and assessments introduced in section I.4 continues to be valid in all essential respects when there is no risk asset and when short selling constraints are effective. Suppose each investor has solved his full set of personal equilibrium equations (23a,b,c,d,e) conditional on any possible vector of market prices $\underline{P}_o$, and has substituted these "conditional solution values" n^o_{ik} back into (23a) and (23g). The conditioning vector of market prices will be an equilibrium vector if and only if $\Sigma_k n^o_{ik} = \Sigma_k n^o_i$, the available supply, for each of the first N securities. We now impose the equivalent requirement $\Sigma_k f^o_{ik} = 1$ (where $f^o_{ik} = n^o_{ik}/n^o_i$, as before) in order to characterize market equilibrium prices.

After multiplying all terms in the investor's equilibrium conditions (23g) by f^o_{ik} and summing over all investors, we have

$$(29a) \qquad (\Sigma_k f^o_{ik}\zeta^o_k)P_{oi} = \Sigma_k f^o_{ik}P^*_{1ik} - \Sigma_k f^o_{ik}a_k v^*_{ik}$$

which is equivalent to

$$(29b) \qquad (\Sigma^i_k f^o_{ik}\zeta^o_k)P_{oi} = \Sigma^i_k f^o_{ik}P^*_{1ik} - \Sigma^i_k f^o_{ik}a_k v^{*o}_{ik}$$

since $f^o_{ik} = 0$ for all investors not included in Σ^i_k.

The absence of a riskless asset makes the market's assessments proportional to (rather than equal to) the simple weighted averages of investor's assessments. Effective short selling constraints limit the effective range of the summation to those investors who hold the stock. But as in the simpler cases, each investor's risk adjustment incorporates both his risk aversion and his assessment of the marginal risk contributed by the i^{th} stock to the real ending valuations of his equilibrium portfolio. And as in the simpler cases, the composite of each investor's preferences and assessments regarding expected ending values and marginal portfolio risks of the stock are weighted by his fractional holding in equilibrium of the outstanding supply of the stock. Not only are all the preferences and assessments of all those who do not hold the stock ignored in the aggregations determining the current price of the stock; the preferences and assessments of those who do hold the stock get greater weight in proportion to the amount of the stock they hold.

III.6 Some Effects of Sheer Ignorance

In section I of this paper, we generalized the usual models of purely competitive stock prices by allowing for the fact that investor's probability judgments differ. In section II, we abandoned the assumption of a riskless asset, and we have just shown certain major consequences of effective restrictions on short selling when investors all enter the market with assessments of $\bar{\underline{P}}_{1k}$ and Z_k with respect to all the $N + 1$ securities available in the market. But in actuality, most investors simply have no judgments whatsoever with respect to most of the stocks available in the market. Even major institutional investors with large staffs only attempt to "follow" two or three hundred stocks out of the many thousands available, and "small" investors are entirely ignorant of all but very small subset of stocks.[54]

This important factual circumstance can be readily introduced into our formal analysis. All we need to do is let $\hat{\bar{\underline{P}}}_{1k}$ and $\hat{Z}_k$ represent the assessments of the k^{th} investor over the subset of stock for which he does have an assessment. In general, each investor's $\hat{\bar{\underline{P}}}_{1k}$ and $\hat{Z}_k$ now have a different rank, and cover a different subset of stocks, and differ in their assessments with respect to any common pair of included stocks.

If there are no short selling constraints, we need only to substitute these vectors and matrices for the $\bar{\underline{P}}^*_{1k}$ and Z^*_k used in section II. The market clearing conditions then immediately give equations (25b) of section III, with a single instead of a double carat over the vectors and matrices. When different investors have judgments or assessments over differing subsets of the stocks being traded in the market, the purely competitive equilibrium vector of current market prices is the unique vector consistent with the satisfaction of (25b) simultaneously for all the N stocks in the market. All our conclusions in section III.4 consequently also apply with full [illegible]

The analysis of sections III.4 and 5 obviously already covers the case where there is <u>both</u> widespread ignorance concerning most stocks on the part of most investors <u>and</u> short selling is also restricted;[55] the qualitative conclusions are the same as when either is present separately, but they are much stronger in empirical significance.

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[55] In this case, the investor's equations (23a–e) for his conditional personal equilibrium are simply solved in terms of $\hat{\underline{P}}^*_{1k}$ and $\hat{Z}^*_k$ instead of $\bar{\underline{P}}^*_{1k}$ and Z^*_k (as in III.4 above), and $\hat{\hat{\underline{P}}}^*_{1k}$ and $\hat{\hat{Z}}^*_k$ will be derived from these conditional solutions, but will have exactly the same interpretation as before.

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