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Super contact and related optimality conditions

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Under transactions costs, and generally any sort of friction, economic agents, acting dynamically in a stochastic environment, may find it optimal, in some region of the state space, to take no action at all. Action is triggered when the state of the economic system reaches the boundary of the 'region of no action'. In this note, first-order conditions for the choice of the region of no action, and for the type of action to be taken, are established, assuming that the uncontrolled state variable follows a diffusion process. The generic name for these conditions is 'smooth pasting' or 'high contact' conditions; they require that *marginal utility* should take the same value before and after the action has been taken. In this note, we show that, in some cases, these conditions involve the first derivatives of the value function of the dynamic program, while, in other cases, they involve the second derivatives and require a higher form of tangency which Dumas (1988) called 'super contact'.

1. Introduction

The stochastic process known as Brownian motion has proved useful in many economic constructs involving uncertainty. In some cases, a boundary behavior is superimposed on the specification of the process; the Brownian motion can at some point be 'reflected' or 'absorbed' or forced to jump back inside a given domain. When the positioning of a point of absorption is a decison variable, the problem is one of 'optimal stopping' of the Brownian motion [Krylov (1981)]. When the Brownian motion is to be reflected at some trigger point, the Brownian motion is said to be subjected to 'instantaneous' control [Harrison and Taylor (1978), Harrison and Taksar (1983), Karatzas

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(1983), Harrison (1985)].[1] When the Brownian motion is forced to jump back from a trigger point to a target point inside a given domain, the problem is one of 'impulse control' [Bensoussan and Lyons (1975), Harrison, Sellke, and Taylor (1983)]. Several applications of these types of regulation have been made in the field of economics.

An American-type option can be exercised at any time before maturity. This leads to an *optimal stopping* problem. Samuelson (1965), McKean (1965), and Merton (1973) established conditions of optimality for this problem, which are known as 'smooth pasting' conditions: the *first derivative* of the option price function (in addition to the price itself) must take the same value before and after the option has been exercised. This generates boundary conditions for the price function of the option at the exercise boundary.

Flood and Garber (1989) regulate an exchange rate process within a target zone by means of *impulse control* applied to the fundamentals (money supplies, etc.). The applicable boundary conditions are simply that the level of the exchange rate does not jump when the control is applied (a 'value matching' condition). *Optimal* impulse control has been used by Brennan and Schwartz (1985) to model the opening and closing of a mine when the price of the ore fluctuates randomly, and by Grossman and Laroque (1990) for the choice of a portfolio in the presence of fixed transactions costs. In both models, a necessary condition of optimality is that the value of the *first* derivative of the value function of the dynamic program remain continuous at the time of regulation, as in the case of optimal stopping. Grossman and Laroque mention specifically that the value of the *second* derivative, even after optimization, typically undergoes a jump when the control is applied, or equivalently that the *second derivative is discontinuous* at points on the trigger boundary.

Absent any optimization, Krugman (1990) has proposed a model of exchange rate intervention within a target zone in which *instantaneous* (infinitesimal) *regulation* leads to a boundary condition involving again the *first derivative* of the exchange rate function. Confusingly, he called this a 'smooth pasting' condition. While it is identical in form to the optimality condition of the problems mentioned earlier, it must evidently be of a different nature. The first goal of this note (sections 2 and 3) is to clarify the respective roles of these two conditions.

Under proportional costs of adjustment, Dumas (1988) has modelled the international physical investment process as an *optimal instantaneous control* problem. He imposes optimality by requiring that the *second derivative* of the value function remain continuous at a regulation point. He proposes to call

[1] In this case, the optimal control is also said to be 'singular': the drift imposed by the reflection on the Brownian motion is infinite or zero (bang-bang).

this condition a 'super contact' or second-order smooth pasting condition, and to use it as the premier optimality condition applicable to all problems involving proportional transactions costs. The condition appears to be new to the field of economics[2] although it has been stated earlier by researchers working in related fields.[3] Our second goal in this note (section 4) is to derive the super contact condition heuristically and in a simplified way and to explain why it holds under proportional costs but fails to hold under fixed costs, as has been pointed out by Grossman and Laroque (1990).

Researchers who have obtained optimal instantaneous control policy without the device of the super contact condition have proposed other conditions of optimality. The third and final goal of the present note (section 5) is to relate these other conditions to super contact.

Consider the variable x which follows an Ito process regulated between two barriers 0 and u. At $x=0$, a costless and rewardless infinitesimal regulator $\mathrm{d}L$ is applied to x and gives it a 'push' upward. At $x=u$ another regulator $\mathrm{d}U$ is applied which instantaneously takes x to a level $v<u$. Overall the stochastic differential equation for x is

$$\mathrm{d}x=\mu(x)\,\mathrm{d}t+\sigma(x)\,\mathrm{d}z+\mathrm{d}L-\mathrm{d}U. \tag{1}$$

L and U are defined as stochastic processes which are right-continuous, nonnegative, and nondecreasing. In addition, U increases (by an amount $u-v$ denoted $\mathrm{d}U$) only when $x=u$; L is continuous and increases only when $x=0$.[4]

Let the upper regulator $\mathrm{d}U$ be operated at a cost $\mathrm{d}C$. C is a right-continuous, nonnegative, nondecreasing process; C increases only when U does, i.e. when $x=u$; when it does, its increase is given by a cost function $c(\ )$: $\mathrm{d}C=c(\mathrm{d}U)$. When we particularize $c(\ )$, we let it be linear: $c(x)=c_0+\rho x$, where c_0 is a fixed cost component and ρ is a coefficient for the proportional cost component.

Finally let $\phi(x)$ be a bounded flow payoff function, and let $\delta>0$ be the discount rate. Define the value function, or the expected discounted payoff,

[2]Constantinides (1986) and Taksar, Klass, and Assaf (1988) consider a portfolio choice problem under proportional transactions cost but obtain an optimal policy by other means. Davis and Norman (1990) state the condition as a necessary implication of optimality.

[3]In the operations research literature, there are mentions of a heuristic 'principle of smooth fit' [Benes et al. (1980), Karatzas and Shreve (1984)] which says that the second derivative of the value function should be continuous across the boundary that demarcates the regions of action and inaction. In Harrison and Taksar (1983), a short discussion and a proposition (numbered 6.1) effectively proves the validity of super contact as an optimality condition.

[4]See Harrison (1985).

or the performance of the u, v policy as

$$F(x; u, v) \equiv \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\delta t}\phi(x_t)\,\mathrm{d}t - \mathrm{d}C | x_0 = x\right], \tag{2}$$

where x follows the stochastic differential equation (1).

Inside the zone of no intervention ($0 < x < u$), the process x moves of its own accord. The expected change in F is brought about by the flow payoff $\phi(x)$ and the effect of discounting:

$$0 \equiv \phi(x) - \delta F(x) + F'(x)\mu(x) + \tfrac{1}{2}F''(x)\sigma(x)^2. \tag{3}$$

Call $V(x)$ a particular solution to (3), and $F_1(x)$ and $F_2(x)$ two solutions of the associated homogeneous equation. Then the general solution is

$$F(x; A_1, A_2) = V(x) + A_1 F_1(x) + A_2 F_2(x), \tag{4}$$

where A_1 and A_2 are two integration constants.

We need a method for calculating $F(x; u, v)$, or equivalently $A_1(u, v)$ and $A_2(u, v)$ (section 2) and, when u and v are choice variable, a method for optimizing them (sections 3 and 4).

2. Calculating $F(x; u, v)$ for given u and v

$F(x)$ is defined as the expected value of an integral the kernel of which is a bounded flow. The trajectory of the associated process F therefore cannot be discontinuous except perhaps at the time when $x = u$ and the regulator is applied. At that time, the behavior of F depends on the cost being incurred. If the cost $\mathrm{d}C$ is finite, a jump $\mathrm{d}F$ occurs, with $\mathrm{d}F = \mathrm{d}C$. If the cost is infinitely small, no jump occurs.

We write this requirement as follows:

$$F(u) = F(v) - c(u - v). \tag{5}$$

This equation has frequently been labelled the 'value matching' condition.

In the case where the regulator $\mathrm{d}U = u - v$ is of small magnitude and the cost $\mathrm{d}C$ is infinitely small [$c(0) = 0$ but $c'(0) > 0$; or, in the linear case, $c_0 = 0$, $\rho > 0$],[5] eq. (5) can be rewritten and expanded as follows:

$$F(u) = F(u - \mathrm{d}U) - c(\mathrm{d}U)$$

or

$$F(u) = F(u) - F'(u)\,\mathrm{d}U - c'(0)\,\mathrm{d}U,$$

[5]Mathematically speaking, the processes R and U are specialized to be continuous.

which yields

$$F'(u) = -c'(0). \tag{5'}$$

We see that, in the special case of infinitesimal moves and costs, the value matching condition [which usually involves the function F itself as in (5)] takes the form of a condition on the first derivative F', without any optimization being involved.[6]

At $x=0$, where the infinitesimal regulator $\mathrm{d}L$ is costless, one must have similarly:

$$F'(0) = 0. \tag{6}$$

The simple technical problem one faces is that of solving the ordinary differential equation (3) subject to boundary conditions (5) [or (5′)] and (6). This would be done by substituting the general solution (4) into the said boundary conditions and solving for A_1 and A_2. See Dixit (this issue).

3. Optimizing the regulator

In this section we seek to optimize the choice of u and v when $c(0) > 0$. It could be shown, as Dixit (this issue) did in a special case, that maximizing the performance $F(x; u, v)$ with respect to u and v is equivalent to maximizing $A_1(u, v)$ or $A_2(u, v)$ with respect to u and v. In other words, the derivative of $F(x; u, v)$ with respect to u or v has the same sign for all values of x. Improving the choice of the boundary increases the value of the performance index everywhere.

Using the fact that the partials of A_1 and A_2 must be zero for the optimum to obtain, eq. (5) (which holds for any u and v) – after substitution of the general solution (4) – can be differentiated with respect to u and v, keeping A_1 and A_2 constant:

$$V'(u) + A_1 F_1'(u) + A_2 F_2'(u) = -c'(u - v), \tag{7}$$

$$0 = V'(v) + A_1 F_1'(v) + A_2 F_2'(v) + c'(u - v), \tag{8}$$

which one can rewrite as

$$F'(u) = -c'(u - v), \tag{9}$$

$$0 = F'(v) + c'(u - v). \tag{10}$$

[6]In reference to Krugman (1990), observe that it is improper to refer to (5′) as a 'smooth pasting condition'. At least, this label would not describe the correct economic meaning of this equation.

In general, the solutions for u and v are not equal to each other, despite the similarity between these two equations. This is because the function $F(x)$ is not generally strictly concave. The strict concavity is lost because of the cost function $c(\ )$ incorporating a fixed cost.

Conditions (9)–(10) are the first-order conditions we were seeking. They serve to determine the two unknowns u and v. They are generally referred to as the 'smooth pasting' conditions. They indicate that, not only is the time path of F continuous as one applies the regulator (see section 2), but also, for optimality, the time path of F' must be continuous: $F'(u) = F'(v)$. The marginal indirect expected payoff function takes the same value at the point one jumps from and at the point one jumps to. Moreover, both are opposite to the marginal cost incurred when going from u to v.

4. The case of a purely proportional cost

We now consider the limiting situation where the cost is strictly proportional to the distance $u - v$. The cost per unit distance is called ρ and is a constant. Moreover $r(0) = 0$; there is no fixed cost of regulation. These conditions plus the assumption that the flow payoff $\phi(x)$ is a strictly concave function are enough to guarantee the strict concavity of the optimized expected discounted payoff $F(x)$.[7]

If $F(x)$ is a strictly concave function, $F'(x)$ is strictly monotonic and eqs. (9)–(10) above, which are still valid, imply that the optimal values of u and v are equal to each other: $u^* = v^*$. In mathematical terms, this means that the optimal process U is continuous. The finding makes intuitive sense: under purely proportional cost to regulation there is no sense in taking discrete actions.[8] With infinitesimal costs (or costs), infinitesimal moves are optimal. This is instantaneous (as opposed to impulse) control.

Now, if $u = v$, not only do conditions (9)–(10) merge into one condition, but we have also seen, under eq. (5′) above, that this same condition holds identically for any choice of the trigger point u. We seem to be left without any condition for the optimal choice of u!

Fortunately, we can regenerate an optimality condition. Noting again that optimizing F is equivalent to finding stationary values for A_1 and A_2, (5′) – after substitution of the general solution (4) – can be differentiated with respect to u, keeping A_1 and A_2 constant, to yield

$$V''(u) + A_1 F_1''(u) + A_2 F_2''(u) = 0. \tag{11}$$

We recognize the left-hand side of this equation as $F''(u)$. Hence, we have as

[7]See Harrison and Taksar (1983) who classify this situation as one of 'instantaneous control'.

[8]Except perhaps at $t = 0$ if $x_0 > u$.

an optimality condition:

$$F''(u) = 0. \tag{12}$$

At the level of interpretation, the super contact condition (12) is none other than the natural extension of smooth pasting conditions (9)–(10) to the limiting case of the infinitesimal regulator. Indeed, rewrite and expand (9)–(10) as follows:

$$F'(u) = -\rho \tag{9'}$$

and

$$0 = \rho + F'(u - \mathrm{d}U),$$

or, after expansion,

$$0 = -\rho + F'(u) - F''(u)\,\mathrm{d}U, \tag{10'}$$

which together yield $0 = F''(u)$ as in (12). We see that, in the special case of infinitesimal costs, where infinitesimal moves are optimal, the smooth pasting conditions [which usually involve the function F' as in (9)–(10)] take the form of a condition involving the second derivative F''.

5. Related conditions

Several authors have proposed conditions of optimality which apply mostly to proportional costs and infinitesimal regulators in the above or similar contexts. While different from those we have advanced above, these conditions are, of course, related.

Dumas (1987) and Dixit (1988) have proposed a condition involving the level reached by the unknown value function $F(\ )$ at the point of intervention (a zeroeth-order condition). Dumas (1987) observed that the value function can be extended to the outside of the zone of no intervention, to reflect the possibility that, in the initial situation, x might be started above the intervention level u. If that were the case, it would be best at that time to move x instantaneously to the level u (this is the only exception to the infinitesimal character of the regulator). Hence the value function outside the area is equal to its value on the boundary, $F(u)$, minus ρ times $x - u$ – a straight line. Since the super contact condition (12) says that the function inside the area has a second derivative equal to zero at the point $x = u$, it also implies that the second derivative of F – in addition to the function itself and its first derivative – is continuous at that point. If that is so, the O.D.E. (3) applies not only inside the area $(0 < x < u)$ but also at the boundary point $x = u$.[9]

[9]This sentence is distinctly incorrect in the case of $c(0) > 0$ and non-infinitesimal regulators (section 3 above). See Grossman and Laroque (1990).

Inserting the values $F''(u)=0$ and $F'(u)=-\rho$ into the O.D.E., one finds that the level of the value function at that point is

$$F(u)=(\phi(u)-\rho\mu)/\delta. \tag{13}$$

At the boundary point the value of staying at the barrier forever is given by capitalizing the flow payoff stream $\phi(u)$ plus the marginal boundary cost, which one expect to receive repeatedly if x has an upward drift $\mu>0$.

Bertola (1987, revised 1989) – studying also the case of proportional costs – defines a function $f(x;u)$:

$$f(x;u)\equiv \mathrm{E}\left[\int_0^{\infty} \mathrm{e}^{-\delta t}\phi'(x_t)\,\mathrm{d}t \,|\, x_0=x\right]. \tag{14}$$

$f(x)$ is interpreted as the expected discounted *marginal* payoff generated by the flow payoff function $\phi(\ )$ only, *not* including the cost of regulation accruing at $x=u$. The fact that x is a Brownian motion reflected at $x=0$ and $x=u$ is part and parcel of the definition (14) of the function $f(\ )$. Because of it, $f(\ )$ mechanically (i.e., prior to any optimization) satisfies the condition[10]

$$f'(0)=f'(u)=0. \tag{15}$$

Bertola then writes an optimality condition for u which reads:[11]

$$0=-f(u)+\rho. \tag{16}$$

Cutting x induces a marginal cost ρ and a marginal benefit loss $-f(u)$.

The marginal loss in benefit should normally be computed as the *marginal present value* of future flow payoffs foregone because of one unit of x removed at the present time [$F'(u)$ in our notation]. Instead, $f(u)$ in (16) is equal to the *present value of the marginal* flow payoff[12] foregone because of one unit of x removed at all future times. Integrating by parts[13] and using the fact that $f(\ )$ satisfies boundary condition (15), one could show that $F'(u)=f(u)$. Hence optimality condition (16) becomes formally identical to the first-degree value matching condition (5′). A similar procedure would show that $F''(u)=f'(u)$ when (15) and (16) hold (and *vice versa*) so that (15) becomes formally identical to the super contact condition (12). Conditions

[10] See above the similar eq. (6) which applied to $F(\ )$.

[11] I am grateful to Francisco Delgado for helpful discussions on the subject matter of this paragraph.

[12] Not including future costs at the upper boundary.

[13] See the appendix to Bertola (1987, 1989).

(15)–(16), considered as a whole, are therefore equivalent to (5′)–(12). Ironically, however, in Bertola's approach the respective roles of the two conditions are interchanged: property (5′) of $F(\ )$, which was a mechanical consequence of the definition (2) and the behavior of x, is assimilated to eq. (16) which is an optimality condition, while the super contact optimality condition (12) is assimilated to a mechanical property (15) of the function $f(\ )$.

Harrison (1985), in his original treatise on regulated Brownian motion (ch. 6), considers, as we have, a two-sided regulator placed at levels 0 and u, of which only the upper level u is being optimized. One difference with our setting is that the lower regulator at 0, while not being optimized, is nonetheless costly with a purely proportional cost. The cost per unit of distance at that point is denoted γ. Furthermore $\phi(x)$ is particularized to being a linear function: $\phi(x)=hx$. Using (5′) and (6) (in which the right-hand side 0 is replaced by γ), Harrison solves explicitly for the function $F(x)$:

$$F(x;A_1,A_2)=\frac{h}{\delta}x+\frac{h\mu}{\delta^2}+A_1\exp(\alpha_1 x)+A_2\exp(\alpha_2 x), \tag{17}$$

where A_1 and A_2 are given by

$$A_1\alpha_1\exp(\alpha_1 u)+A_2\alpha_2\exp(\alpha_2 u)=\rho-h/\delta, \tag{18}$$

$$A_1\alpha_1+A_2\alpha_2=\gamma-h/\delta. \tag{19}$$

Now, impose the super contact condition at $x=u$:

$$A_1\alpha_1^2\exp(\alpha_1 u)+A_2\alpha_2^2\exp(\alpha_2 u)=0. \tag{20}$$

Eliminating A_1 and A_2 between (18)–(20) leads to

$$\frac{\alpha_1-\alpha_2}{\alpha_1\exp(-\alpha_2 u)-\alpha_2\exp(-\alpha_1 u)}=\frac{\rho-h/\delta}{\gamma-h/\delta}, \tag{21}$$

which is precisely the optimality condition which Harrison obtained (his eq. 6.3.13, p. 108) by a completely different 'policy improvement' arguement. In the reverse, Harrison himself (see the proof of corollary 6.3.15, p. 108) establishes that (21) implies super contact: $F''(u)=0$. The two conditions are therefore equivalent in his special case.

6. Conclusion

This note put forth two basic ideas:

- Value matching is not an optimality condition, smooth pasting is.
- In the case of a discrete regulator, value matching is expressed in terms of the performance function itself and smooth pasting in terms of its first derivative. But in the case of an infinitesimal regulator the order of differentiation moves up one notch: the value matching condition involves the first derivative of the performance function (which misleadingly makes it look like a smooth pasting condition) and the smooth pasting condition now involves the second derivative, leading to 'super contact'.

References

Benes, V.E., L.A. Shepp, and H.S. Witsenhausen, 1980, Some solvable stochastic control problems, Stochastics 4, 134–160.

Brennan, M.J. and E.S. Schwartz, 1985, Evaluating natural resource investments, Journal of Business 58, 135–157.

Bensoussan, A. and J.-L. Lyons, 1975, Nouvelles méthodes en controle impulsionnel, Applied Mathematics and Optimization Quarterly 1, 289–312.

Bertola, G., 1987, Irreversible investment, Working paper, July (MIT, Cambridge, MA), revised in 1989 (Princeton University, Princeton, NJ).

Constantinides, G.M., 1986, Capital market equilibrium with transactions costs, Journal of Political Economy 94, 842–862.

Davis, M.H.A. and A.R. Norman, 1990, Portfolio selection with transactions costs, Working paper (Imperial College, London), forthcoming in: Mathematics of Operations Research.

Dixit, A.K., 1988, A simplified exposition of some results concerning regulated Brownian motion, Working paper, Aug. (Princeton University, Princeton, NJ), revised in 1990 as: A simplified treatment of the theory of optimal regulation of Brownian motion, Journal of Economic Dynamics and Control, this issue.

Dumas, B., 1987, Time to ship and the dynamic behavior of deviations from the law of one price, Working paper, July (University of Pennsylvania, Philadelphia, PA), revised in March 1988 as: Pricing physical assets internationally, forthcoming in: Review of Financial Studies.

Dumas, B. and E. Luciano, 1989, An exact solution to the portfolio choice problem under transactions costs, Working paper, Dec. (University of Pennsylvania, Philadelphia, PA).

Flood, R.P. and P.M. Garber, 1989, The linkage between speculative attack and target zone models of exchange rates, NBER working paper 2918.

Grossman, S.J. and G. Laroque, 1990, Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods, Econometrica 58, 25–52.

Harrison, J.M. and A.J. Taylor, 1978, Optimal control of a Brownian storage system, Stochastic Processes and Their Applications 6, 179–194.

Harrison, J.M. and M.L. Taksar, 1983, Instantaneous control of Brownian motion, Mathematics of Operations Research 8, 439–453.

Harrison, J.M., T.M. Sellke, and A.J. Taylor, 1983, Impulse control of Brownian motion, Mathematics of Operations Research 8, 454–466.

Harrison, J.M., 1985, Brownian motion and stochastic flow systems (Wiley, New York, NY).

Karatzas, I. and S.E. Shreve, 1984, Connections between optimal stopping and singular stochastic control, I: Monotone follower problem, SIAM Journal of Control and Optimization 22, 856–877.

McKean, H.P., Jr., 1965, Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics, Industrial Management Review 6, no. 2, 32–39.

Krugman, P., 1990, Target zones and exchange rate dynamics, Quarterly Journal of Economics, forthcoming.
Merton, R.C., 1973, Theory of rational option pricing, Bell Journal of Economics and Management Science 4, 141–183.
Samuelson, P.A., 1965, Rational theory of warrant pricing, Industrial Management Review 6, no. 2, 13–31.
Taksar, M., M.J. Klass, and D. Assaf, 1988, A diffusion model for optimal portfolio selection in the presence of brokerage fees, Mathematics of Operations Research 13, 277–294.